$$\overline{F}^i_{jk}=F^i_{jk}+\varphi_k\delta^i_j-g_{jk}\varphi^i+\frac{1}{2}(C^i_{jb}A^b_k+C^i_{kb}A^b_j)-\Omega^{ir}_{sj}(C^s_{rb}A^b_k-\tag{2.19}$$

$$-C^i_{kb}A^b_r)-\Omega^{ir}_{sk}(C^s_{rb}A^b_j-C^s_{jb}A^b_r)$$

$$\overline{C}^i_{jk}=C^i_{jk}+\omega_j\delta^i_k-g_{jk}\omega^i.\tag{2.20}$$

If $\overline{N}=N$ $(t\in\mathcal{T}_N)$, we obtain

$$\underset{1}{I}=0\ \&\ \underset{2}{I}=0\Rightarrow\underset{1}{\overline{I}}=0\ \&\ \underset{2}{\overline{I}}=0,\ \text{or}$$

Theorem (2.5). *If $F\Gamma$ is a semi-symmetric Finsler connections then the set of all semi-symmetric metrical Finsler connections is given by* (2.15) (2.16) *and* (2.17).

Theorem (2.6). *If F is a semi-symmetric metric Finsler connections, then the set of all semi-symmetric metrical Finsler connections is given by* (2.18) (2.19) *and* (2.20)

a) *If $\overline{N}=N$ $(t\in\mathcal{T}_N)$ from* (2.18) (2.19) *and* (2.20) *we obtain Miron's result* [3].

b) *If $\overline{T}^i_{jk}=\overline{T}^i_{jk}$, $\overline{S}^i_{jk}=S^i_{jk}$ since $\underset{1}{\overline{I}}=\underset{2}{I}$; $\underset{1}{\overline{I}}=\underset{2}{I}$, from* (2.15)—(2.17) *we obtain* $\varphi=0$, $\omega=0$. *We have*

Theorem (2.7). 1) *If $\overline{T}^i_{jk}=T^i_{jk}$ and $\overline{S}^i_{jk}=S^i_{jk}$ then the set of all metrical Finsler connections $F\overline{\Gamma}$ is given by* (2.15) (2.16) (2.17) *with* $\varphi=0$, $\omega=0$ 2) *If $F\Gamma$ is a fixed metrical Finsler connection, and $\overline{T}^i_{jk}=T^i_{jk}$, $\overline{S}^i_{jk}=S^i_{jk}$, then $t\in\in\mathcal{T}_{Cl}(\overline{C}^i_{jk}=C^i_{jk})$* 3) *For* 2), *if $\overline{N}=N$ then*

$$t:F\Gamma\to F\overline{\Gamma},$$

is an affine transformation $(\overline{N}=N,\ \overline{F}=F,\ \overline{C}=C)$.

From 3) it result that there exists only one metrical Finsler connections $F\Gamma(N)$ with the torsion tensor fields T^i_{jk} and S^i_{jk} or $T^i_{jk}=0$, $S^i_{jk}=0$ (In this case, this connection is $(N, \overset{\subset}{F}, \overset{\subset}{C})$ [2] [3], or the Cartan connection if $N=\overset{\subset}{N}$ [2] [3].

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CONVEXITY OF FINSLER ENERGIES

BY

CONSTANTIN UDRIŞTE

This paper has as a starting point the following conjecture of M a t s u m o t o (7): the absolute length $L(x, X)$ of a nonzero vector X in a Finsler space (M, L), with fundamental function L which satisfies desirable assumptions, is equal to or less than relative length $\|X\|_y=\sqrt{g_{ij}(x,y)X^iX^j}$.

§1. Riemannian manifold $(T_xM-\{0\}, g)$. Let (M, L) be an n-dimensional Finsler space and

$$L^2(x,y)=\begin{cases} g_{ij}(x,y)y^iy^j & \text{if } x\in M,\ y\in T_xM-\{O\}\\ 0 & \text{if } x\in M,\ y=0.\end{cases}$$

The partial function $y\to L^2(x, y)$ is called the *absolute Finsler energy* on T_xM and does not possess critical points on $T_xM-\{O\}$. The partial functions $y\to g_{ij}(x, y)$ are the components of a Riemannian metric g on $T_xM-\{O\}$.

1.1. **Theorem.** *Every element of $T_xM-\{O\}$ is irrotational with respect to the Riemannian connection determined by g.*

§2. Convexity of the absolute energy on $T_xM-\{0\}$. The definition of a Finsler space implies that the energy L^2 is a strongly convex function on $(T_xM-\{0\},\ g_0=[\delta_{ij}])$.

2.1. **Theorem.** *The changing of metric $g_0\to g$ on $T_xM-\{0\}$ keeps the Hessian of the absolute energy L^2.*

2.2. **Corollary.** *The energy L^2 is a strongly convex function on the Riemannian manifold $(T_xM-\{0\}, g)$.*

2.3. **Corollary.** *The Riemannian manifold $(T_xM-\{0\}, g)$ contains no closed geodesics.*

§3. Indicatrix in $(T_xM-\{0\}, g)$. First we formulate the following Corollaries pf Theorem 2.1.

3.1. **Corollaries.** 1) *The sublevel set $N_c: L^2(x, y)\leqslant c^2$ contains the geodesic segments in $T_xM-\{0\}, g)$ whose ends are in N_c. Particularly N_c is conex.*

2) *The indicatrix $I_x: L^2(x, y)=1$ is a totally umbilical strictly convex hypersurface in $(T_xM-\{0\}, g)$.*

3.2. **Theorem.** *The orbits of* grad L^2 *are geodesics in $(T_xM-\{0\}, g)$ reparametrized by $s=a+be^{2t}$, $t\in R$, $a,b=const.$*

§4. Critical points of the relative energy on $T_xM-\{0\}$. Define the Riemannian manifold $(T_xM-\{0\}, g)$. Let $L^2(x, X)=g_{ij}(x, X)X^iX^j$ be the *absolute Finsler energy* of a nonzero vector X tangent to M at a point x.

The *relative Finsler energy* of the same vector is the positively homogeneous function of degree zero,

$$f : T_xM-\{0\}\to R,\ f(y)=\|X\|_y^2=g_{ij}(x,\ y)X^iX^j.$$

4.1. **Theorem.** *Critical points of the relative energy f are degenerate.*

4.2. **Theorem.** *$X\in T_xM$-$\{0\}$ is a critical point of the relative energy f at which Hess f vanishes identically. Hence $f(X)=L^2(x,\ X)$ is a critical value of f.*

4.3. **Conclusion.** In order to decide if the critical point X is or not an extremum point of the relative energy f we must resort either to (partial) covariant derivatives of the order three at least at X, or to the direct study of function variation round X, or to the study of f convexity over the whole domain.

4.4. **Theorem.** *Let the Riemannian manifold $(T_xM-\{0\},\ g)$. If the tensor $C_{mkn,l}X^mX^n$ is positive semidefinite, then the relative energy f attached to vector X is a convex function.*

4.5. **Corollaries.** *If on the Riemannian manifold $(T_xM-\{0\},\ g)$ the tensor $C_{mkh,l}X^mX^n$ is positive semidefinite, then*

1) *the set of critical points (global minimum points) of the relative energy f attached to X contains the geodesic segments in $(T_xM-\{0\},\ g)$ for which the ends are critical points of f;*

2) *(critical value) absolute energy $f(X)=L^2(x,\ X)$ of the vector X is a global minimum of the relative energy.*

4.7. **Theorem.** *Critical points of the restriction of relative energy f to $N_c : L^2(x,\ y)\leqslant c^2$ coincide with critical points of f on $T_xM-\{0\}$ which belong to N_c.*

§5. **Complete Riemannian manifold** $(T_xM,\ \tilde{g})$. In case that the absolute energy is not differentiable for $y=0$ we are led to the Riemannian manifold $(T_xM-\{0\},\ g)$ which cannot be complete. In order to eliminate this disadvantage, we start as indicated by R. M i r o n (9). Let g be a Finsler tensor field of type (0, 2) which is symmetric and positive definite and having local components $g_{ij}(x,\ y)$ on $\pi^{-1}(U)$. The tensor field g is called *metrical Finsler structure* on the manifold M. To this structure one associates the absolute Finsler energy $L^2 : TM\to R,\ L^2(x,\ y)=g_{ij}(x,\ y)y^iy^j$.

Let the Riemannian manifold $(T_xM,\ g)$. Having in mind the Riemannian model, we suppose that the restriction $L^2 : T_xM\to[0,\infty)\subset\mathbb{R}$ is an exhaustion function, i.e., $\forall\lambda\in R$, the subset $(L^2)^{-1}(-\infty,\ \lambda)]$ of T_xM is compact. The Riemannian manifold $(T_xM,\ g)$ is not necessarily complete, but the presence of the exhaustion function L^2 allows for a change of the metric g so that the new Riemannian manifold is complete.

