Analele ştiinţifice
ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXX, s. I a, Matematică, 1984 — 4

Romanian-Japanese Colloquium
on Finsler Geometry, Iaşi,
Romania, 16—25 August, 1984

ON FINSLER SPACES OF CONSTANT POSITIVE CURVATURE

BY

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In last time we had several interesting papers appeared concerning Finsler spaces of scalar curvature and of constant curvature ([4], [5], [10]), concepts which Funk [2] and Berwald [1] introduced in the early years of Finsler geometry.

We consider some examples of Finsler spaces of constant positive curvature, and study Finsler spaces of constant curvature which are not necessarily projectively flat.

The terminology and notations are referred to Matsumoto [3].

§1. **The torsion tensor** R^i_{jk}. We consider two Finsler spaces $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ on a common underlying manifold M^n. Let $F\Gamma = (N, F, C)$ (resp. $F\bar{\Gamma} = (\bar{N}, \bar{F}, \bar{C})$ be the Finsler connection of F^n (resp. $\bar{F}^n$). Then we have a (0) p-homogeneous tensor $Y_j{}^i{}_k$ satisfying

$$\bar{F}_j{}^i{}_k - F_j{}^i{}_k = Y_j{}^i{}_k . \tag{1.1}$$

Assume that both of the deflection tensors $D^i{}_j$ of F^n and $\bar{D}^i{}_j$ of $\bar{F}^n$ vanish identically. Contracting (1.1) by $y^j y^k$, we have

$$\bar{G}^i - G^i = Y^i, \quad (F_0{}^i{}_0 = 2G^i, \quad Y_0{}^i{}_0 = 2Y^i). \tag{1.2}$$

Here if we treat the Y^i written in the decomposed form $Y^i = p y^i + X^i$, where p is a (1) p-homogeneous scalar and X^i is a (2) p-homogeneous vector, then (1.2) is rewritten in the form

$$\bar{G}^i = G^i + p y^i + X^i. \tag{1.2$'$}$$

We shall restrict our consideration to the Berwald connection $B\Gamma = (G^i{}_j, G_j{}^i{}_k, 0)$. Since $\bar{G}^i{}_j = \dot{\partial}_j \bar{G}^i$ and $\bar{G}_j{}^i{}_k = \dot{\partial}_k \bar{G}^i{}_j$, (1.2) yields

$$\begin{aligned} &\bar{G}^i{}_j = G^i_j + p_j y^i + p\delta^i{}_j + X^i{}_j, \quad p_j = \dot{\partial}_j p, \quad X^i{}_j = \dot{\partial}_j X^i, \\ &\bar{G}_j{}^i{}_k = G_j{}^i{}_k + p_{jk} y^i + p_j \delta^i{}_k + p_k \delta^i{}_j + X_j{}^i{}_k, \quad X_j{}^i{}_k = \dot{\partial}_k X^i{}_j . \end{aligned} \tag{1.3}$$

Therefore the (v) h-torsion tensor $\bar{R}_j{}^i{}_k$ of $B\bar{\Gamma}$ is given by

$$\begin{aligned} R_j{}^i{}_k = R_j{}^i{}_k + \mathfrak{A}_{(jk)} \{ y^i (P_{j;k} + p_{kr} X^r{}_j) + \delta^i_j (p_{;k} - p p_k - \\ - P_r X^r{}_k) + X^i{}_{j;k} + X_k{}^i{}_r X^r{}_j \}, \end{aligned} \tag{1.4}$$

where we denote by (;) the h-covariant differentiation with respect to $B\Gamma$. From (1.4) we obtain

$$\text{(1.5)}\quad \begin{aligned} &1)\ R_0{}^i{}_k-R_{00}h^i_k/(n-1)=\{\bar{R}_0{}^r{}_k-(\bar{R}_{00}-Z^m{}_m)\delta^r_k/(n-1)-Z^r_k\}h^i{}_r,\\ &2)\ \bar{R}_0{}^i{}_k-\bar{R}_{00}\bar{h}^i_k/(n-1)=\{R_0{}^r{}_k-(R_{00}+Z^m{}_m)\delta^r_k/(n-1)+Z^r_k\}\bar{h}^i{}_r,\end{aligned}$$

