Ladies and Gentlemen,
Dear Colleagues

By the endeavour of some enthusiastic finslerists, this Romanian-Japanese Colloquium on Finsler geometry is finally under way.

Our guests, famous scientists, professors: Makoto Matsumoto, Masao Hashiguchi, Masaki Fukui, Shun-ichi Hojo, Yoshihiro Ichijyo, Fumio Ikeda, Satoshi Ikeda, Hideo Izumi, Toshio Sakaguchi, Choko Shibata, Hideo Shimada, Shoji Watanabe, Hiroshi Yasuda as well as Mrs Hashiguchi and Mrs Watanabe, travelled a long and tiresome way in order to participate in the first international meeting of large audience devoted to pure Finsler geometry.

We wish them welcome to Romania and great success in the works of the Colloquium. I address my heartful thanks to Professor Masao Hashiguchi, the main supporter of our scientific meeting.

Romanian geometers feel honored by the presence of some members of best school of Finsler geometry in the whole world, the Japanese one. They consider that direct dialogue with the Japanese colleagues in the oldest Romanian University, the „Al. I. Cuza" University followed by the discussions at the Universities of Braşov and Bucureşti, may determine a real progress in a field of the geometry with a brilliant theoretic and applicative perspective.

Much honoured guests from Japan! We know very well your fruitful achievements. They are fundamental in Finsler geometry and its applications. They culminate with the illustrious professor Makoto Matsumoto's monograph Foundations of Finsler Geometry and Special Finsler Spaces. *We highly appreciate them.*

No doubt you are informed that the study of Finsler geometry in Romania has a tradition of half a century. The first paper written by a Romanian is Formules fondamentales dans la théorie des hypersurfaces d'un espace de Finsler *and was published in Comptes Rendus de l'Acad. Sci., Paris, 1934. It represents the beginnings of subspaces of Finsler space theory. We express our grief that the author of this paper, „the father of Romanian finslerists", Academician Mendel Haimovici died 11 years ago. He would have reached today an age allowing him to organise personally this Romanian-Japanese Colloquium. I am sure that he would have done it with great pleasure and competence.*

In the last 50 years, the study of Finsler geometry in Romania followed an ascending trajectory, tangent to some new directions and tendencies, marked by three National Seminars and by the profitable collaboration of the Japanese School.

All these will be common bridges of approaching modern problems interesting the community of the finslerists over the world.

Today, when the mankind looks anxiously to the future, let us hope that Romanian-Japanese Colloquium of Finsler geometry will be a flower of the friendship of our people, in a better and more generous world.

Radu Miron

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ON GENERALIZED MIRON-BERWALD CONNECTIONS

BY

TADASHI AIKOU and MASAO HASHIGUCHI

In the spring last year the second author enjoyed studying in Iaşi at the invitation of Prof. R. Miron. Then he was invited to the various conferences and honored to have opportunities to talk about topics of our mutual interest [3]. The significance of axioms in Finsler geometry was especially emphasized by taking account of generalized Berwald spaces [2, 5] and generalized Finsler spaces [9,4].

By Prof. M. Matsumoto [7], generalized Berwald spaces are concerned with Finsler connections of Berwald type with surviving (h) h-torsion $T_j{}^i{}_k$. On the other hand, in his recent paper [8], which is read to this Colloquium, Prof. Matsumoto has treated Finsler connections induced on a hypersurface of a Finsler space and then obtained interesting Finsler connections of Berwald type with surviving (v) hv-torsion $P^i{}_{jk}$.

In the present lecture, continued from [3], we would like to discuss Finsler connections of Berwald type generally (§1) and to consider what kind of Finsler connection should be reasonable as a generalization of the Berwald connection (§2). For fuller particulars refer to our appearing paper [1]. And lastly (§3) we shall notice that the results are applicable to the case of generalized Finsler spaces. The terminology and notations are referred to Matsumoto's monograph [6].

§1. L-Metrical Finsler Connections. We are first concerned with a Finsler space. Let L and g_{ij} denote the fundamental function and the fundamental tensor respectively. And we shall express a Finsler connection by its coefficients $(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$.

