$$\beta_g' Z(V) = Z(g^{-1}V), \quad S_v' Z(V) = Z(V). \tag{16}$$

Un champ fondamental $Z(V)$ est un champ fondamental *induit* $Y(v)$, considéré en [1], *Definition* 7.3.

4. Sections globales. *Remarque* 5. L'isomorphisme

$$I : F \to L \times V^n | I(u) = (z, z^{-1}y) \tag{17}$$

considéré en [1]. *Proposition* 6.1. est une conséquence de la proposition suivante :

L'application

$$\sigma_L^o : L \to F | \sigma_L^o(z) = (0, z) \tag{18}$$

est une *section globale* de fibré trivial $F_o(M)$. L'espace total F est par conséquent trivial et difféomorph avec la variété produit $L \times V^n$.

Remarque 6. Le groupe affine GA du fibré principal $A(M)$ est réductible au groupe lineaire G.

Cette afirmation est le résultat des propositions :

a) Le fibré tangent $T(M)$ est un *fibré associé* du fibré $A(M)$.

b) La section nulle

$$\sigma_T^o : x \in M \to \sigma_T^o(x) = 0 \in M_x \tag{19}$$

est une *section globale* du fibré tangent $T(M)$.

Pour chaque $x \in M$, un rèpère affine $u = (y, z) \in \pi_A^{-1}(x)$, détermine un isomorphisme

${}_u\bar{\alpha} : V^n \to M_x | {}_u\bar{\alpha}(v) = y + zv$, son inverse étant ${}_u\bar{\alpha}^{-1} : M_x \to V_n | {}_u\bar{\alpha}^{-1}(X) = z^{-1}(X - y)$, $X \in M_x$. Nous avons défini ainsi l'application $\bar{\alpha} : F \times V^n \to T | \bar{\alpha}(u, v) = y + zv$, d'où on déduit $\bar{\alpha}_v : A \to T | \bar{\alpha}_v(u) = y + zv$ et par conséquent $\bar{\alpha}_v = \pi_1 \circ S_v$.

Remarque 7. L'application

$$\sigma^o = (\sigma_T^o, \sigma_T^o) ; \quad L(M) \to T(M)$$

représente un plongement de fibré $L(M)$ dans le fibré de Finsler $F(M)$. L'image $\sigma^o(L(M))$ est un *sous-fibré fermé* de $F(M)$ et son complémentaire $F(M) - \sigma^o(L(M))$ est un *sous-fibré ouvert* de $F(M)$.

Ces sous-fibrés jouent un rôle important dans la géométrie de Finsler de la variété M, qui constitue un problème d'une étude future.

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THE GROUP OF TRANSFORMATIONS OF FINSLER CONNECTIONS SPECIAL CASES

BY

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§0. Introduction. Let M be an n-dimensional differentiable manifold, and (E, π, M) a vector bundle over M with fibre dimension m. Denote by $\mathcal{X}(E)$ the algebra of vector fields over E, by $\mathcal{X}^*(E)$ the algebra of covector fields on E and by $\Lambda_1^1(E)$ the algebra of the linear mappings of $\mathcal{X}(E)$ on $\mathcal{X}(E)$. Further we shall consider any metric tensor field on E as a section G in $L(TE, T^*E)$, $G : \mathcal{X}^*(E)$ and $G^{-1} : \mathcal{X}^*(E) \to \mathcal{X}(E)$ be the inverse of G such that $G^{-1}(G(X)) = X$ for all ω and X in $\mathcal{X}^*(E)$ respectively in $\mathcal{X}(E)$. In the following sections we give an invariant form to the properties of the Finsler geometric objects, to make possible a global view upon how useful the method of Finsler geometry can be in the study of the geometry of a vector bundle endowed with remarkable geometric structures.

§1. The group of transformations of Finsler connections. Let N be a nonlinear connection and ∇ a Finsler connection on (E, π, M). A transformation of Finsler connection $(N_1 \nabla) \to (\bar{N}_1 \bar{\nabla})$ is given by

$$\bar{N} = N - A, \quad \bar{\nabla}_X^H = \nabla_X^H - D_X^H + \nabla_{A(X)}^V, \quad \bar{\nabla}_X^V = \nabla_X^V - D_X^V. \tag{1.1}$$

Where A is a Finsler tensor field of type $\binom{0\ 1}{1\ 0}$ and

$$D : \mathcal{X}(E) \to \Lambda_1^1(E), \quad X \to D_X$$

such that

$$(D_{X^H} Y^H)^V = 0 ; \quad (D_{X^V} Y^V)^H = 0. \quad (D_{X^V} Y^H)^V = 0 ; \ (D_{X^H} Y^V)^H = 0.$$

If we denote that $D_X^H = D_{X^H}$, $D_X^V = D_{X^V}$ we get simmilar propertis for D_X as for ∇_X. If we denote by D^H, $D^H : \mathcal{X}(E) \to \Lambda_1^1(E)$ and by $D^V : \mathcal{X}(E) \to \Lambda_1^1(E)$ it is possible to rewrite the above conditions : the vertical and horizontal distributions are parallel with respect to D_X^H and D_X^V for any X in $\mathcal{X}(E)$.

Theorem 1.2. *The set of all transformation of Finsler connections of type* (1.1) *and the composition of mappings has a group structure. The group acts effectively and transitively on the set of Finsler connections.*

Further we give the transformation laws for the torsion and curvature tensor fields when $A = 0$.

$$\begin{cases} \bar{T}(X, Y) = T(X, Y) - (D_X Y - D_Y X), \\ \bar{R}(X, Y) = R(X, Y) - (\tilde{\nabla}_X D)_Y + (\tilde{\nabla}_Y D)_X + D_X \circ D_Y - D_Y \circ D_X - D_{T(X,Y)} \end{cases}$$

Where
$$\begin{cases} \widetilde{\nabla}_X D = {}^t\nabla_X D - D\circ\nabla_X \\ {}^t\nabla_X D(Y) = \nabla_X\circ D_Y - D_Y\circ D_X \end{cases}$$
for any X, Y in $\mathcal{X}(E)$

We recall the classic definition of the Obata operators and extend it to our case by the equality:

$$\Lambda(u) = 1/2(u - G^{-1}\circ{}^t u\circ G), \quad \Lambda^*(u) = 1/2(u + G^{-1}\circ{}^t u\circ G),$$

where $u\in\Lambda_1^1(E)$ and ${}^t u$ is the adjoint of the operator of u.

Proposition 1.3. *The curvature tensor field R, associated to a Finsler connection and the Obata operator Λ^* satisfy the following identity:*

$$(1.4)\quad \begin{cases} {}^t R(X,Y)\circ G = -G\circ R(X,Y), \ \Lambda^*(R(X,Y)) = 0 \\ \text{for any } X, Y\in\mathcal{X}(E). \end{cases}$$

§2. Metric Finsler connections. Definition 2.1. *A Finsler connection is a metric Finsler connection if the following conditions hold:*

(1) $\nabla_X(V)\subset V$ for any X in $\mathcal{X}(E)$,

(2) $\widetilde{\nabla}_X G = 0$ for any X in $\mathcal{X}(t)$ and $\widetilde{\nabla}_X G = {}^t\nabla_X G - G\circ\nabla_X$.

Let $(\mathring{\nabla}, \mathring{N})$ be a Finsler connection. A transformation of Finsler metric connection is given by:

$$(2.2)\quad \begin{cases} N = \mathring{N} - A, \ \nabla_X^H = \mathring{\nabla}_X^H + \mathring{\nabla}^V_{A(X^H)} + 1/2G^{-1}(\tilde{\mathring{\nabla}}_X^H G + \tilde{\mathring{\nabla}}^V_{A(X^H)}G) + \Lambda(W^H(X)), \\ \nabla_X^V = \mathring{\nabla}_X^V + 1/2G^{-1}(\tilde{\mathring{\nabla}}_X^V G) + \Lambda(W^V(X)). \end{cases}$$

where $W : \mathcal{X}(E)\to\Lambda_1^1(E)$, $W^H(X) = W(X^H)$, and $W^V(X) = W(X^V)$ and the vertical and horizontal distributions are parallel with respect to W_X^V and W_X^H. Given a linear connection ∇ on E one can parallelly propagate frames. For any path τ between two points in E the parallel transport along τ defines a linear mapping $L(\tau)$ between tangent spaces at any two points of the path. $L(\tau)$ is an isometry if there is a metric structure G such that $\tilde{\nabla}_X G = 0$ for all X in $\mathcal{X}(E)$.

The holonomy group $\Phi(w)$ of a point $w\in E$ is the group of linear transformations in the tangent spaces at W defined by parallel transport along loops starting at w.

If ∇ is a Finsler metric connection with respect to a metric structure G, then its holonomy group is a subgroup of the generalized orthogonal group corresponding to the signature of G.

Theorem 2.3. *Let ∇ be a linear connection on E, whose holonomy group keeps a nondegenerate quadratic form G_0 and the vertical subspace of the tangent space invariant. Then ∇ is a metric Finsler connection with respect to G which has the same signature. as G.*

Remark: If E is simply connected than the thorem above can be reformulated in terms of the Lie algebra of the holonomy group $\Phi(w)$.

§3. Metric semi-symmetric Finsler connections. Definition 3.1. A Finsler connection is called semi-symmetric if the torsion field satisfy the following relations

$$(3.2)\quad \begin{cases} T^H(X^H, Y^H) = 1/(n-1)[t(Y^H)X^H - t(X^H)Y^H], \\ T^V(X^V, Y^V) = 1/(m-1)[t(Y^V)X^V - t(X^V)Y^V], \end{cases}$$

and denote by $\eta = 1/(n-1)t^H$, $t^H(X) = t(X^H)$ and $\sigma = 1/(m-1)t^V$, $t^V(X) = t(X^V)$, $t^H(X) = \lambda^i(T(X_i, X^H))$; $t^V(X) = \lambda^a(T(X_a, X^V))$, where $(\lambda^i)_{i=1,n}$; $(\lambda^a)_{a=1,m}$ and $(X_i)_{i=1,n}$, $(X_a)_{a=1,m}$ are dual bases in HT^*E, VT^*E, HTE, VTE respectively. Hence, t is the extension of the Vrănceanu 1-torsion form.

The relation (3.2) becomes:

$$(3.2)\quad T^H(X^H, Y^H) = \eta(Y^H)X^H - \eta(X^H)Y^H, \quad T^V(X^V, Y^V) = \sigma(Y^V)X^V - \sigma(X^V)Y^V,$$

where $\sigma, \eta\in\mathcal{X}^*(E)$ and $\eta^V = 0$; $\sigma^H = 0$. The metric Finsler connection ∇, given in [8] will be considered semi-symmetric with $\overset{c}{T}{}^H(X^H, Y^H) = 0$ and $\overset{c}{T}{}^V(X^V, Y^V) = 0$.

Theorem 3.3. The set of all semisymmetric metric Finsler connections is given by

$$(3.3)\quad \nabla_X^H = \overset{c}{\nabla}{}_X^H + 2\Lambda(\eta_{X^H}), \quad \nabla_X^V = \overset{c}{\nabla}{}_X^V + 2\Lambda(\sigma_{X^V})$$

with

$$\eta_{X^H}(Y) = \eta(Y)X^H; \quad \sigma_{X^V}(Y) = \sigma(Y)X^V.$$

Further we shall define a tensor field H invariant under the transformations of the form above. First define:

$$\boldsymbol{K}(X^H, Y^H) = R(X^H, Y^H) - \nabla_{[X^H, Y^H]^V},$$

$$K(X^V, Y^V) = R(X^V, Y^V), \quad K(X^H, Y^V) = R(X^H, Y^V) - \nabla_{T(X^H, Y^V)},$$

which have the transformation laws derived from (3.3) given below

$$\overline{\boldsymbol{K}}(X^H, Y^H) = \boldsymbol{K}(X^H, Y^H) + 2\Lambda((\mu_{X^H})_{Y^H} - (\mu_{Y^H})_{X^H},$$

$$\overline{\boldsymbol{K}}(X^V, Y^V) = \boldsymbol{K}(X^V, Y^V) + 2\Lambda((\sigma_{X^V})_{Y^V} - (\sigma_{Y^V})_{X^V}),$$

$$\overline{\boldsymbol{K}}(X^H, Y^V) = K(X^H, Y^V) + 2\Lambda((\nabla_{X^H}\sigma)_{Y^V} - (\nabla_{Y^V}\eta)_{X^H}).$$

Theorem 3.4. *For $n>2$ and $m>2$ the following Finsler tensor field is an invariant of the group of transformation of Finsler metric semisymmetric connections.*

$$H(X^H, Y^H) = \boldsymbol{K}(X^H, Y^H) + 2/(n-2)\Lambda((K_{X^H})_{Y^H} - (K_{Y^H})_{X^H}) + \mathcal{K}^H/[(n-1)(n-2)].$$
$$.\Lambda((G_{X^H})_{Y^H} - (G_{Y^H})_{X^H}).$$

$$H(X^V, Y^V) = K(X^V, Y^V) + 2/(m-2)\Lambda((K_{X^V})_{Y^V} - (K_{Y^V})_{X^V}) + \mathcal{K}^V/[(m-1)$$
$$(m-2)].\Lambda((G_{X^V})_{Y^V} - (G_{Y^V})_{X^V}).$$

$$H(X^H, Y^V) = K(X^H, Y^V) + 2/[(m-1)(n-1)]((m-1)(\tilde{K}_{X^H})_{Y^V} - (n-1)((\tilde{K}_{Y^V})_{X^H})$$

where we have denoted :

$$K_{X^H}(Z)=G(K(v_h, X^H), v_h),\ K_{X^V}(Z)=G(K(v_a, X^V), v_a)$$

when (v_h) and (v_a) are orthonormal basis in HTE respectively VTE, and

$$\mathcal{K}^H=K(v_h, v_h)\ ;\ \mathcal{K}^V=K(v_a, v_a)$$

and

$$\tilde{K}_{X^H}(z)=G(K(v_a, X^H)Z, v_a),\quad \tilde{K}_{X^V}(z)=G(K(v_i, X^V)z, v_i).$$

By direct computation one gets $\overline{H}=H$.

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ON FINSLER SPACES WITH PROPERTY $\mathcal{H}$

BY

TOSHIO SAKAGUCHI

Introduction. Recently H. I z u m i and M. Y o s h i d a [4] introduced the notion of property $\mathcal{H}$ and proved that a Finsler space of scalar curvature (accordingly of *s-ps* curvature), an $R3$-like Finsler space and a Finsler space with $F_h{}^i{}_{jk}=0$ have the property $\mathcal{H}$. Also they proved that the notion of property $\mathcal{H}$ is preserved under any h-conformal transformation.

The purpose of the present paper is to investigate the Finsler space with property $\mathcal{H}$ in detail. In § 1 we give the fundamental properties of the Finsler space with property $\mathcal{H}$. §2 is devoted to find the conditions that the Finsler space with property $\mathcal{H}$ turns into the $R3$-like Finsler space or the Finsler space with $F_h{}^i{}_{jk}=0$. Also in §2 we consider the Finsler space with property $\mathcal{H}$ which satisfies $P.\ P_{hij|k}-j|k=0$.

The notations and definitions in the present paper are referred to the paper of H. I z u m i and T. S a k a g u c h i [2].

§1. Fundamental properties of a Finsler space with property $\mathcal{H}$. First we know the following

Theorem 1.1. (H. Izumi and T. N. Srivastava [3]). *If in a Finsler space we denote* $\alpha^i_k:=F^{-2}H^i_k$ *and* $\alpha^i_{jk}:=FP.\ \alpha^i_{k(j)}$, *then we have*

$$(1.1)\qquad H^i_{jk}=F(1_j\alpha^i_k+(1/3)\alpha^i_{jk})-j|k,\ \textit{where}\ \alpha_{ik}=\alpha_{ki},\ \alpha_{ik}=\alpha^h_k g_{hi}.$$

Definition (H. Izumi and M. Yoshida [4]). In a Finsler space, if the indicatrix tensor α^i_{jk} satisfies the condition of the form

$$(1.2)\qquad \alpha^i_{jk}-\alpha^i_{kj}=\beta_j h^i_k-\beta_k h^i_j,$$

then it is called that the Finsler space has the *property* $\mathcal{H}$.

The following two propositions are proved by H. I z u m i and M. Y o s h i d a [4].

Proposition 1.2. *In a Finsler space* F_n $(n>2)$, *the property* $\mathcal{H}$ *is characterized by*

$$(1.3)\qquad \mathcal{H}^i_{jk}:=Z^i_{jk}-[1/(n-2)](Z_jh^i_k-Z_kh^i_j)=Z^i_{jk}-(F/3)(\beta_jh^i_k-\beta_kh^i_j)=0$$

where $Z^i_{jk}:=P.\ H^i_{jk}$ *and* $Z_j:=Z^m_{jm}$.

Proposition 1.3. *A Finsler space with property* $\mathcal{H}$ *satisfies*

$$(1.4)\qquad Z^m_{hj}C_{mik}+Z^m_{ik}C_{mhj}-j|k=0.$$

From (1.1) and (1.2) we have