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Analele ştiinţifice
ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXX, s. I a. Matematică, 1984 — 4

Romanian-Japanese Colloquium
on Finsler Geometry, Iaşi,
Romania, 16—25 August, 1984

ON *TM*-CONNECTIONS OF A FINSLER SPACE AND THE INDUCED *TM*-CONNECTIONS ON ITS HYPERSURFACES

BY

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TM-connections on a Finsler space are discussed axiomatically and the induced *TM*-connections on its hypersurface are considered.

§1. TM-Connections. Let M_n be an n-dimensional Finsler space with a fundamental function $L(x, y)$ and be endowed with a *TM*-connection $TM\Gamma=(\Gamma_j{}^i{}_k, \Gamma^i{}_k, C_j{}^i{}_k)$. This connection is defined as follows:

Firstly the v-connection is given by

$$C_j{}^i{}_k=g^{ih}C_{jhk}, \quad C_{jhk}=\tfrac{1}{2}\partial g_{jh}/\partial y^k, \tag{1.1}$$

where $g_{jh}=\frac{1}{2}\partial^2L^2/\partial y^j\partial y^h$. Secondly the non-linear connection is

$$\Gamma^i{}_k=y^j\Gamma_j{}^i{}_k=G^i{}_j+T^i{}_j\,, \tag{1.2}$$

where $G^i{}_j$ is the non-linear connection of Cartan (or Berwald) and (a) $T^i{}_j$ is a positively homogeneous tensor of degree 1 in y^i such that $y^kT^i{}_k=y_iT^i{}_k=0$.

Thirdly the h-connection is given by

$$\Gamma_j{}^i{}_k=\partial\Gamma^i{}_k/\partial y^j+Q_j{}^i{}_k=G_j{}^i{}_k+T_j{}^i{}_k+Q_j{}^i{}_k\,, \tag{1.3}$$

where $G_j{}^i{}_k$ is the h-connection of Berwald, $T_j{}^i{}_k=\partial T^i_k/\partial y^j$ and (b) $Q_j{}^i{}_k$ is a positively homogeneous tensor of degree 0 in y^i such that $y_iQ_j{}^i{}_k=y^jQ_j{}^i{}_k=0$.

Theorem 1. *A TM-connection Γ is characterized by the following six axioms (TM1)—(TM6):*

(TM1) *The Γ is metrical, i.e.,* $L_{|k}=\partial L/\partial x^k-\Gamma^j{}_k\partial L/\partial y^j=0$.
(TM2) *The deflection tensor vanishes, namely* $y^j\Gamma_j{}^i{}_k-\Gamma^i{}_k=0$.
(TM3) *The Γ is v-metrical, i.e.,* $g_{ij}|_k=\partial g_{ij}/\partial y^k-C_i{}^r{}_kg_{rj}-C_j{}^r{}_kg_{ir}=0$.
(TM4) *The v-torsion tensor vanishes, namely* $C_j{}^i{}_k-C_k{}^i{}_j=0$.
(TM5) *The absolute differential of* $y_i(=g_{ij}y^j)$ *is* $Dy^i=g_{ij}Dy^j$,
or
(TM5)′ *The absolute differential of* g_{ij} *is indicatric, i.e.,* $y^iDg_{ij}=0$.
(TM6) *The paths with respect to the Γ are always geodesics of* M_n.

Corollary 1.1. *Given tensors* $T^i{}_j$ *and* $Q_j{}^i{}_k$ *satisfying* (a) *and* (b) *respectively, a TM-connection is uniquely determined by* (TM1)—(TM6).

We shall call a *TM*-connection Γ an *RTM-connection* if the Γ is r-metrical, that is, the following equation holds for the Γ, $(Dg_{ij}=0)$:

$$T_{ijk}+T_{jik}+Q_{ijk}+Q_{jik}+2(C_{ijr}T^r_k+P_{ijk})=0, \tag{1.4}$$

where $T_{ijk}=g_{jr}T^r_{ik}$ and $Q_{ijk}=g_{jr}Q^r_{ik}$.

Theorem 2. *An RTM-connection* Γ *is characterized by the following five axioms* (RTM1)—(RTM5):

(RTM1) *The* Γ *is h-metrical, i.e.,* $g_{ij|k}=0$.

(RTMi)=(TMi) ($i=2, 3, 4$) (RTM5)=(TM6).

If we put $Z_{ijk}=Q_{ijk}-Q_{jik}$, then the following condition holds:

(d) Z_{ijk} is a positively homogeneous tensor of degree 0 in y^i such that $Z_{ijk}y^i=Z_{ijk}y^j=0$ and $Z_{ijk}+Z_{jik}=0$.

$$\Gamma^i_{jk}=\Gamma^{*i}_{jk}-C^i_{jr}T^r_k+\tfrac{1}{2}g^{ih}(Z_{jrk}+T_{jrk}-T_{rjk}), \tag{1.5}$$

where Γ^{*i}_{jk} is the h-connection of Cartan.

Corollary 2.1. *Given tensors* T^i_k *and* Z_{ijk} *satisfying* (a) *and* (d) *respectively, an RTM-connection is uniquely determined by* (RTM1)—(RTM5). *In this case, the h-connection is given by* (1.5).

$$\tau_j{}^i{}_k=T_j{}^i{}_k+Q_j{}^i{}_k-(T_k{}^i{}_j+Q_k{}^i{}_j), \quad \tau_{jik}=g_{ir}\tau_j{}^r{}_k. \tag{1.6}$$

$$\Gamma^i_{jk}=\Gamma^{*i}_{jk}+\tfrac{1}{2}g^{ir}(\tau_{jkr}+\tau_{jrk}+\tau_{kjr})+g^{ih}C_{jkr}T^r_h-C^i_{jr}T^r_k-C^i_{kr}T^r_j. \tag{1.7}$$

Corollary 2.2. *Given the h-torsion tensor* $\tau_j{}^i{}_k$ *satisfying* $\tau_0{}^0{}_k=0$, *an RTM-connection is uniquely determined by* (RTM1)—(RTM5). *In this case, the h-connection is given by* (1.7).

We shall call a TM-connection $\Gamma=(\Gamma^i_{jk}, \Gamma^i_k, 0)$ a *TM(0)-connection*.

Theorem 3. *A* $TM(0)$*-connection is characterized by the following five axioms* (T1)—(T5): (Ti)=(TMi) ($i=1, 2$), (T3)=(TM5), (T4)=(TM6),

(T5) *The v-connection parameters vanish, i.e.,* $C^i_{jk}=0$.

Given tensors T^i_k and $Q_j{}^i{}_k$ satisfying (a) and (b) respectively, a $TM(0)$-connection is uniquely determined by (T1)—(T5).

We shall call a $TM(0)$-connection an $STM(0)$-connection if the h-torsion tensor $\tau_j{}^i{}_k$ vanishes.

Theorem 4. *An* $STM(0)$*-connection is characterized by the following five axioms:* (T1), (T2), (T3), (T5) *and*

(STM6) *The h-torsion tensor vanishes,* i.e., $\tau_j{}^i{}_k=0$.

$$\Gamma^i_k=G^i_k+T^i_k, \quad T^i_k=\tfrac{1}{2}Q_k{}^i{}_0, \quad T_j{}^i{}_k=\tfrac{1}{2}Q_k{}^i{}_{0\|j}. \tag{1.8}$$

$$\Gamma^i_{jk}=G_j{}^i{}_k+\tfrac{1}{2}(Q_j{}^i{}_k+Q_k{}^i{}_j)+\tfrac{1}{4}(Q_k{}^i{}_{0\|j}+Q_j{}^i{}_{0\|k}). \tag{1.9}$$

$$y^r(Q_j{}^i{}_{r\|k}-Q_k{}^i{}_{0\|j})=0. \tag{1.10}$$

$$\Gamma^i_{jk}=G_j{}^i{}_k+\tfrac{1}{2}Q_k{}^i{}_{0\|j}+Q_j{}^i{}_k. \tag{1.11}$$

Corollary 4.1. *An* $STM(0)$*-connection is uniquely determined by* (T1) (T2), (T5) *and* (STM6) *if there is given a tensor* $Q_j{}^i{}_k$ *such that* (i) $Q_j{}^i{}_k$ *satisfies* (b) *or* (ii) $Q_j{}^i{}_k$ *satisfies* (b) *and* (1.10). *In this case, the h-connection is given by* (1.9) *or* (1.11) *respectively.*

§2. The induced TM-connections on its hipersurface. Let M_{n-1} be a hypersurface of M_n represented parametrically by the equation

$$x^i=x^i(u^\alpha) \quad (i=1, 2, \ldots, n;\ \alpha=1, 2, \ldots, n-1), \tag{2.1}$$

where we suppose that the variables u^α form a coordinate system of M_{n-1}. In the following, latin indices run from 1 to n, while greek indices from 1 to $n-1$. We put $B^i_\alpha=\partial x^i/\partial u^\alpha$ and assume that the matrix (B^i_α) is of rank $n-1$. If we denote the components of a vector y^i tangent to a curve C contained in M_{n-1} by y^α in terms of u^α-system, we have

$$y^i=B^i_\alpha y^\alpha, \quad \partial y^i/\partial y^\alpha=B^i_\alpha. \tag{2.2}$$

The fundamental function $\overline{L}(u^\alpha, y^\alpha)$ on M_{n-1} induced from $L(x^i, y^i)$ of M_n is $\overline{L}(u^\alpha, y^\alpha)=L(x^i(u^\alpha), B^i_\alpha y^\alpha)$. Then the r-metric tensor $g_{\alpha\beta}(u^\alpha, y^\alpha)=\frac{1}{2}\partial^2L^2/\partial y^\alpha\partial y^\beta$ is expressible in

$$g_{\alpha\beta}(u^\alpha, y^\alpha)=g_{ij}(x^i, y^i)B^{ij}_{\alpha\beta}, \quad B^{ij}_{\alpha\beta}=B^i_\beta B^i_\beta. \tag{2.3}$$

With respect to g_{ij}, we can choose a unit normal vector $N^i(u^\alpha, y^\alpha)$ at each point (u^α) of M_{n-1} such that $g_{ij}N^iB^j_\alpha=0$ and $g_{ij}N^iN^j=1$.

Let (B^α_i, N_i) be the inverse matrix of (B^i_α, N^i). Then we have

$$\begin{gathered}\mu_{\alpha\beta}:=C_{ijk}B^{ij}_{\alpha\beta}N^k, \quad \mu_\alpha:=C_{ijk}B^i_\alpha N^jN^k, \quad \mu:=C_{ijk}N^iN^jN^k,\\ \partial B^\alpha_i/\partial y^\beta=B^\alpha_{i|\!|\beta}=2\mu^\alpha_\beta N_i, \quad \mu^\alpha_\beta=g^{\alpha\gamma}\mu_{\gamma\beta}.\end{gathered} \tag{2.4}$$

When $X^i=B^i_\alpha X^\alpha$, we define the absolute differentials as follows:

$$Dy^\alpha=B^\alpha_iDy^i, \; DX^\alpha=B^\alpha_iDX^i, \; \overset{0}{D}X^\alpha=B^\alpha_iDX^i. \tag{2.5}$$

Then on M_{n-1} we can define connections $\Gamma=(\Gamma_\alpha{}^\beta{}_\gamma, \Gamma^\alpha_\gamma, C_\beta{}^\alpha{}_\gamma)$ and $\overline{\Gamma}_0=(\overset{0}{\Gamma}_\beta{}^\alpha{}_\gamma, \Gamma^\alpha_\gamma, 0)$ by means of (2.5). In this case, these are as follows:

$$C_\beta{}^\alpha{}_\gamma=C_j{}^i{}_kB^{\alpha jk}_{i\beta\gamma}, \quad B^{\alpha jk}_{i\beta\gamma}=B^\alpha_iB^j_\beta B^k_\gamma, \tag{2.6}$$

$$\Gamma^\alpha_\gamma=B^\alpha_i(B_0{}^i{}_\gamma+\Gamma^i_kB^k_\gamma), \quad B_0{}^i{}_\gamma=(\partial B^i_\gamma/\partial u^\beta)y^\beta, \tag{2.7}$$

$$\Gamma_\beta{}^\alpha{}_\gamma=B^\alpha_i(B_\beta{}^i{}_\gamma+\Gamma_j{}^i{}_kB^{jk}_{\beta\gamma}+C_\beta{}^i{}_kN^kH_\gamma), \; C_\beta{}^i{}_k=C_j{}^i{}_kB^j_\beta, \tag{2.8}$$

$$H_\gamma=N_i(B_0{}^i{}_\gamma+\Gamma^i_hB^h_\gamma), \tag{2.9}$$

$$H_{\beta\gamma}=N_i(B_\beta{}^i{}_\gamma+\Gamma_j{}^i{}_kB^{jk}_{\beta\gamma}+C_\beta{}^i{}_kN^kH_\gamma), \tag{2.10}$$

$$\overset{0}{\Gamma}_\beta{}^\alpha{}_\gamma=B^\alpha_i(B_\beta{}^i{}_\gamma+\Gamma_j{}^i{}_kB^{jk}_{\beta\gamma}). \; \overset{0}{H}_{\beta\gamma}=N_i(B_\beta{}^i{}_\gamma+\Gamma_j{}^i{}_kB^{jk}_{\beta\gamma}). \tag{2.11}$$

H_γ is the normal curvature vector, while $H_{\beta\gamma}$(or $\overset{0}{H}_{\beta\gamma}$) is the second fundamental tensor. We shall call $\overline{\Gamma}$ and $\overline{\Gamma}_0$ the *induced TM-connection* and the *induced TM(0)-connection* respectively.

If we denote the Christoffel symbols of the second kind formed with $g_{\alpha\beta}$ by $\gamma_\alpha{}^\gamma{}_\beta$, then we have

$$2G^\gamma=\gamma_\alpha{}^\gamma{}_\beta y^\alpha y^\beta=B^\gamma_i(B_0{}^i{}_0+2G^i). \tag{2.12}$$

Put

$$\overset{b}{\Gamma}{}^\alpha_\gamma=B^\alpha_i(B_0{}^i{}_\gamma+G^i_kB^k_\gamma), \quad T^\alpha_\gamma=T^i_kB^{\alpha k}_{i\gamma}, \quad Q_\beta{}^\alpha{}_\gamma=Q_j{}^i{}_kB^{\alpha jk}_{i\beta\gamma}. \tag{2.13}$$

Differentiating (2.12) and making use of (2.4) and (2.9), we have

$$G^{\alpha}{}_{\gamma}=G^{\alpha}{}_{\|\gamma}=\mu^{\alpha}{}_{\gamma}H_0+\overset{b}{\Gamma}{}^{\alpha}{}_{\gamma},\quad H_0=H_{\gamma}y^{\gamma}=N_i(B_0{}^i{}_0+2G^i). \tag{2.14}$$

From (2.4), (2.7), (2.8), (2.9), (2.13), and (2.14), we have

$$\tilde{T}^{\alpha}{}_{\gamma}=\Gamma^{\alpha}{}_{\gamma}-G^{\alpha}{}_{\gamma}=-\mu^{\alpha}{}_{\gamma}H_0+T^{\alpha}{}_{\gamma}. \tag{2.15}$$

$$\tilde{Q}_{\beta}{}^{\alpha}{}_{\gamma}=\Gamma_{\beta}{}^{\alpha}{}_{\gamma}-\Gamma^{\alpha}{}_{\gamma\|\beta}=-\mu^{\alpha}{}_{\beta}H_{\gamma}+Q_{\beta}{}^{\alpha}{}_{\gamma}. \tag{2.16}$$

Theorem 5. *The induced TM-connection is the TM-connection on M_{n-1} determined by tensors $\tilde{T}^{\alpha}{}_{\gamma}$ and $\tilde{Q}_{\beta}{}^{\alpha}{}_{\gamma}$ in* (2.15) *and* (2.16).

We shall call the connection on M_{n-1} induced from an *RTM*-connection on M_n the *induced RTM-connection.* Then we have

$$g_{\alpha\beta|\gamma}=g_{ij|k}B^{ijk}_{\alpha\beta\gamma}=0,\quad g_{\alpha\beta}|_{\gamma}=g_{ij}|_kB^{ijk}_{\alpha\beta\gamma}=0. \tag{2.17}$$

$$\tilde{Z}_{\alpha\beta\gamma}=\tilde{Q}_{\alpha\beta\gamma}-\tilde{Q}_{\beta\alpha\gamma}=Q_{\alpha\beta\gamma}-Q_{\beta\alpha\gamma}=Z_{ijk}B^{ijk}_{\alpha\beta\gamma}, \tag{2.18}$$

$$\Gamma_{\beta}{}^{\alpha}{}_{\gamma}=\Gamma^{*\alpha}_{\beta\gamma}-C^{\alpha}_{\beta\varepsilon}\tilde{T}^{\varepsilon}_{\gamma}+\tfrac{1}{2}g^{\alpha\varepsilon}(\tilde{Z}_{\beta\varepsilon\gamma}+\tilde{T}_{\beta\varepsilon\gamma}-\tilde{T}_{\varepsilon\beta\gamma}), \tag{2.19}$$

where $\Gamma^{*\alpha}_{\beta\gamma}$ is the intrinsic Cartan Connection formed with $\bar{L}(u^{\alpha}, y^{\alpha})$

Theorem 6. *The induced RTM-connection is the RTM-connection on M_{n-1} determined by tensors $\tilde{T}^{\alpha}_{\gamma}$ and $\tilde{Z}_{\alpha\beta\gamma}$ in* (2.15) *and* (2.18). *In this case, the h-connection is expressible in* (2.19).

Note. If the *h*-torsion tensor $\tilde{\tau}_{\beta}{}^{\alpha}{}_{\gamma}$ satisfying $\tilde{\tau}_0{}^0{}_{\gamma}=0$ is given, then the induced *RTM*-connection is determined by $\tilde{\tau}_{\beta}{}^{\alpha}{}_{\gamma}$ alone according to Corollary 2.2.

From (2.7), (2.9), (2.11) and (2.14), we have

$$\tilde{T}^{\alpha}{}_{\gamma}=\mu^{\alpha}{}_{\gamma}H_0+T^{\alpha}{}_{\gamma},\quad \tilde{Q}_{\beta}{}^{\alpha}{}_{\gamma}=-2\mu^{\alpha}{}_{\beta}H_{\gamma}+Q_{\beta}{}^{\alpha}{}_{\gamma}, \tag{2.20}$$

$$\tilde{\tau}_{\beta}{}^{\alpha}{}_{\gamma}=\tau_j{}^i{}_kB^{\alpha jk}_{i\beta\gamma},\quad \tilde{P}_{\beta}{}^{\alpha}{}_{\gamma}=2\mu^{\alpha}{}_{\beta}H_{\gamma}-Q_{\beta}{}^{\alpha}{}_{\gamma}. \tag{2.21}$$

Theorem 7. *The induced TM(0)-connection is the TM(0)-connection on M_{n-1} determined by tensors $\tilde{T}^{\alpha}{}_{\gamma}$ and $\tilde{Q}_{\beta}{}^{\alpha}{}_{\gamma}$ in* (2.20), *and the h-and hv-torsion tensors $\tilde{\tau}_{\beta}{}^{\alpha}{}_{\gamma}$ and $\tilde{P}_{\beta}{}^{\alpha}{}_{\gamma}$ are given by* (2.21).

We shall call the connection on M_{n-1} induced from an STM(0)-connection on M_n the *induced STM(0)-connection.*

$$\tfrac{1}{2}\tilde{Q}_{\gamma}{}^{\alpha}{}_0=-\mu^{\alpha}{}_{\gamma}H_0+\tfrac{1}{2}Q_{\gamma}{}^{\alpha}{}_0=-\mu^{\alpha}{}_{\gamma}H_0+T^{\alpha}{}_{\gamma}=\tilde{T}^{\alpha}{}_{\gamma}, \tag{2.22}$$

$$\overset{0}{\Gamma}{}_{\beta}{}^{\alpha}{}_{\gamma}=G_{\beta}{}^{\alpha}{}_{\gamma}+\tfrac{1}{2}(\tilde{Q}_{\beta}{}^{\alpha}{}_{\gamma}+\tilde{Q}_{\gamma}{}^{\alpha}{}_{\beta})+\tfrac{1}{4}(\tilde{Q}_{\gamma}{}^{\alpha}{}_{0\|\beta}+\tilde{Q}_{\beta}{}^{\alpha}{}_{0\|\gamma}),\quad G_{\beta}{}^{\alpha}{}_{\gamma}=G^{\alpha}{}_{\gamma\|\beta}. \tag{2.23}$$

Theorem 8. *The induced STM(0)-connection is the STM(0)-connection on M_{n-1} determined by the tensor $\tilde{Q}_{\beta}{}^{\alpha}{}_{\gamma}$ in* (2.20) *and the h-connection is expressible in* (2.23).

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