In a Finsler space there are known two canonical Finsler connections, that is, the Cartan one, $C\Gamma$, and the Berwald one, $B\Gamma$. They are uniquely determined by the following systems of axioms:

$C\Gamma$ (Matsumoto [6])	$B\Gamma$ (Okada [11])
(C1) $g_{ij\mid k}=0$,	(B1) $L_{\mid k}=0$,
(C2) $D^i{}_k \equiv F_0{}^i{}_k - N^i{}_k = 0$,	(B2) $D^i{}_k=0$
(C3) $T_j{}^i{}_k \equiv F_j{}^i{}_k - F_k{}^i{}_j = 0$,	(B3) $T_j{}^i{}_k=0$,
(C4) $S_j{}^i{}_k \equiv C_j{}^i{}_k - C_k{}^i{}_j = 0$,	(B4) $P^i{}_{jk} \equiv \dot\partial_k N^i{}_j - F_k{}^i{}_j = 0$,
(C5) $g_{ij\|k}=0$,	(B5) $C_j{}^i{}_k=0$.

Omitting some axioms from each of the systems above we have various Finsler connections of Cartan type and of Berwald type. Especially, a Finsler connection satisfying (C1), (C5) is called *metrical*. So, we shall call a Finsler connection satisfying (B1), (B5) *L-metrical*. We have their general forms.

Theorem 1. (Miron-Hashiguchi [10]). *A metrical Finsler connection can be expressed in terms of its D^i_k, $T_j{}^i{}_k$ and $S_j{}^i{}_k$:*

$$N^i_k = G^i_k - X^i_k\,,$$

$$F_j{}^i{}_k = \Gamma_j{}^i{}_k + g_j{}^i{}_m X^m{}_k + g_k{}^i{}_m X^m{}_j - g_j{}^m{}_k X_m{}^i + T^*{}_j{}^i{}_k,$$

$$C_j{}^i{}_k = g_j{}^i{}_k + S^*{}_j{}^i{}_k\,,$$

where $C\Gamma = (\Gamma_j{}^i{}_k, G^i_k, g_j{}^i{}_k)$ and

$$X^i_k = D^i_k + g^i_{km}(T_{00}{}^m - D^m{}_0) - T^*{}_0{}^i{}_k, \quad T^*{}_j{}^i{}_k = (T_j{}^i{}_k - T^i{}_{jk} + T_{jk}{}^i)/2,$$

$$S^*{}_j{}^i{}_k = (S_j{}^i{}_k - S^i{}_{jk} + S_{jk}{}^i)/2.$$

Theorem 2. (Aikou-Hashiguchi [1]). *An L-metrical Finsler connection can be expressed in terms of its D^i_k, $T_j{}^i{}_k$ and $P^i{}_{jk}$:*

$$N^i_k = G^i_k - ((P^i{}_{0k} + T_k{}^i{}_0 + D^i_k) + \dot\partial_k(P_{00}{}^i + T^i{}_{00} + D_0^i))/2,$$

$$\overline{F}_j{}^i{}_k = \dot\partial_j N^i{}_k - P^i{}_{kj}, \quad C_j{}^i{}_k = 0.$$

§2. Generalized Berwald Connections. Tensors D^i_k, $T_j{}^i{}_k(=-T_k{}^i{}_j)$ and $S_j{}^i{}_k(=-S_k{}^i{}_j)$ in Theorem 1 can be given arbitrarily. In Theorem 2, however, matters are different. In the following, we shall assume Finsler connections to be p-homogeneous. Then any Finsler connection satisfies

$$D^i_k + P^i{}_{k0} = 0, \tag{1}$$

$$y^r(\dot\partial_k Q_j{}^i{}_r - \dot\partial_j Q_k{}^i{}_r) = 0. \tag{2}$$

where $Q_j{}^i{}_k = T_j{}^i{}_k - (P^i{}_{jk} - P^i{}_{kj})$. The converse of Theorem 2 is given as

Theorem 3. *Given a (1) p-homogeneous tensor D^i_k and (0) p-homogeneous tensors $T_j{}^i{}_k(=-T_k{}^i{}_j)$, $P^i{}_{jk}$ satisfying the conditions (1), (2), there exists a unique L-metrical Finsler connection whose deflection, (h) h- and (v) hv-torsion tensors are the given D^i_k, $T_j{}^i{}_k$ and $P^i{}_{jk}$ respectively.*

A Finsler connection satisfying (C1) ~ (C5) but (C3) is called a *generalized Cartan connection* $GC\Gamma$ (Hashiguchi-Ichijyō [5]), and a Finsler connection satisfying (B1) ~ (B5) but (B3) is called a *Berwald connection with torsion* $B\Gamma T$ (Matsumoto [7]). Such Finsler connections with surviving $T_j{}^i{}_k$ are very important, because the remarkable class of Finsler spaces called generalized Berwald spaces is defined with respect to each of these connections.