where $\bar{h}_{ij}$ is the angular metric tensor of the Finsler space $\bar{F}^n$ and we put

$$\text{(1.6)}\quad Z^i_k=2X^i{}_{;k}-X^i_{k;0}+2X^i_{k\,r}X^r-X^i{}_rX^r_k\,.$$

§2. Finsler Spaces of Scalar Curvature. A Finsler space $F^n=(M^n, L)$ of scalar curvature, is characterized by the following condition

$$\text{(2.1)}\quad R_{h0k}=KL^2h_{hk},\quad K=R_{00}/(n-1)L^2,$$

and it is of constant curvature if, furthermore, the scalar K is constant. From (1.5) and (2.1) we have

Proposition 1. *Assume that the tensor Z^i_k given by (1.6) vanishes. Then the Finsler space $\bar{F}^n$ equipped with the relation (1.2′) is of scalar curvature if and only if the Finsler space F^n is also of scalar curvature.*

Here if the equation (1.2′) is obtained from a projective change of the metric, then the vector X^i vanishes, and Z^i_k vanishes also. Thus

Theorem (Szabó). *Let F^n be a Finsler space of scalar curvature. Then the Finsler space $\bar{F}^n$, obtained from F^n by a projective change of the metric, is also of scalar curvature.*

Next, we assume that the Finsler space $\bar{F}^n$ is of scalar curvature. Then (1.5) 1) yields

$$\text{(2.2)}\quad R_0{}^i{}_k=(P_{;0}-p^2-2P_rX^r+K\bar{L}^2)h^i_k+Y^iZ_{0k}/L^2-Z^i_k\,.$$

Proposition 2. *The Finsler space $\bar{F}^n=(M^n, \bar{L})$ equipped with the relation (1.2′) is of scalar curvature if and only if the condition (2.2) holds.*

Here we shall restrict our consideration to the case where the F^n is a Riemannian space $(M^n, \alpha=(a_{ij}(x)y^iy^j)^{1/2})$. Then (2.2) implies

$$\text{(2.3)}\quad R_{0h0k}=B(a_{hk}-a_ha_k/\alpha^2)+H_{hk},$$

where we put

$$\text{(2.3)}\quad \begin{aligned} &1)\quad B=K\bar{L}^2+p_{;0}-P^2-2P_rX^r,\quad H_{hk}=Z_{hk}+a_hZ_{0k}/\alpha^2,\\ &\qquad a_h=a_{hm}y^m,\quad Z_{hk}=a_{hr}Z^r_k\,.\end{aligned}$$

Moreover the curvature tensor of a Riemannian space F^n is given by

$$\text{(2.4)}\quad \begin{aligned} R_{hijk}=&B(a_{hj}a_{ik}-a_{hk}a_{ij})/3\alpha^2+\mathfrak{A}_{(hi)}\{\mathfrak{A}_{(jk)}\{B_{hj}h_{ik}+B_j(a_{ik}a_h-\\ &-a_{hk}a_i)/\alpha+2Ba_{ij}a_ha_k/\alpha^2\}+H_{ikhj}-H_{ijhk}\}/6,\end{aligned}$$

where we put $H_{ijhk}=\dot{\partial}_h\dot{\partial}_kH_{ij}$, $B_j=\dot{\partial}_jB$, $B_{ji}=\dot{\partial}_iB_j$.

Theorem 1. *Let $F^n=(M^n, L=\alpha)$ be a Riemannian space. Then the Finsler space $\bar{F}^n=(M^n, \bar{L})$ equipped with the relation (1.2′) is of scalar curvature K if and only if the Riemannian curvature tensor R_{hijk} is written as (2.4).*

Corollary 1.1. *Let a space F^n be Riemannian.*

(1) Assume that the vector X^i vanishes. Then the Finsler space $\bar{F}^n=(M^n, \bar{L})$ equipped with the relation (1.2′) is of constant curvature K if and only if the Riemannian space F^n is of constant curvature, provided $n>2$.

(2) Assume that the scalar p vanishes. Then the Finsler space $\bar{F}^n$ equipped with the relation (1.2′) is of constant curvature K if and only if the Riemannian curvature tensor of F^n is given by

$$\text{(2.5)}\quad \begin{aligned} R_{hijk}=&K\bar{L}^2(a_{hj}a_{ik}-a_{hk}a_{ij})/3\alpha^2+\mathfrak{A}_{(hi)}\{2K(h_{ik}\bar{g}_{hj}-\\ &-h_{ij}\bar{g}_{hk}-M_{ijk}y_h/\alpha^2+\bar{L}^2M_{hkj}a_i/\alpha^4-a_i(a_{hk}y_j-a_{jh}y_k)/\alpha^2)+\\ &+H_{ikhj}-H_{ijhk}\}/6,\quad M_{ijk}=a_{ij}a_k-a_{ik}a_j.\end{aligned}$$

§3. Randers spaces of scalar curvature. It is well-known [6] that the scalar p and the vector X^i of a Randers space $\bar{F}^n=(M^n, \bar{L}=\alpha+\beta)$ are given by

$$\text{(3.1)}\quad P=E_{00}/2\bar{L}-F_{\beta 0}/\tau,\quad X^i=\alpha F^i_0\,,$$

where

$$\text{(3.2)}\quad 2E_{ij}=b_{i;j}+b_{j;i},\quad 2F_{ij}=b_{i;j}-b_{j;i},\quad \tau=\bar{L}/\alpha,\quad F_{i\beta}=F_{im}b^m,$$

and (;) is the covariant differentiation with respect to the associated Riemannian connection.

Assume that a Randers space $\bar{F}^n$ is of constant curvature K. Then the equation (2.3) becomes

$$\text{(3.3)}\quad \begin{aligned} R_{0h0k}=&Bh_{hk}+(F_{h00}a_k+F_{k00}a_h)/\alpha-\alpha(F_{h0k}+F_{k0h})+\\ &+\alpha^2V_{hk}-3M_{hk}+V_{00}a_ha_k/\alpha^2-(V_{h0}a_k+V_{k0}a_h).\end{aligned}$$

Hence we have

Theorem 2. *A Randers space $\bar{F}^n$ is of constant curvature K if and only if one of the following facts hold good:*

(1) The associated Riemannian space $F^n=(M^n, \alpha)$ is of constant curvature $k=-s^2$ ($s=$const.) and the covariant vector b_i satisfies $b_{i;j}=s(a_{ij}-b_ib_j)$. In this case the curvature of $\bar{F}^n=(M^n, \bar{L})$ is equal to $K=-s^2/4$.

(2) The covariant vector b_i is a translation with respect to α, i.e., $b_{i;j}+b_{j;i}=0$ and $(a^{ij}b_ib_j)_{;k}=0$ and the curvature tensor R_{hijk} of the associated Riemannian space F^n is written as

$$\text{(3.4)}\quad \begin{aligned} R_{hijk}=&-2(2b_{hi}b_{jk}-b_{hj}b_{ik}+b_{hk}b_{ij}+K\mathfrak{A}_{(jk)}\{b_hb_ja_{ik}+\\ &+b_ib_ka_{hj}-(a^{lm}b_lb_m-1)a_{hj}a_{ik}\}.\end{aligned}$$

In this case the curvature of $\bar{F}^n$ is equal to $K=b_{ij}b_{hk}a^{hi}a^{jk}/(n-1)$.

§4. Kropina space of constant curvature. The quantities p and X^i of a Kropina space $\bar{F}^n$ are given by

$$(4.1)\qquad p=-(E_{00}/\overline{L}+F\beta_0)/\rho,\quad X^i=-\overline{L}(F^i_0+pb^i)/2.$$

By the similar way as in shown in §3 we have

Theorem 3. *A Kropina space $\overline{F}^n$ is of scalar curvature K if and only if a covariant vector b_i is conformally Killing on F^n and the following condition holds*

$$(4.2)\qquad \begin{aligned} R_{0k0k}=&Ah_{hk}+\nu\omega_h\omega_k-\overline{L}^2V_{hk}/4-2(V_{00}+F_{0\beta}p)a_ha_k/\beta^2+\\ &+P(F_{h\beta}a_k+F_{k\beta}a_h)/2-\overline{L}^2M_{hk}/4+(U_{hk}+U_{kh})/2.\end{aligned}$$

Corollary 3.1. *Assume that b_i is a translation with respect to α. The Kropina space $\overline{F}^n$ is of constant curvature K if and only if the following condition*

$$(4.3)\qquad R_{0h0k}=Ah_{hk}+\nu\omega_h\omega_k-\overline{L}^2V_{hk}/4-2V_{00}a_ha_k/\beta^2-\overline{L}^2M_{hk}+(U_{hk}+U_{kh})/2$$

holds. In this case the curvature K is equal to $F_{mr}F^{mr}/4(n-1)$, provided that α is positive definite.

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ON FINSLER SPACES OF CONSTANT CURVATURE WHICH ARE CONFORMAL TO A LOCALLY MINKOWSKI SPACE

BY

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§1. Conformally Minkowski space. A Finsler space F^n is called a locally Minkowski, if there exists a coordinate system (x^i) in which components of the fundamental tensor of F^n are functions of y^i alone. Such a coordinate system (x^i) is called adapted in F^n. If a Finsler space $F^n=(M^n, L)$ is conformal to a locally Minkowski space $\overline{F}^n=(M^n, M)$, the Finsler space is called a conformally Minkowski space. In an adapted coordinate system (x^p_0) in M^n the fundamental tensor $g_{pq}(x^p_0, y^p_0)$ of F^n is written in the form

$$(1.1)\qquad g_{pq}(x^p_0, y^p_0)=\exp(2\rho_0)\bar{g}_{pq}(y^p_0),$$

where ρ_0 is a function of x^p_0 and $\bar{g}_{pq}$ is the fundamental tensor of $\overline{F}^n$.

In an adapted coordinate system (x^p_0) it holds

$$(1.2)\qquad L(x^p_0, y^p_0)=\exp(\rho_0(x^p_0))M(y^p_0)\ ;$$

therefore in any coordinate (x^i) we have

$$(1.3)\qquad L(x^i, y^i)=L(x^p_0, y^p_0)=\exp(\rho_0(x^p_0(x^i)))M(y^p_0)=\exp(\rho(x))M(X^p_iy^i),$$

where we put $\rho(x^i)=\rho_0(x^p_0(x^i))$ and $X^p_i=\partial x^p_0/\partial x^i$. From (1.3) we get

$$(1.4)\qquad L(x, y)=M(a^p),\quad a^p=\exp(\rho(x))X^p_iy^i.$$

Thus F^n is considered as Finsler spaces with the 1-form metric [5] and we can construct a 1-form connection $'\Gamma$ [7]:

$$(1.5)\qquad {}'\Gamma^i_{jk}=\Gamma^i_{jk}+\delta^i_j\rho_k,\quad \rho_k=\partial\rho/\partial x^k,$$

where $\Gamma^i_{jk}=X^i_pX_j{}^p{}_k$, $X^i_p=\partial x^i/\partial x^p_0$, $X_j{}^p{}_k=\partial^2x^p_0/\partial x^j\partial x^k$.

A Wagner connection $W\Gamma=(F^i_{jk}, N^i_j, C^i_{jk})$ is uniquely determined by the following axioms:

(1) $g_{ij|k}=0$, $g_{ij}|_k=0$,
(2) $F^i_{jk}-F^i_{kj}=\delta^i_js_k-\delta^i_ks_j$, where s_i is some vector field,
(3) $C^i_{jk}=C^i_{kj}$, (4) $y^jF^i_{jk}-N^i_k=0$.

The 1-form connection $('\Gamma^i_{jk}, '\Gamma^i_k, C^i_{jk})$ of the conformally Minkowski space is a Wagner connection [7]. Hence we denote it by $W\Gamma(\rho_k)=('\Gamma^i_{jk}, '\Gamma^i_k, C^i_{jk})$. The coefficients are given by