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METRICAL FINSLER CONNECTIONS $\mathcal{T}_I(g)$

BY

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In the present paper we determine the transformations group $\mathcal{T}_I(g)$, of all metrical Finsler connections, which admit the invariants I_1, I_2.

§0. Introduction. Let M_n be a differentiable manifold of class C^∞, (U_a, φ_a) a local chart of M arround $x \in M$, $(\partial/\partial x^i)$ $i=\overline{1,n}$, the natural basis of the linear space M_x on this, chart, $T(M)$ the tangent bundle to M, (x^i, y^i) the canonical coordinates of the point $y \in T(M)$ $(\pi(y)=x)$, $T(M)_y$ the tangent spaces to $T(M)$ and $(\partial/\partial x^i, \partial/\partial y^i)$ the natural basis of $T(M)_y$. A non-linear connection on $T(M)$ is a regular distribution $N: y \in T(M) \to \to N_y \subset T(M)_y$ such that $T(M)_y = N_y \oplus T(M)_y^v$ ([1] [2]) Let:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^s \frac{\partial}{\partial y^s}, \quad i, s = \overline{1, n}, \tag{0.1}$$

be a local basis of N (N_i^s-are called the coefficients of the nonlinear connection N). By the Matsumoto theory a Finsler connection is denoted by $F\Gamma$. Throughout the paper the Finsler connection $F\Gamma$ is denoted $F\Gamma = (N, F, C)$ having N_j^i, F_{jk}^i, C_{jk}^i as the coefficients (Miron R. [2]).

All metrical connections are determined by R. Miron, and Hashiguchi M. [3].

§1. The general Finsler transformations, $\mathcal{T}_I$. Let $\mathcal{T}$ be the set of the transformations of Finsler connections, where $t \in \mathcal{T}$, $t: (N, F, C) \to (\overline{N}, \overline{F}, \overline{C})$ is ([2]);

$$\overline{N}_j^i = N_j^i - A_j^i; \quad \overline{F}_{jk}^i = F_{jk}^i + C_{js}^i A_k^s - B_{jk}^i; \quad \overline{C}_{jk}^i = C_{jk}^i - D_{jk}^i \tag{1.1}$$

The transformation t is denoted by $t\ (A, B, D)$ and

$$\underset{1}{I}_{jk}^i \overset{\text{def}}{=} T_{jk}^i - \frac{1}{n-1}(\delta_j^i T_k - \delta_k^i T_j); \quad \underset{2}{I}_{jk}^i \overset{\text{def}}{=} S_{jk}^i - \frac{1}{n-1}(\delta_j^i S_k - \delta_k^i S_j), \tag{1.2}$$

where $T_{jk}^i = F_{jk}^i - F_{jk}^i$; $T_k = T_{ik}^i$; $S_{jk}^i = C_{jk}^i - C_{kj}^i$; $S_k = S_{ik}^i$.

Let $\mathcal{T}_I$ be the subgroup of $\mathcal{T}$ defined by the transformations $t \in \mathcal{T}$ which are characterized by

$$\underset{1}{\overline{I}}_{jk}^i = \underset{1}{I}_{jk}^i; \quad \underset{2}{\overline{I}}_{jk}^i = \underset{2}{I}_{jk}^i. \tag{1.3}$$

The set of all Finsler connections $\mathcal{T}_I$ we given in [7]:

§2. **The metrical Finsler connection,** $\mathcal{T}_l(g)$. We put

$$\beta_j = \tau_j - 1/(n-1)(A_j - \bar{A}_j). \tag{2.1}$$

One obtains

$$\bar{N}^i_j = N^i_j - A^i_j\ ;\ \bar{F}^i_{jk} = F^i_{jk} + \beta_j\delta^i_k - \beta_k\delta^i_j + A^i_{jk} - U^i_{jk} \tag{2.2}$$

$$\bar{C}^i_{jk} = C^i_{jk} + \alpha_j\delta^i_k - \alpha_k\delta^i_j - V^i_{jk}. \tag{2.3}$$

Let $g_{ij}(x, y)$ be a metrical Finsler structure ([2]) [3]), $F\Gamma = (N, F, C)$ a fixed Finsler connection and $(\bar{N}, \bar{F}, \bar{C})$ the Finsler connection given by (2.2) and (2.3). If $F\bar{\Gamma} = (\bar{N}\bar{F}\bar{C})$ is metrical, i.e.,

$$g_{ij\|k} = 0\ ;\quad g_{ij}\|_k = 0, \tag{2.4}$$

where (‖) and (‖) are the h-and v-covariant derivative with respect to $F\bar{\Gamma}$, then from (2.2)—(2.4) and

$$\frac{\bar{\delta}}{\delta x^i} = \frac{\delta}{\delta x^i} + A^k_i \frac{\partial}{\partial y^k} \tag{2.5}$$

we obtain

$$\begin{aligned}\overset{*}{\Omega}{}^{ir}_{sj}(U^s_{rk} + 1/2C^s_{ra}A^a_k - 1/2C^s_{ka}A^a_r) = 1/2(\beta_j\delta^i_k - 2\beta_k\delta^i_j + g_{jk}\beta^i -\\ - g^{ia}g_{aj|k} - g^{ia}g_{aj}|_b A^b_k)\end{aligned} \tag{2.6}$$

where (|) and (/) are the h-and v-covariant derivative with respect to $F\Gamma$
Since

$$\Omega^{ir}_{sj}(\beta_r\delta^s_k - 2\beta_k\delta^s_r + g_{rk}\beta^s - g^{sa}g_{ar|k} - g^{sa}g_{ar}|_b A^b_k) = 0 \tag{2.7}$$

where Ω, $\overset{*}{\Omega}$ are Obata's operators ([3]), the system of tensor equations (2.6) has solutions. The general solutions, of (2.6), are

$$\begin{aligned}U^i_{jk} = \frac{1}{2}\{\beta_j\delta^i_k - 2\beta_k\delta^i_j + g_{jk}\beta^i - g^{ia}g_{aj|k} - g^{ia}g_{aj}|_b A^b_k - \frac{1}{2}C^i_{ja}A^a_k +\\ + \frac{1}{2}C^i_{ka}A^a_j + \Omega^{ir}_{sj}X^s_{rk}\ .\end{aligned} \tag{2.8}$$

From $g_{ij}\|_k = 0$, we obtain

$$\overset{*}{\Omega}{}^{ir}_{sj}V^s_{rk} = 1/2(\alpha_j\delta^i_k - 2\alpha_k\delta^i_j + g_{jk}\alpha^i - g^{ia}g_{aj}|_k). \tag{2.9}$$

The equations (2.9) has the general solutions :

$$V^i_{jk} = \frac{1}{2}(\alpha_j\delta^i_k - 2\alpha_k\delta^i_j + g_{jk}\alpha^i - g^{ia}g_{aj}|_k) + \Omega^{ir}_{sj}Y^s_{rk}. \tag{2.10}$$

We have
Theorem 2.1. *Let $F\Gamma$ be a fixed Finsler connection. Then the set of all metrical Finsler connections $F\bar{\Gamma}$ where $t \in \mathcal{T}_l$ is given by*

$$\bar{N}^i_j = N^i_j - A^i_j. \tag{2.11}$$

$$\bar{F}^i_{jk} = F^i_{jk} + \frac{1}{2}\beta_j\delta^i_k - \frac{1}{2}g_{jk}\beta^i + C^i_{ja}A^a_k + \frac{1}{2}g^{ia}g_{aj|k} + \frac{1}{2}g^{ia}g_{aj}|_b A^b_k - \Omega^{ir}_{sj}X^s_{rk} \tag{2.12}$$

$$\bar{C}^i_{jk} = C^i_{jk} + \frac{1}{2}\alpha_j\delta^i_k - \frac{1}{2}g_{jk}\alpha^i + \frac{1}{2}g^{ia}g_{aj}|_k - \Omega^{ir}_{sj}Y^s_{rk}, \tag{2.13}$$

wehre A^i_j, X^i_{jk}, Y^i_{jk}, are arbitrary Finsler tensors field: α_k, τ_k are arbitrary $\alpha, \tau \in Z^0_1(M)$.

Theorem 2.2 *Let $F\Gamma$ be a fixed metrical Finsler connection. Then the set $\mathcal{T}_l(g)$, of all metrical Finsler connections $F\bar{\Gamma}$, is given by*

$$\bar{N}^i_j = N^i_j - A^i_j \tag{2.11'}$$

$$\bar{F}^i_{jk} = F^i_{jk} + (1/2)\beta_j\delta^i_k - (1/2)g_{jk}\beta^i + C^i_{ja}A^a_k - \Omega^{ir}_{sj}X^s_{rk} \tag{2.12'}$$

$$\bar{C}^i_{jk} = C^i_{jk} + \frac{1}{2}\alpha_j\delta^i_k - \frac{1}{2}g_{jk}\alpha^i - \Omega^{ir}_{sj}Y^s_{rk}. \tag{2.13'}$$

α, τ, X, Y satisfying

$$3\beta_k\delta^i_j - 3\beta_j\delta^i_k + 2(C^i_{ja}A^a_k - C^i_{ka}A^a_j) = 2(\Omega^{ir}_{sj}X^s_{rk} - \Omega^{ir}_{sk}X^s_{rj})$$

$$3\alpha_k\delta^i_j - 3\alpha_j\delta^i_k = 2(\Omega^{ir}_{sj}Y^s_{rk} - \Omega^{ir}_{sk}Y^s_{rj}).$$

$\Omega^{ir}_{sj}X^s_{ik}$ and $\Omega^{ir}_{sj}X^s_{rk}$ are given by (2.14) (2.15)

or

$$\begin{cases} -\Omega^{ir}_{sj}X^s_{rk} = \frac{3}{2}\beta_j\delta^i_k - \frac{3}{2}g_{jk}\beta^i + g^{ia}(g_{ak}|_j - g_{jk|a}) + \frac{1}{2}g^{ia}(g_{ka}|_b A^b_j - \\ \qquad - g_{jk}|_b A^b_a - \\ -\Omega^{ir}_{sj}(C^s_{rb}A^b_k - C^s_{kb}A^b_r) - \Omega^{ir}_{sk}(C^s_{rb}A^b_j - C^s_{jb}A^b_r) + \frac{1}{2}C^i_{kb}A^b_j - \frac{1}{2}C^i_{jb}A^b_k. \end{cases} \tag{2.14}$$

Analogus we obtain

$$-\Omega^{ir}_{sj}Y^s_{rk} = \frac{3}{2}\alpha_j\delta^i_k - \frac{3}{2}g_{jk}\alpha^i + g^{ia}(g_{ak}|_j - g_{jk}|_a). \tag{2.14'}$$

We have
Theorem (2.3). *Let $F\Gamma$ be a fixed Finsler connection. Then the set of all metrical Finsler connections $F\bar{\Gamma}$, where $t : F\Gamma \to F\bar{\Gamma}$, and $t \in \mathcal{T}_l$, is given by*

$$\bar{N}^i_j = N^i_j - A^i_j \tag{2.15}$$

$$\begin{aligned}\bar{F}^i_{jk} = F^i_{jk} + \varphi_k\delta^i_j - g_{jk}\varphi^i + 1/2g^{ir}(g_{rk|j} + g_{rj|k} - g_{jk|r}) +\\ + 1/2g^{ir}(g_{rk}|_b A^b_j + g_{rj}|_b A^b_k - g_{jk}|_b A^b_r) + 1/2(C^i_{jb}A^b_k +\\ + C^i_{kb}A^b_j) - \Omega^{ir}_{sj}(C^s_{rb}A^b_k - C^s_{kb}A^b_r) - \Omega^{ir}_{sk}(C^s_{rb}A^b_j - C^s_{jb}A^b_r).\end{aligned} \tag{2.16}$$

$$\bar{C}^i_{jk} = C^i_{jk} + \omega_j\delta^i_k - g_{jk}\omega^i + \frac{1}{2}g^{ir}(g_{rk}|_j + g_{rj}|_k - g_{jk}|_r). \tag{2.17}$$

where $\varphi_k = 2\beta_k$, $\omega_k = 2\alpha_k$ and β_k is (2.1).

Theorem (2.4.). *The set $\mathcal{T}_l(g)$ of all metrical Finsler connections is given by*

$$\bar{N}^i_j = N^i_j - A^i_j, \tag{2.18}$$

$$\overline{F}^i_{jk}=F^i_{jk}+\varphi_k\delta^i_j-g_{jk}\varphi^i+\frac{1}{2}(C^i_{jb}A^b_k+C^i_{kb}A^b_j)-\Omega^{ir}_{sj}(C^s_{rb}A^b_k-$$
(2.19)
$$-C^i_{kb}A^b_r)-\Omega^{ir}_{sk}(C^s_{rb}A^b_j-C^s_{jb}A^b_r)$$

$$\overline{C}^i_{jk}=C^i_{jk}+\omega_j\delta^i_k-g_{jk}\omega^i. \tag{2.20}$$

If $\overline{N}=N$ $(t\in\mathcal{T}_N)$, we obtain

$$\underset{1}{I}=0\ \&\ \underset{2}{I}=0\Rightarrow\underset{1}{\overline{I}}=0\ \&\ \underset{2}{\overline{I}}=0, \text{ or}$$

Theorem (2.5). *If $F\Gamma$ is a semi-symmetric Finsler connections then the set of all semi-symmetric metrical Finsler connections is given by* (2.15) (2.16) *and* (2.17).

Theorem (2.6). *If F is a semi-symmetric metric Finsler connections, then the set of all semi-symmetric metrical Finsler connections is given by* (2.18) (2.19) *and* (2.20)

a) *If $\overline{N}=N$ $(t\in\mathcal{T}_N)$ from* (2.18) (2.19) *and* (2.20) *we obtain Miron's result* [3].

b) *If $\overline{T}^i_{jk}=\overline{T}^i_{jk}$, $\overline{S}^i_{jk}=S^i_{jk}$ since $\underset{1}{\overline{I}}=\underset{2}{I}$; $\underset{1}{\overline{I}}=\underset{2}{I}$, from* (2.15)—(2.17) *we obtain* $\varphi=0$, $\omega=0$. *We have*

Theorem (2.7). 1) *If $\overline{T}^i_{jk}=T^i_{jk}$ and $\overline{S}^i_{jk}=S^i_{jk}$ then the set of all metrical Finsler connections $F\overline{\Gamma}$ is given by* (2.15) (2.16) (2.17) *with* $\varphi=0$, $\omega=0$ 2) *If $F\Gamma$ is a fixed metrical Finsler connection, and $\overline{T}^i_{jk}=T^i_{jk}$, $\overline{S}^i_{jk}=S^i_{jk}$, then $t\in\in\mathcal{T}_{CI}(\overline{C}^i_{jk}=C^i_{jk})$* 3) *For* 2), *if $\overline{N}=N$ then*

$$t:F\Gamma\to F\overline{\Gamma},$$

is an affine transformation $(\overline{N}=N,\ \overline{F}=F,\ \overline{C}=C)$.

From 3) it result that there exists only one metrical Finsler connections $F\Gamma(N)$ with the torsion tensor fields T^i_{jk} and S^i_{jk} or $T^i_{jk}=0$, $S^i_{jk}=0$ (In this case, this connection is $(N,\overset{\subset}{F},\overset{\subset}{C})$ [2] [3], or the Cartan connection if $N=\overset{\subset}{N}$ [2] [3].

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7. S t a v r e, P. and K l e p p, F r. — *The general Finsler transformations with the invariants $I_l{}^i{}_k$* (to appear)

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CONVEXITY OF FINSLER ENERGIES

BY

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This paper has as a starting point the following conjecture of M a t s u m o t o (7) : the absolute length $L(x, X)$ of a nonzero vector X in a Finsler space (M, L), with fundamental function L which satisfies desirable assumptions, is equal to or less than relative length $\|X\|_y=\sqrt{g_{ij}(x,y)X^iX^j}$.

§1. Riemannian manifold $(T_xM-\{0\}, g)$. Let (M, L) be an n-dimensional Finsler space and

$$L^2(x,y)=\begin{cases} g_{ij}(x,y)y^iy^j & \text{if } x\in M,\ y\in T_xM-\{O\}\\ 0 & \text{if } x\in M,\ y=0.\end{cases}$$

The partial function $y\to L^2(x,y)$ is called the *absolute Finsler energy* on T_xM and does not possess critical points on $T_xM-\{O\}$. The partial functions $y\to g_{ij}(x,y)$ are the components of a Riemannian metric g on $T_xM-\{O\}$.

1.1. **Theorem.** *Every element of $T_xM-\{O\}$ is irrotational with respect to the Riemannian connection determined by g.*

§2. Convexity of the absolute energy on $T_xM-\{0\}$. The definition of a Finsler space implies that the energy L^2 is a strongly convex function on $(T_xM-\{0\},\ g_0=[\delta_{ij}])$.

2.1. **Theorem.** *The changing of metric $g_0\to g$ on $T_xM-\{0\}$ keeps the Hessian of the absolute energy L^2.*

2.2. **Corollary.** *The energy L^2 is a strongly convex function on the Riemannian manifold $(T_xM-\{0\}, g)$.*

2.3. **Corollary.** *The Riemannian manifold $(T_xM-\{0\}, g)$ contains no closed geodesics.*

§3. Indicatrix in $(T_xM-\{0\}, g)$. First we formulate the following Corollaries pf Theorem 2.1.

3.1. **Corollaries.** 1) *The sublevel set $N_c: L^2(x,y)\leq c^2$ contains the geodesic segments in $T_xM-\{0\}, g)$ whose ends are in N_c. Particularly N_c is conex.*

2) *The indicatrix $I_x: L^2(x,y)=1$ is a totally umbilical strictly convex hypersurface in $(T_xM-\{0\}, g)$.*

3.2. **Theorem.** *The orbits of grad L^2 are geodesics in $(T_xM-\{0\}, g)$ reparametrized by $s=a+be^{2t}$, $t\in R$, $a,b=const.$*

§4. Critical points of the relative energy on $T_xM-\{0\}$. Define the Riemannian manifold $(T_xM-\{0\}, g)$. Let $L^2(x,X)=g_{ij}(x,X)X^iX^j$ be the *absolute Finsler energy* of a nonzero vector X tangent to M at a point x.