$$(4.1)\qquad p=-(E_{00}/L+F\beta_0)/\rho,\quad X^i=-L(F_0^i+pb^i)/2.$$

By the similar way as in shown in §3 we have

Theorem 3. *A Kropina space $\bar{F}^n$ is of scalar curvature K if and only if a covariant vector b_i is conformally Killing on F^n and the following condition holds*

$$(4.2)\qquad \begin{aligned} R_{0k0k}=&Ah_{hk}+\nu\omega_h\omega_k-\bar{L}^2V_{hk}/4-2(V_{00}+F_{0\beta}p)a_ha_k/\beta^2+\\ &+P(F_{h\beta}a_k+F_{k\beta}a_h)/2-\bar{L}^2M_{hk}/4+(U_{hk}+U_{kh})/2. \end{aligned}$$

Corollary 3.1. *Assume that b_i is a translation with respect to α. The Kropina space $\bar{F}^n$ is of constant curvature K if and only if the following condition*

$$(4.3)\qquad R_{0h0k}=Ah_{hk}+\nu\omega_h\omega_k-\bar{L}^2V_{hk}/4-2V_{00}a_ha_k/\beta^2-\bar{L}^2M_{hk}+(U_{hk}+U_{kh})/2$$

holds. In this case the curvature K is equal to $F_{mr}F^{mr}/4(n-1)$, provided that α is positive definite.

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ON FINSLER SPACES OF CONSTANT CURVATURE WHICH ARE CONFORMAL TO A LOCALLY MINKOWSKI SPACE

BY

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§1. Conformally Minkowski space. A Finsler space F^n is called a locally Minkowski, if there exists a coordinate system (x^i) in which components of the fundamental tensor of F^n are functions of y^i alone. Such a coordinate system (x^i) is called adapted in F^n. If a Finsler space $F^n=(M^n, L)$ is conformal to a locally Minkowski space $\bar{F}^n=(M^n, M)$, the Finsler space is called a conformally Minkowski space. In an adapted coordinate system (x_0^p) in M^n the fundamental tensor $g_{pq}(x_0^p, y_0^p)$ of F^n is written in the form

$$(1.1)\qquad g_{pq}(x_0^p, y_0^p)=\exp(2\rho_0)\bar{g}_{pq}(y_0^p),$$

where ρ_0 is a function of x_0^p and $\bar{g}_{pq}$ is the fundamental tensor of $\bar{F}^n$.

In an adapted coordinate system (x_0^p) it holds

$$(1.2)\qquad L(x_0^p, y_0^p)=\exp(\rho_0(x_0^p))M(y_0^p)\ ;$$

therefore in any coordinate (x^i) we have

$$(1.3)\qquad L(x^i, y^i)=L(x_0^p, y_0^p)=\exp(\rho_0(x_0^p(x^i)))M(y_0^p)=\exp(\rho(x))M(X_i^py^i),$$

where we put $\rho(x^i)=\rho_0(x_0^p(x^i))$ and $X_i^p=\partial x_0^p/\partial x^i$. From (1.3) we get

$$(1.4)\qquad L(x, y)=M(a^p),\quad a^p=\exp(\rho(x))X_i^py^i.$$

Thus F^n is considered as Finsler spaces with the 1-form metric [5] and we can construct a 1-form connection $'\Gamma$ [7]:

$$(1.5)\qquad {}'\Gamma_{jk}^i=\Gamma_{jk}^i+\delta_j^i\rho_k,\quad \rho_k=\partial\rho/\partial x^k,$$

where $\Gamma_{jk}^i=X_p^iX_{j}{}^p{}_k$, $X_p^i=\partial x^i/\partial x_0^p$, $X_{jk}^p=\partial^2x_0^p/\partial x^j\partial x^k$.

A Wagner connection $W\Gamma=(F_{jk}^i, N_j^i, C_{jk}^i)$ is uniquely determined by the following axioms:

(1) $g_{ij|k}=0$, $g_{ij}|_k=0$,
(2) $F_{jk}^i-F_{kj}^i=\delta_j^is_k-\delta_k^is_j$, where s_i is some vector field,
(3) $C_{jk}^i=C_{kj}^i$, (4) $y^jF_{jk}^i-N_k^i=0$.

The 1-form connection $('\Gamma_{jk}^i, '\Gamma_k^i, C_{jk}^i)$ of the conformally Minkowski space is a Wagner connection [7]. Hence we denote it by $W\Gamma(\rho_k)=('\Gamma_{jk}^i, '\Gamma_k^i, C_{jk}^i)$. The coefficients are given by

$$(1.6)\qquad {}'\Gamma_j{}^i{}_k = F_j{}^i{}_k + D^{ir}_{jk}\rho_r,\quad {}'\Gamma^i_k = N^i_k + D^{ir}_k\rho_r,\quad C_j{}^i{}_k = g^{ir}\dot{\partial}_r g_{jk}/2$$

where $C\Gamma = (F_j{}^i{}_k, N^i_k, C_j{}^i{}_k)$ is Cartan connection and D^{ir}_{jk}, D^{ir}_k, which are conformal invariants are given respectively in the following:

$$D^{ir}_{jk} = U^{ir}_{jk} + \delta^i_j\delta^r_k,$$

$$U^{ir}_{jk} := g_{jk}g^{ir} - \delta^i_j\delta^r_k - \delta^r_j\delta^i_k - C^{ir}_j y_k - C^{ir}_k y_j + C_j{}^i{}_k y^r + C^r_{jk}y^i +$$

$$+ L^2(C^{mr}_j C_m{}^i{}_k + C^{im}_j C_m{}^r{}_k - C^{ir}_m C_j{}^m{}_k).$$

$$D^{ir}_k = B^{ir}_k + y^i\delta^r_k,$$

$$B^{ir}_k := y_k g^{ir} - \delta^i_k y^r - \delta^r_k y^i - L^2 C^{ir}_k.$$

Theorem (H a s h i g u c h i and I c h i j y ō [2]). *A Finsler space is conformally Minkowski if and only if it is a Wagner space with respect to a gradient vector field and its h-curvature tensor vanishes.*

If we put the difference $C\Gamma$ and $W\Gamma(\rho_k)$ by $D_j{}^i{}_k$, we have from (1.6)

$$(1.7)\qquad D_j{}^i{}_k = F_j{}^i{}_k - {}'\Gamma_j{}^i{}_k = -D^{ir}_{jk}\rho_r.$$

The h-curvature tensor is given, from (1.7), by

$$(1.8)\qquad \begin{aligned} R_j{}^i{}_{hk} &= D_j{}^i{}_{h|k} - D_j{}^i{}_{k|h} + D_m{}^i{}_h D_j{}^m{}_k - D_m{}^i{}_k D_j{}^m{}_h + \\ &+ LC_j{}^i{}_m(D_0{}^m{}_{h|k} - D_0{}^m{}_{k|h} + D_s{}^m{}_h D_0{}^s{}_k - D_s{}^m{}_k D_0{}^s{}_h), \end{aligned}$$

where the index 0 means the contraction by the element of support y^i and we used the h-curvature tensor of Wagner space with respect to a gradient vector field vanishes.

From (1.8) we have

$$(1.9)\qquad R_0{}^i{}_{0k} = D_0{}^i{}_{0|k} - D_0{}^i{}_{k|0} + D_m{}^i{}_0 D_0{}^m{}_k - D_m{}^i{}_k D_0{}^m{}_0.$$

A Finsler space F^n is said to be of scalar curvature K if the equation

$$(1.10)\qquad R_{0i0j} = KL^2h_{ij}$$

holds good at any (x, y) of F^n, and to be of constant curvature K if, furthermore, the scalar K is constant. It is well-known that if F^n is of scalar curvature K and $K_{|i}=0$ then K is constant [6].

From (1.7), (1.9) and (1.10) we have

$$(1.11)\qquad \begin{aligned} K &= R^i_{00i}/(n-1)L^2 = [1/(n-1)][-h^{ir}\rho_{r|i} - P^{ir}_i\rho_r - ((n-1)y^r/L^2 + \\ &+ C^{ir}_i)\rho_{r|0} + ((n-2)h^{rs} + L^2(C^{ir}_m C^{ms}_i - C^{sr}_m C^{im}_i))\rho_r\rho_s]. \end{aligned}$$

If $\rho_{r|i}=0$, from (1.11) we have

$$(1.12)\qquad K = Q^r\rho_r/(n-1),\quad Q^r = -P^{ir}_i + ((n-2)h^{rs} + L^2(C^{ir}_m C^{ms}_i - C^{im}_i C^{sr}_m))\rho_s.$$

And further differentiating h-covariantly we obtain

$$(1.13)\qquad K_{|i} = Q^r{}_{|i}\rho_r/(n-1),\quad Q^r{}_{|i} = -P^{sr}_{s|i} + L^2(C^{jr}_m C^{ms}_j - C^{jm}_j C^{sr}_m)_{|i}\rho_s.$$

Hence we have

Theorem 1. *Let F^n be conformally Minkowski Finsler space of scalar curvature K. If ρ_r is h-covariant constant with respect to the Cartan connection $C\Gamma$ and $Q^r{}_{|i}=0$, then the F^n is of constant curvature.*

Corollary. *Let F^n be conformally Minkowski Finsler space of scalar curvature K. If ρ_r and v-Ricci tensor S_{ij} are h-covariant constant with respect to the Cartan connection $C\Gamma$ and $C_{i|0|j}=0$, then the F^n is of constant curvature.*

§2. RCG-space. We consider a Randers space $(M^n, \alpha+\beta)$ $(\alpha = (a_{ij}(x)y^iy^j)^{1/2},\ \beta = b_i(x)y^i)$.

Theorem (Y a s u d a and S h i m a d a [9]). *A Randers space is of constant curvature K if and only if it is a: (1) RCG-space or (2) RCΓ-space.*

We shall prove the following

Theorem 2. *If the corresponding Riemannian space (M^n, α) of the RCG-space is flat, then the RCG-space is conformally Minkowski.*

Proof. From the Theorem (T a m á s s y and M a t s u m o t o [7]) a Finsler space is conformally Minkowski, if and only if there exists a linear, symmetric and flat connection satisfying $\nabla_k g_{ij} = 2b_k g_{ij}$, where ∇ is the covariant differentiation and $b_k(x)$ is a gradient vector field. If we take $\{{}^i_{jk}\}$ the Christoffel symbol for the Riemannian space (M^n, α), we easily see that the integrability condition of the above recurrent differential equation satisfies identically.

Remark. It is remarked that such an RCG-space is a locally Minkowski space.

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