Now, let $T_j{}^i{}_k(=-T_k{}^i{}_j)$ be an arbitrary (0) p-homogeneous tensor. There exists a $GC\Gamma$ whose (h) h-torsion is $T_j{}^i{}_k$, but we can not necessarily get a $B\Gamma T$ whose (h) h-torsion is $T_j{}^i{}_k$. If we abandon (B4), it is shown that there exists a Finsler connection of Berwald type whose (h) h-torsion is $T_j{}^i{}_k$. Such Finsler connections with surviving $T_j{}^i{}_k$ and (or) $P^i{}_{jk}$ appear in Prof.

Matsumoto's theory of hypersurfaces [8]. So, as a typical generalization of $B\Gamma$ we shall call a Finsler connection satisfying (B1), (B2), (B5) a *generalized Berwald connection* $GB\Gamma$.

This definition has various advantages. For example,

Theorem 4. *A $GC\Gamma$ is characterized as a Finsler connection $(F_j{}^i{}_k, F_0{}^i{}_k, g_j{}^i{}_k)$ satisfying $g_{ij|k}=0$, and a $GB\Gamma$ is characterized as a Finsler connection $(F_j{}^i{}_k, F_0{}^i{}_k, 0)$ satisfying $L_{|k}=0$.*

The so-called C-zero connection of a $GC\Gamma$ is a $GB\Gamma$. Especially, the Rund connection is a $GB\Gamma$ thus obtained from $C\Gamma$.

A Finsler connection is called *linear*, if $F_j{}^i{}_k$ depend on position alone. A Finsler space is called a *generalized Berwald space*, if we can introduce a linear $GC\Gamma$ [5]. By Prof. Matsumoto [7], it is also defined as a Finsler space such that we can introduce a linear $B\Gamma T$. If a $GB\Gamma$ is linear, then $P^i{}_{jk}=0$. So, we can define such a space in terms of a $GB\Gamma$ with weaker conditions.

Theorem 5. *A Finsler space is a generalized Berwald space, if we can introduce a linear $GB\Gamma$.*

§3. Generalized Miron-Berwald Connections. In his paper [9] Prof. Miron wrote, „A Finsler space is sometimes adopted as a basic concept in the theoretical physics. However, the fact that the fundamental tensor field of a Finsler space is provided from a positively homogeneous function is not always desirable for physicists, as pointed out by several authors". And he treated a generalized Finsler metric g_{ij}, which is a symmetric and non-degenerate tensor field of type (0, 2). We shall assume that g_{ij} is (0) p-homogeneous and satisfies his regularity conditions:

(a) $(\dot\partial_k g_{ij})y^i y^j = 0$, (b) $\det(A^i_k) \neq 0$,

where $A^i_k = \delta^i_k + g^{im}(\dot\partial_k g_{jm})y^j$. We shall call a space M with such a g_{ij} a *generalized Finsler space.*

Since (A^i_k) has the inverse (B^i_k), we can put $G^i = B^i_k \gamma_0{}^h{}_0/2$, where $\gamma_j{}^i{}_k$ denote the Christoffel symbols formed with respect to g_{ij}. Prof. Miron gave two canonical Finsler connections. The one is uniquely determined by the system of axioms for $C\Gamma$ in which (C2)is replaced by $N^i_k = \dot\partial_k G^i$. So we shall call this connection the *Miron-Cartan connection* $MC\Gamma$. The other is given by $(\dot\partial_j \dot\partial_k G^i, \dot\partial_j G^i, 0)$. We shall call this the *Miron-Berwald connection* $MB\Gamma$. Putting $L=\sqrt{(g_{ij}y^i y^j)}$, we have a Finsler space (M, L). Then we have

Theorem 6 [4]. *$MB\Gamma$ of (M, g_{ij}) is just $B\Gamma$ of (M, L).*

Thus $MB\Gamma$ is characterized by the same system of axioms for $B\Gamma$ of a Finsler space. Therefore, we can define a *generalized* $MB\Gamma$ of (M, g_{ij}) as a $GB\Gamma$ of (M, L) and apply the results about $GB\Gamma$.

In a forthcoming paper we should discuss the relations between $MC\Gamma$ and $MB\Gamma$ and their generalizations. The peculiarity $D^i_k \neq 0$ of $MC\Gamma$ is at once its strength and weakness.

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FINSLER HYPERCONNECTIONS

BY

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The theory of the hypertangent bundle and linear hyperconnections was considered in [2], for their application in the so-called unified field theory of gravitation and electromagnetism. In this paper, we define and study the principal properties of the Finsler hypertangent bundle and the Finsler hyperconnections.

§1. Finsler hypertangent bundle. Let M be an n-dimensional differentiable manifold and $T=(TM, \pi_T, M)$ the tangent bundle of M (with fibre type R^n). Let $J=(JM, \pi_J, M)$ be an N-dimensional differentiable vector bundle over M with fibre type $R^N (N>n)$. We suppose that there exists in J a subbundle $T'=(T'M, \pi_{T'}, M)$ which is differentiably equivalent to T. Since T is imbedded in J as a subbundle, there exists a subbundle $\bar{T}=(\bar{T}M, \pi_{\bar{T}}, M)$ with fibre type R^{N-n}, which is complementary to T in J. The vector bundle J is called an N-dimensional *hypertangent bundle* over M and the subbundle $\bar{T}$ is called a *vertical vector bundle*.

Let J be an hypertangent bundle over M. We call the *Finsler hypertangent bundle*, the Finsler bundle associated to the pair of vector bundle (T, J), [1], i.e., the vector bundle $FJ=(\pi_T^* JM, \pi_{FJ}, TM)$. If T is imbedded in J, then the Finsler tangent bundle of M, $FT=(\pi_T^* TM, \pi_{FT}, TM)$, is imbedded in FJ and there exists a subbundle $F\bar{T}$ of FJ which is complementary to FT. $F\bar{T}$ is the Finsler bundle associated to $(T, \bar{T})$, namely, $F\bar{T}=(\pi_T^* \bar{T}M, \pi_{F\bar{T}})$ with fibre type R^m, $m=N-n$. $F\bar{T}$ is called a *Finsler vertical bundle*.

Let $(\tilde{U}, \varphi)$ be a structural chart of FJ, $\tilde{x}=(x^i, y^i)$, $i=\overline{1, n}$, the local coordinates of TM in $\tilde{U}$ and (x^i, y^i, z^α), $\alpha=\overline{1, N}$, the local coordinates of FJ in $\pi_{FJ}^{-1}(\tilde{U})$. Let $s_i(\tilde{x})=\varphi^{-1}(\tilde{x}, e_i)$, $i=\overline{1, n}$, $\tilde{x}\in\tilde{U}$ be n linearly independent cross-sections of FT, where $\{e_i\}$ is the canonical base of R^n and $s_a(\tilde{x})=\varphi^{-1}(\tilde{x}, e_a)$, $a=\overline{n+1, N}$, $N=n+m$, m linearly independent cross-sections of $F\bar{T}$, where $\{e_a\}$ is a complementary base of $\{e_i\}$ in R^N. We put $E_\alpha(\tilde{x})=\varphi^{-1}(\tilde{x}, e_\alpha)$, $\alpha=\overline{1, N}$, $\forall \tilde{x}\in U$, where $\{e_\alpha\}$ is the base $\{e_i, e_a\}$ of R^N. We obtain N linearly independent cross-sections of FJ on $\tilde{U}$. If $(\tilde{U}', \varphi')$ is another structural chart of FJ with $\tilde{U}\cap\tilde{U}'\neq\varnothing$, then we have the following transformation rules:
$x^{i'}=x^{i'}(x^1,\dots,x^n)$, $y^{i'}=(\partial x^{i'}/\partial x^i)y^i$, $z^{\alpha'}=M_\alpha^{\alpha'}z^\alpha$, $E_\alpha(\tilde{x})=M_\alpha^{\alpha'}(\tilde{x})E_{\alpha'}(\tilde{x})$, where $M: \tilde{U}\cap\tilde{U}'\to Gl(n, R)$. We have: