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GENERALIZED FINSLER SPACES CONFORMAL TO A RIEMANNIAN SPACE AND THE CARTAN-LIKE CONNECTIONS

BY

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Let us consider an n-dimensional manifold M^n on which a symmetric and nondegenerate Finsler tensor field $g_{ij}(x, y)$ of type (0, 2) is defined. We shall call (M^n, g_{ij}) a *generalized Finsler space*. Many authors studied generalized Finsler spaces.

A *Cartan-like connection* $*C\Gamma=(F_j{}^i{}_k, N^i_k, C_j{}^i{}_k)$ on (M^n, g_{ij}) is a Finsler connection satisfying the following axioms [5], [7]:

(C1) $g_{ij|k}=\delta g_{ij}/\delta x^k-g_{rj}F_i{}^r{}_k-g_{ir}F_j{}^r{}_k=0,$

(C2) $g_{ij}|_k=\partial g_{ij}/\partial y^k-g_{rj}C_i{}^r{}_k-g_{ir}C_j{}^r{}_k=0,$

(C3) $T_j{}^i{}_k=F_j{}^i{}_k-F_k{}^i{}_j=0,$

(C4) $S_j{}^i{}_k=C_j{}^i{}_k-C_k{}^i{}_j=0,$

(C5) $y^i|_j=-N^i_j+F_0{}^i{}_j=0,$

where $\delta/\delta x^k=\partial/\partial x^k-N^r_k\partial/\partial y^r$ and the suffix 0 means the contraction by the supporting element y. Supposing some conditions, the Cartan-like connection $(F_j{}^i{}_k, N^i_k, C_j{}^i{}_k)$ can be uniquely determined by the system of the above axioms [7]. In his paper [4], M o ó r obtained the coefficients $F_j{}^i{}_k$. And we can easily see

$$C_j{}^i{}_k=1/2\, g^{ir}(\partial g_{jr}/\partial y^k+\partial g_{kr}/\partial y^j-\partial g_{jk}/\partial y^r).$$

In the present paper, we shall deal with generalized Finsler spaces (M^n, g_{ij}) with

$$g_{ij}(x, y)=e^{2\sigma(x,y)}\gamma_{ij}(x),$$

and the Cartan-like connections on the generalized Finsler spaces, where $\gamma_{ij}(x)$ is a Riemannian metric tensor and $\sigma(x, y)$ is a function of x and y but is not necessarily positively homogeneous of degree 0 with respect to y.

The terminologies and notations are referred to M a t s u m o t o's book [2].

1. A generalized Finsler space (M^n, g_{ij}) with $g_{ij}=e^{2\sigma}\gamma_{ij}$ and the Cartan-like connection. In a generalized Finsler space (M^n, g_{ij}) with $g_{ij}=e^{2\sigma}\gamma_{ij}$, Theorem 6 in the previous paper [8] (or Theorem 1 in N u m a t a's paper [5]) is generalized as follows:

Theorem 1. *Let (M^n, g_{ij}), where $g_{ij}=e^{2\sigma}\gamma_{ij}$, be a generalized Finsler space. If the function σ satisfies $1+\sigma_{\|0}\neq 0$, where $\sigma_{\|i}=\partial\sigma/\partial y^i$, then the Cartan-like connection ${}^*C\Gamma=(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ on (M^n, g_{ij}) is uniquely determined. The connection coefficients $F_j{}^i{}_k$, $N^i{}_k$ and $C_j{}^i{}_k$ are given by*

$$(1)\qquad \begin{aligned} F_j{}^i{}_k&=\{{}^{\,i}_{jk}\}+\delta^i_j\sigma_{|k}+\delta^i_k\sigma_{|j}-\gamma_{jk}\gamma^{ir}\sigma_{|r},\\ N^i{}_k&=F_j{}^i{}_k y^j,\quad C_j{}^i{}_k=\sigma_{\|k}\delta^i_j+\sigma_{\|j}\delta^i_k-\sigma_{\|}{}^i g_{jk},\end{aligned}$$

where

$$\sigma_{|i}=(\sigma_{;i}-\sigma_{|0}\sigma_{\|i}+\sigma_{|}y_i)/(1+\sigma_{\|0}),\quad y_i=g_{ij}y^j,$$

$$\sigma_{;i}=\partial\sigma/\partial x^i-\{{}^{\,r}_{0i}\}\sigma_{\|r},$$

$$\sigma_{|0}=(\sigma_{;0}+L^2\sigma_{;})/(1+2\sigma_{\|0}+\sigma_{\|}{}^2L^2),$$

$$\sigma_{;}=\sigma_{;r}\sigma_{\|}{}^r=\sigma_{;r}\sigma_{\|i}g^{ir},\quad \sigma_{\|}{}^2=\sigma_{\|i}\sigma_{\|}{}^i,$$

$$\sigma_{|}=\sigma_{;}-\sigma_{|0}\sigma_{\|}{}^2,\quad L^2=g_{ij}y^iy^j$$

and $\{{}^{\,i}_{jk}\}$ is the Christoffel symbol formed with respect to γ_{ij}.

Remark. (i) In the proof of the above theorem, the same method that Numata treates in his paper [5] is used.

(ii) Under the assumption $1+\sigma_{\|0}\neq 0$, we have $1+2\sigma_{\|0}+\sigma_{\|}{}^2L^2\neq 0$. Because, if we assume that $1+2\sigma_{\|0}+\sigma_{\|}{}^2L^2=0$, then we have $l_i+L\sigma_{\|i}=y_i/L+L\sigma_{\|i}=0$ from $1+2\sigma_{\|0}+\sigma_{\|}{}^2L^2=g^{ij}(l_i+L\sigma_{\|i})(l_j+L\sigma_{\|j})=0$, and then we have $g^{ij}(l_i+L\sigma_{\|i})l_j=1+\sigma_{\|0}=0$.

(iii) In the case σ is a function of position only, we have $\sigma_{|i}=\partial\sigma/\partial x^i$. Thus the equation (1) gives the familiar equation for the conformal change of Riemannian metrics.

Under the same notations in Theorem 1, $(\{{}^{\,i}_{jk}\}, \{{}^{\,i}_{0k}\}, C_j{}^i{}_k)$ is a Finsler connection on (M^n, g_{ij}) but is not necessarily a Cartan-like connection. Then, we have easily

Theorem 2. *Under the same notations in Theorem 1, a necessary and sufficient condition for the Finsler connection $(\{{}^{\,i}_{jk}\}, \{{}^{\,i}_{0k}\}, C_j{}^i{}_k)$ on (M^n, g_{ij}) to be the Cartan-like connection is that the h-covariant differentiation of σ with respect to the Finsler connection $(\{{}^{\,i}_{jk}\}, \{{}^{\,i}_{0k}\}, C_j{}^i{}_k)$ vanish identically.*

From Theorem 1 and Theorem 2, we have

Corollary 3. *There exists a unique Cartan-like connection on $(M^n, e^{2\sigma}\gamma_{ij})$ with $\sigma=\sqrt{\gamma_{rs}y^ry^s}$. The connection is given by $(\{{}^{\,i}_{jk}\}, \{{}^{\,i}_{0k}\}, C_j{}^i{}_k)$.*

2. (v) hv-torsion tensor P^i_{jk} and hv-curvature tensor $P_h{}^i{}_{jk}$. In this section, we use the following theorem in the previous paper [9].

Theorem. 3. *A necessary and sufficient condition for $F_j{}^i{}_k$ of the Cartan-like connection $(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ on a generalized Finsler space to be functions of position only is that the following equations be satisfied:*

$$(2)\qquad P^i{}_{jk}=\partial N^i{}_j/\partial y^k-F_k{}^i{}_j=0,\quad C_{khi|j}+C_{hki|j}=0,$$

where P^i_{jk} is the (v) hv-torsion tensor of the space and $C_{khi}=g_{hr}C_k{}^r{}_i$.

Remark. The above equation (2) is equivalent to $C_{khi|j}=0$. This fact was pointed out by Hashiguchi.

In the subsequent considerations, we shall deal with the generalized Finsler space in Corollary 3 and denote by M^n it. And we shall consider the Cartan-like connection on M^n.

From the above theorem and Corollary 3, we have

Theorem 4. In M^n, we have

$$P^i{}_{jk}=0,\qquad P_h{}^i{}_{jk}=\partial\{{}^{\,i}_{hj}\}/\partial y^k-C_h{}^i{}_{k|j}+C_h{}^i{}_rP^r{}_{jk}=0.$$

3. (v) h-torsion tensor R^i_{jk} and h-curvature tensor $R_h{}^i{}_{jk}$. The torsion tensor $C_j{}^i{}_k$ is rewritten as follows:

$$(3)\qquad C_j{}^i{}_k=1/\sigma(\gamma_{kr}\delta^i_j+\gamma_{jr}\delta^i_k-\gamma_{jk}\delta^i_r)y^r.$$

From (3), we have

$$(4)\qquad C_0{}^i{}_k=\sigma\delta^i_k.$$

From the definitions of R^i_{jk} and R^i_{hjk}, using (3) and (4), we have the following equations in M^n:

$$(5)\qquad \begin{aligned} R^i{}_{jk}&=\mathfrak{A}_{(jk)}\{\delta\{{}^{\,i}_{0j}\}/\delta x^k\}=\overset{(r)}{R}{}_0{}^i{}_{jk}=R^i_{0jk/(1+\sigma)},\\ R_h{}^i{}_{jk}&=\mathfrak{A}_{(jk)}\{\delta\{{}^{\,i}_{hj}\}/\delta x^k+\{{}^{\,r}_{hj}\}\{{}^{\,i}_{rk}\}\}+C_h{}^i{}_rR^r_{jk}=\overset{(r)}{R}{}_h{}^i{}_{jk}+\gamma_h{}^i{}_{rs}\overset{(r)}{R}{}_t{}^r{}_{jk}y^sy^t,\end{aligned}$$

where $\mathfrak{A}_{(jk)}$ means the interchange of the indices j, k in the terms enclosed with the parentheses {} and subtraction, and

$$\overset{(r)}{R}{}_h{}^i{}_{jk}=\mathfrak{A}_{(jk)}\{\partial\{{}^{\,i}_{hj}\}/\partial x^k+\{{}^{\,r}_{hj}\}\{{}^{\,i}_{rk}\}\},\quad \gamma_h{}^i{}_{rs}=(\gamma_{hs}\delta^i_r+\gamma_{rs}\delta^i_h-\gamma_{hr}\delta^i_s)/\sigma.$$

Thus, from (5), we have

Theorem 5. *In M^n, two conditions $R_h{}^i{}_{jk}=0$ and $\overset{(r)}{R}{}_h{}^i{}_{jk}=0$ are equivalent.*

4. v-curvature tensor $S_h{}^i{}_{jk}$. Let us introduce a tensor defined by

$$h_{ij}=g_{ij}-l_il_j.$$

Then we have

$$(6)\qquad h_{ij}y^i=0.$$

From the definition of $S_h{}^i{}_{jk}$ and (3), in M^n we have

$$(7)\qquad \begin{aligned} S_{hijk}&=g_{is}S_h{}^s{}_{jk}=g_{is}\mathfrak{A}_{(jk)}\{\partial C_h{}^s{}_j/\partial y^k+C_h{}^t{}_jC_t{}^s{}_k\}=\\ &=(1+\sigma^{-1})e^{2\sigma}(h_{hk}h_{ij}-h_{hj}h_{ik}).\end{aligned}$$

In the case of two dimensions, from (6) and (7), we have

Theorem 6. *In M^2, we have $S_{hijk}=0$.*

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ON TM-CONNECTIONS OF A FINSLER SPACE AND THE INDUCED TM-CONNECTIONS ON ITS HYPERSURFACES

BY

HIROSHI YASUDA

TM-connections on a Finsler space are discussed axiomatically and the induced TM-connections on its hypersurface are considered.

§1. TM-Connections. Let M_n be an n-dimensional Finsler space with a fundamental function $L(x, y)$ and be endowed with a TM-connection $TM\Gamma=(\Gamma_j{}^i{}_k, \Gamma^i{}_k, C_j{}^i{}_k)$. This connection is defined as follows:

Firstly the v-connection is given by

$$C_j{}^i{}_k=g^{ih}C_{jhk},\quad C_{jhk}=\tfrac{1}{2}\partial g_{jh}/\partial y^k, \tag{1.1}$$

where $g_{jh}=\frac{1}{2}\partial^2L^2/\partial y^j\partial y^h$. Secondly the non-linear connection is

$$\Gamma^i{}_k=y^j\Gamma_j{}^i{}_k=G^i{}_j+T^i{}_j\,, \tag{1.2}$$

where $G^i{}_j$ is the non-linear connection of Cartan (or Berwald) and (a) $T^i{}_j$ is a positively homogeneous tensor of degree 1 in y^i such that $y^kT^i{}_k=y_iT^i{}_k=0$.

Thirdly the h-connection is given by

$$\Gamma_j{}^i{}_k=\partial\Gamma^i{}_k/\partial y^j+Q_j{}^i{}_k=G_j{}^i{}_k+T_j{}^i{}_k+Q_j{}^i{}_k\,, \tag{1.3}$$

where $G_j{}^i{}_k$ is the h-connection of Berwald, $T_j{}^i{}_k=\partial T^i_k/\partial y^j$ and (b) $Q_j{}^i{}_k$ is a positively homogeneous tensor of degree 0 in y^i such that $y_iQ_j{}^i{}_k=y^jQ_j{}^i{}_k=0$.

Theorem 1. *A TM-connection Γ is characterized by the following six axioms $(TM1)$—$(TM6)$:*

(TM1) *The Γ is metrical, i.e., $L_{|k}=\partial L/\partial x^k-\Gamma^j{}_k\partial L/\partial y^j=0$.*
(TM2) *The deflection tensor vanishes, namely $y^j\Gamma_j{}^i{}_k-\Gamma^i{}_k=0$.*
(TM3) *The Γ is v-metrical, i.e., $g_{ij}|_k=\partial g_{ij}/\partial y^k-C_i{}^r{}_kg_{rj}-C_j{}^r{}_kg_{ir}=0$.*
(TM4) *The v-torsion tensor vanishes, namely $C_j{}^i{}_k-C_k{}^i{}_j=0$.*
(TM5) *The absolute differential of $y_i(=g_{ij}y^j)$ is $Dy^i=g_{ij}Dy^j$,*
or
(TM5)′ *The absolute differential of g_{ij} is indicatric, i.e., $y^iDg_{ij}=0$.*
(TM6) *The paths with respect to the Γ are always geodesics of M_n.*

Corollary 1.1. *Given tensors $T^i{}_j$ and $Q_j{}^i{}_k$ satisfying* (a) *and* (b) *respectively, a TM-connection is uniquely determined by* (TM1)—(TM6).

We shall call a TM-connection Γ an *RTM-connection* if the Γ is r-metrical, that is, the following equation holds for the Γ, $(Dg_{ij}=0)$:

$$T_{ijk}+T_{jik}+Q_{ijk}+Q_{jik}+2(C_{ijr}T^r_k+P_{ijk})=0, \tag{1.4}$$