5.1. **Theorem.** *Let $\tilde{g}$ be the Riemannian metric of the components $\tilde{g}_{ij}=g_{ij}+L^2{}_{,i}L^2{}_{,j}$ on T_xM. If L^2 is an exhaustion function, then the Riemannian manifold $(T_xM,\ \tilde{g})$ is complete.*

5.2. **Theorem.** *The change of the Riemannian metric $g_{ij}\to\tilde{g}_{ij}=g_{ij}+L^2_{,i}L^2_{,j}$ on T_xM implies*

$$\mathrm{Hess}_{\tilde{g}}L^2=\frac{1}{1+\|\mathrm{grad}\ L^2\|}\mathrm{Hess}_gL^2.$$

Theorem 5.2 shows that L^2 is convex (strictly convex) with respect to the metric $\tilde{g}$ iff it is convex (strictly convex) with respect to the metric g.

5.3. **Corollaries.** *Suppose that $L^2 : T_xM\to R$ is an exhaustion function and strictly convex with respect to g.*

1) *The complete Riemannian manifold $(T_xM,\ \tilde{g})$ does not possess closed geodesics.*

2) *The complete Riemannian manifold $(T_xM,\ \tilde{g})$ is diffeomorphic to an Euclidean space.*

3) *The sublevel set $N_c : L^2(x,\ y)\leqslant c^2$ (compact set) is totally convex both in $(T_xM,\ g)$ and in $(T_xM,\ \tilde{g})$.*

4) *The hypersurface $\partial N_c : L^2(x,\ y)=c^2$ is strictly convex both in $(T_xM,\ g)$ and in $(T_xM,\ \tilde{g})$.*

§6. **Distribution** L_x^2. In order to eliminate the disadvantage that sometimes we are led to the Riemannian manifold $(T_xM-\{0\},\ g)$ which is not complete we can go on in the following way. We replace the restriction of L^2 to T_xM by the distribution L_x^2 defined so

$$(L_x^2,\ \varphi)=\int_{T_xM}L^2(x,\ y)\varphi(y)dv_y,\ \varphi\in K,$$

where the integration is actually effected on a compact such that the test functions φ vanishe identically in outside. The properties of function $y\to\to L^2(x,\ y)$ are transferred into properties of the distribution L_x^2.

For details, see Proceedings of the National Seminar on Finsler Spaces, Braşov, February, 1984.

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GENERALIZED FINSLER SPACES CONFORMAL TO A RIEMANNIAN SPACE AND THE CARTAN-LIKE CONNECTIONS

BY

SHÔJI WATANABE

Let us consider an n-dimensional manifold M^n on which a symmetric and nondegenerate Finsler tensor field $g_{ij}(x, y)$ of type (0, 2) is defined. We shall call (M^n, g_{ij}) a *generalized Finsler space*. Many authors studied generalized Finsler spaces.

A *Cartan-like connection* $*C\Gamma=(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ on (M^n, g_{ij}) is a Finsler connection satisfying the following axioms [5], [7]:

(C1) $g_{ij|k}=\delta g_{ij}/\delta x^k-g_{rj}F_i{}^r{}_k-g_{ir}F_j{}^r{}_k=0,$

(C2) $g_{ij}|_k=\partial g_{ij}/\partial y^k-g_{rj}C_i{}^r{}_k-g_{ir}C_j{}^r{}_k=0,$

(C3) $T_j{}^i{}_k=F_j{}^i{}_k-F_k{}^i{}_j=0,$

(C4) $S_j{}^i{}_k=C_j{}^i{}_k-C_k{}^i{}_j=0,$

(C5) $y^i|_j=-N^i{}_j+F_0{}^i{}_j=0,$

where $\delta/\delta x^k=\partial/\partial x^k-N^r{}_k\partial/\partial y^r$ and the suffix 0 means the contraction by the supporting element y. Supposing some conditions, the Cartan-like connection $(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ can be uniquely determined by the system of the above axioms [7]. In his paper [4], M o ó r obtained the coefficients $F_j{}^i{}_k$. And we can easily see

$$C_j{}^i{}_k=1/2\, g^{ir}(\partial g_{jr}/\partial y^k+\partial g_{kr}/\partial y^j-\partial g_{jk}/\partial y^r).$$

In the present paper, we shall deal with generalized Finsler spaces (M^n, g_{ij}) with

$$g_{ij}(x, y)=e^{2\sigma(x,y)}\gamma_{ij}(x),$$

and the Cartan-like connections on the generalized Finsler spaces, where $\gamma_{ij}(x)$ is a Riemannian metric tensor and $\sigma(x, y)$ is a function of x and y but is not necessarily positively homogeneous of degree 0 with respect to y.

The terminologies and notations are referred to M a t s u m o t o's book [2].

1. A generalized Finsler space (M^n, g_{ij}) with $g_{ij}=e^{2\sigma}\gamma_{ij}$ and the Cartan-like connection. In a generalized Finsler space (M^n, g_{ij}) with $g_{ij}=e^{2\sigma}\gamma_{ij}$, Theorem 6 in the previous paper [8] (or Theorem 1 in N u m a t a's paper [5]) is generalized as follows: