3. Hashiguchi, M. — *Lecture notes in Romania and Hungary*, 1983 (unpublished)
4. Hashiguchi, M.—*On generalized Finsler spaces*, An. şt. Univ. Iaşi, S. I a, 30 (1984), 69—73
5. Hashiguchi, M. and Ichijyo, Y. — *On generalized Berwald spaces*, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) 15 (1982), 19—32
6. Matsumoto, M. — *Foundations of Finsler geometry and special Finsler spaces*, 1977 (unpublished), 373 pp.
7. Matsumoto, M. — *Berwald connections with (h) h-torsion and generalized Berwald spaces*, Tensor, N.S. 35 (1981), 223—229
8. Matsumoto, M. — *The induced and intrinsic Finsler connections of a hypersurface and Finslerian projective geometry*, to appear in J. Math. Kyoto Univ.
9. Miron, R. — *Metrical Finsler structures and metrical Finsler connections*, J. Math. Kyoto Univ. 23 (1983), 219—224
10. Miron, R. and Hashiguchi, M. — *Metrical Finsler connections*, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) 12 (1979), 21—35
11. Okada, T. — *Minkowskian product of Finsler spaces and Berwald connections*, J. Math. Kyoto Univ. 22 (1982), 323—332

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FINSLER HYPERCONNECTIONS

BY

I. D. ALBU and D. OPRIŞ

The theory of the hypertangent bundle and linear hyperconnections was considered in [2], for their application in the so-called unified field theory of gravitation and electromagnetism. In this paper, we define and study the principal properties of the Finsler hypertangent bundle and the Finsler hyperconnections.

§1. Finsler hypertangent bundle. Let M be an n-dimensional differentiable manifold and $T=(TM, \pi_T, M)$ the tangent bundle of M (with fibre type R^n). Let $J=(JM, \pi_J, M)$ be an N-dimensional differentiable vector bundle over M with fibre type $R^N (N>n)$. We suppose that there exists in J a subbundle $T'=(T'M, \pi_{T'}, M)$ which is differentiably equivalent to T. Since T is imbedded in J as a subbundle, there exists a subbundle $\bar{T}=(\bar{T}M, \pi_{\bar{T}}, M)$ with fibre type R^{N-n}, which is complementary to T in J. The vector bundle J is called an N-dimensional *hypertanget bundle* over M and the subbundle T is called a *vertical vector bundle.*

Let J be an hypertangent bundle over M. We call the *Finsler hypertangent bundle*, the Finsler bundle associated to the pair of vector bundle (T, J), [1], i.e., the vector bundle $FJ=(\pi_T^* JM, \pi_{FJ}, TM)$. If T is imbedded in J, then the Finsler tangent bundle of M, $FT=(\pi_T^* TM, \pi_{FT}, TM)$, is imbedded in FJ and there exists a subbundle $F\bar{T}$ of FJ which is complementary to FT. $F\bar{T}$ is the Finsler bundle associated to $(T, \bar{T})$, namely, $F\bar{T}=(\pi_T^* \bar{T}M, \pi_{F\bar{T}})$ with fibre type R^m, $m=N-n$. $F\bar{T}$ is called a *Finsler vertical bundle.*

Let $(\tilde{U}, \varphi)$ be a structural chart of FJ, $\tilde{x}=(x^i, y^i)$, $i=\overline{1, n}$, the local coordinates of TM in $\bar{U}$ and (x^i, y^i, z^α), $\alpha=\overline{1, N}$, the local coordinates of FJ in $\pi_{FJ}^{-1}(\tilde{U})$. Let $s_i(\tilde{x})=\varphi^{-1}(\tilde{x}, e_i)$, $i=\overline{1, n}$, $\tilde{x}\in\tilde{U}$ be n linearly independent cross-sections of FT, where $\{e_i\}$ is the canonical base of R^n and $s_a(\tilde{x})=\varphi^{-1}(\tilde{x}, e_a)$, $a=\overline{n+1, N}$, $N=n+m$, m linearly independent cross-sections of $F\bar{T}$, where $\{e_a\}$ is a complementary base of $\{e_i\}$ in R^N. We put $E_\alpha(\tilde{x})=\varphi^{-1}(\tilde{x}, e_\alpha)$, $\alpha=\overline{1, N}$, $\forall\tilde{x}\in U$, where $\{e_\alpha\}$ is the base $\{e_i, c_a\}$ of R^N. We obtain N linearly independent cross-sections of FJ on $\tilde{U}$. If $(\tilde{U}', \varphi')$ is another structural chart of FJ with $\tilde{U}\cap\tilde{U}'\neq\varnothing$, then we have the following transformation rules:

$x^{i'}=x^{i'}(x^1,\dots,x^n)$, $y^{i'}=(\partial x^{i'}/\partial x^i)y^i, z^{\alpha'}=M_\alpha^{\alpha'}z^\alpha, E_\alpha(\tilde{x})=M_\alpha^{\alpha'}(\tilde{x})E_{\alpha'}(\tilde{x})$, where M: $\tilde{U}\cap\tilde{U}'\to Gl(n, R)$. We have:

(1) $s_i(\tilde{x})=B_i^\alpha(\tilde{x})E_\alpha(\tilde{x}),\ s_a(\tilde{x})=D_a^\alpha(\tilde{x})E_\alpha(\tilde{x}),\ E_\alpha(\tilde{x})=B_\alpha^i(\tilde{x})s_i(\tilde{x})+D_\alpha^a(\tilde{x})s_a(\tilde{x})$, $\forall \tilde{x}\in\tilde{U}$, where $B_i^\alpha D_\alpha^a=0$, $D_a^\alpha D_\alpha^b=\delta_a^b$, $B_i^\alpha B_\beta^i+D_a^\alpha D_\beta^a=\delta_\beta^\alpha$, $B_i^\alpha B_\alpha^j=\delta_i^j$, $B_\alpha^i D_a^\alpha=0$.

For $\hat{X}\in\mathrm{Sect}\,(FT)$, $\hat{Y}\in\mathrm{Sect}\,(F\bar{T})$, $\hat{Z}\in\mathrm{Sect}\,(FJ)$, we have:

$$\hat{X}=\hat{X}^i s_i,\ \hat{Y}=\hat{Y}^a s_a,\ \hat{Z}=\hat{Z}^\alpha E_\alpha=\hat{Z}^\alpha B_\alpha^i s_i+\hat{Z}^\alpha D_\alpha^a s_a. \tag{2}$$

§2. Finsler hyperconnections. *A Finsler hyperconnection* is defined by a pair $FJ\Gamma(\Gamma^{FJ},\Gamma^T)$, where Γ^{FJ} is a linear connection in FJ and Γ^T is a non-linear connection in T. The components of Γ^{FJ}, $\Gamma_{i\beta}^\alpha(\tilde{x})$, $C_{i\beta}^\alpha(\tilde{x})$, verify the following transformation rule, at a change of local coordinate system:

$$\begin{aligned}\Gamma_{i\beta}^\alpha&=\Gamma_{i'\beta'}^{\alpha'}\frac{\partial x^{i'}}{\partial x^i}M_{\alpha'}^\alpha M_\beta^{\beta'}+C_{i'\beta'}^{\alpha'}\frac{\partial^2 x^{i'}}{\partial x^i\partial x^j}y^j M_{\alpha'}^\alpha M_\beta^{\beta'}+M_{\alpha'}^\alpha\frac{\partial M_\beta^{\alpha'}}{\partial x^i},\\ C_{i\beta}^\alpha&=C_{i'\beta'}^{\alpha'}\frac{\partial x^{i'}}{\partial x^i}M_{\alpha'}^\alpha M_\beta^{\beta'},\end{aligned}\tag{3}$$

while the components of Γ^T, $N_j^i(\tilde{x})$, satisfy the rule:

$$N_j^i=N_{j'}^{i'}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial x^i}{\partial x^{i'}}+\frac{\partial^2 x^i}{\partial x^{i'}\partial x^{j'}}\frac{\partial x^{i'}}{\partial x^i}y^{j'}. \tag{4}$$

For $X=X^i\,\partial/\partial x^i+\bar{X}^i\,\partial/\partial y^i\in\mathcal{X}(TM)$ and $\hat{Y}=\hat{Y}^\alpha E_\alpha$, the covariant derivative is given by

$$D_X^{FJ}\hat{Y}=\left(X^i\left(\frac{\partial\hat{Y}^\alpha}{\partial x^i}+\Gamma_{i\beta}^\alpha\hat{Y}^\beta\right)+\bar{X}^i\left(\frac{\partial\hat{Y}^\alpha}{\partial y^i}+C_{i\beta}^\alpha\hat{Y}^\beta\right)\right)E_\alpha \tag{5}$$

Since Γ^T admits the local base $\{\delta/\delta x^i=\partial/\partial x^i-N_i^j\partial/\partial y^j,\ \partial/\partial y^i\}$, we have

$$D^{FJ}_{\frac{\delta}{\delta x^i}}E_\alpha=(\Gamma_{i\alpha}^\beta-N_i^jC_{j\alpha}^\beta)E_\beta=F_{i\alpha}^\beta E_\beta,\ D^{FJ}_{\frac{\partial}{\partial y^i}}E_\alpha=C_{i\alpha}^\beta E_\beta. \tag{6}$$

It results that a Finsler hyperconnection is determined by the triad $(N_i^j, N_i^j, F_{i\beta}^\alpha, C_{i\beta}^\alpha)$.

§3. Induced Finsler connection. For $s_i\in\mathrm{Sect}\,(FT)$, $i=\overline{1,n}$, defined in §1, we have

$$D^{FJ}_{\frac{\delta}{\delta x^i}}s_j=\mathring{F}_{ij}^h s_h+\mathring{H}_{ij}^a s_a,\ D^{FJ}_{\frac{\partial}{\partial y^i}}s_j=\mathring{C}_{ij}^h s_h+\mathring{L}_{ij}^a s_a, \tag{7}$$

where

$$\begin{aligned}\mathring{F}_{ij}^h&=B_\beta^h\left(\frac{\delta B_j^\beta}{\delta x^i}+F_{i\alpha}^\beta B_j^\alpha\right),\quad \mathring{C}_{ij}^h=B_\beta^h\left(\frac{\partial B_j^\beta}{\partial y^i}+C_{i\alpha}^\beta B_j\right),\\ \mathring{H}_{ij}^a&=D_\beta^a\left(\frac{\delta B_j^\beta}{\delta x^i}+F_{i\alpha}^\beta B_j^\alpha\right),\quad \mathring{L}_{ij}^a=D_\beta^a\left(\frac{\partial B_j^\beta}{\partial y^i}+C_{i\alpha}^\beta B_j^\alpha\right).\end{aligned}\tag{8}$$

Proposition 1. *The functions $(N_j^i, \mathring{F}_{ij}^h, \mathring{C}_{ij}^h)$ define a Finsler connection $FT\mathring{\Gamma}$ in FT, which is called the induced Finsler connection.*

Remark. The induced Finsler connection $FT\mathring{\Gamma}$ is determined by the given Finsler hyperconnection $FJ\Gamma$ and the choice of the Finsler vertical bundle $F\bar{T}$.

Let $\hat{X}\in\mathrm{Sect}\,(FT)$. We have

$$D^{FJ}_{\frac{\delta}{\delta x^i}}\hat{X}=(B_j^\alpha\hat{X}^j{}_{\mathring{|}i}+D_a^\alpha\mathring{H}_{ij}^a\hat{X}^j)E_\alpha,\quad D^{FJ}_{\frac{\partial}{\partial y^i}}\hat{X}=(B_j^\alpha\hat{X}^j\mathring{|}_i+D_a^\alpha\mathring{L}_{ij}^a\hat{X}^j)E_\alpha, \tag{9}$$

where

$$\begin{aligned}\hat{X}^j{}_{\mathring{|}i}&=\frac{\delta\hat{X}^j}{\delta x^i}+\mathring{F}_{ih}^j\hat{X}^h,\quad \hat{X}^j\mathring{|}_i=\frac{\partial\hat{X}^j}{\partial y^i}+\mathring{C}_{ih}^j\hat{X}^h,\\ \mathring{H}_{ij}^a&=D_\beta^a\left(\frac{\delta B_j^\beta}{\delta x_\beta^i}+F_{i\alpha}^\beta B_j^\alpha-\mathring{F}_{ij}^h B_h^\beta\right)=D_\beta^a B_j^\beta{}_{\underset{m}{|}i},\\ \mathring{L}_{ij}^a&=D_\beta^a\left(\frac{\partial B_j^\beta}{\partial y^i}+C_{i\alpha}^\beta B_j^\alpha-\mathring{C}_{ij}^h B_h^\beta\right)=D_\beta^a B_j^\beta\overset{m}{|}_i.\end{aligned}\tag{10}$$

$\underset{m}{|}$ and $\overset{m}{|}$ denote the mixed covariant derivations with respect to the Finsler connections $FJ\Gamma$ and $FT\mathring{\Gamma}$, [1].

Let $s_a\in\mathrm{Sect}(F\bar{T})$, $a=\overline{n+1,n+m}$, be the cross-sections defined in §1. We have

$$D^{FJ}_{\frac{\delta}{\delta x^i}}s_a=\dot{H}_{ia}^h s_h+\dot{F}_{ia}^b s_b,\ D^{FJ}_{\frac{\partial}{\partial y^i}}s_a=\dot{L}_{ia}^h s_h+\dot{C}_{ia}^b s_b, \tag{11}$$

where

$$\begin{aligned}\dot{H}_{ia}^h&=B_\beta^h\left(\frac{\delta D_a^\beta}{\delta x^i}+F_{i\alpha}^\beta D_a^\alpha\right),\quad \dot{F}_{ia}^b=D_\beta^b\left(\frac{\delta D_a^\beta}{\delta x^i}+F_{i\alpha}^\beta D_a^\alpha\right),\\ \dot{L}_{ia}^h&=B_\beta^h\left(\frac{\partial D_a^\beta}{\partial y^i}+C_{i\alpha}^\beta D_a^\alpha\right),\quad \dot{C}_{ia}^b=D_\beta^b\left(\frac{\partial D_a^\beta}{\partial y^i}+C_{i\alpha}^\beta D_a^\alpha\right).\end{aligned}\tag{12}$$

Proposition 2. *The functions $(N_i^j, \dot{F}_{ib}^a, \dot{C}_{ib}^a)$ define a Finsler connection $F\bar{T}\dot{\Gamma}$ in $F\bar{T}$, which is called the vertical Finsler connection.*

Remark. The vertical Finsler connection $F\bar{T}\dot{\Gamma}$ is determined by the given Finsler hyperconnection $FJ\Gamma$ and the choice of the Finsler vertical bundle $F\bar{T}$.

If $\hat{Y}\in\mathrm{Sect}(F\bar{T})$, we obtain:

$$D^{FJ}_{\frac{\delta}{\delta x^i}}\hat{Y}=(B_j^\alpha\dot{H}_{ia}^j\hat{Y}^a+D_a^\alpha\hat{Y}^a{}_{\dot{|}i})E_\alpha,\ D^{FJ}_{\frac{\partial}{\partial y^i}}\hat{Y}=(B_j^\alpha\dot{L}_{ia}^j\hat{Y}^a+D_a^\alpha\hat{Y}^a\dot{|}_i)E_\alpha, \tag{13}$$

where

$$\hat{Y}^{a}{}_{|i}=\frac{\delta \hat{Y}^{a}}{\delta x^{i}}+\overset{\circ}{F}{}^{a}_{ib}\hat{Y}^{b},\ \hat{Y}^{a}|_{i}=\frac{\partial \hat{Y}^{a}}{\partial y^{i}}+\overset{\circ}{C}{}^{a}_{ib}\hat{Y}^{b},$$

(14)
$$\overset{*}{H}{}^{h}_{ia}=B^{h}_{\beta}\left(\frac{\delta D^{\beta}_{a}}{\delta x^{i}}+F^{\beta}_{i\alpha}D^{\alpha}_{a}-\overset{\circ}{F}{}^{b}_{ia}D^{\beta}_{b}\right)=B^{h}_{\beta}D^{\beta}_{a}{}^{\mathrm{v}}_{|i},$$
$$\overset{*}{L}{}^{h}_{ia}=B^{h}_{\beta}\left(\frac{\partial D^{\beta}_{a}}{\partial y^{i}}+C^{\beta}_{i\alpha}D^{\alpha}_{a}-\overset{\circ}{C}{}^{b}_{ia}D^{\beta}_{b}\right)=B^{h}_{\beta}D^{\beta}_{a}{}^{\mathrm{v}}|_{i}.$$

$\overset{\mathrm{v}}{|}$ and $\overset{\mathrm{v}}{|}$ denote the mixed covariant derivations with respect to $FJ\Gamma$ and $F\overline{T}\overset{\circ}{\Gamma}$,

Remark. If $\hat{Z}\in \mathrm{Sect}(FJ)$, then

(15)
$$D^{FJ}_{\frac{\delta}{\delta x^{i}}}\hat{Z}=(B^{\alpha}_{h}(\hat{Z}^{h}{}_{|i}+\overset{*}{H}{}^{h}_{ia}\hat{Z}^{a})+D^{\alpha}_{a}(\overset{*}{H}{}^{a}_{ij}\hat{Z}^{j}+\hat{Z}^{a}{}_{|i}))E_{\alpha},$$
$$D^{FJ}_{\frac{\partial}{\partial y^{i}}}\hat{Z}=(B^{\alpha}_{h}(\hat{Z}^{h}|_{i}+\overset{*}{L}{}^{h}_{ia}\hat{Z}^{a})+D^{\alpha}_{a}(\overset{*}{H}{}^{a}_{ij}\hat{Z}^{j}+\hat{Z}^{a}|_{i}))E_{\alpha}.$$

Proposition 3. *Let $F\overline{T}$ be a fixed Finsler vertical bundle in FJ, $(N^{i}_{j}, \overset{\circ}{F}{}^{i}_{jk}, \overset{\circ}{C}{}^{i}_{jk})$ a Finsler connection in FT, $(N^{i}_{j}, \overset{\circ}{F}{}^{a}_{ib}, \overset{\circ}{C}{}^{a}_{ib})$ a Finsler connection in $F\overline{T}$ and $\overset{\circ}{H}{}^{a}_{ij}$, $\overset{\circ}{L}{}^{a}_{ij}$, $\overset{*}{H}{}^{h}_{ia}$, $\overset{*}{L}{}^{h}_{ia}$, mixed Finsler tensor fields. Then the functions $(N^{i}_{j}, F^{\alpha}_{i\beta}, C^{\alpha}_{i\beta})$, given by*

(16)
$$F^{i}_{jk}=\overset{\circ}{F}{}^{i}_{jk},\quad F^{i}_{ja}=\overset{*}{H}{}^{i}_{ja},\quad F^{a}_{ij}=\overset{\circ}{H}{}^{a}_{ij},\quad F^{a}_{ib}=\overset{\circ}{F}{}^{a}_{ib},$$
$$C^{i}_{jk}=\overset{\circ}{C}{}^{i}_{jk},\quad C^{i}_{ja}=\overset{*}{L}{}^{i}_{ja},\quad C^{a}_{ij}=\overset{\circ}{L}{}^{a}_{ij},\quad C^{a}_{ib}=\overset{\circ}{C}{}^{a}_{ib},$$

determine a Finsler connection in FJ (with respect to the cross-sections s_i, s_a).

Remark. With respect to $\{E_{\alpha}\}$, we have:

(17)
$$F^{\alpha}_{i\beta}=B^{\alpha}_{h}\left(\frac{\delta B^{h}_{\beta}}{\delta x^{i}}+\overset{\circ}{F}{}^{h}_{ik}B^{k}_{\beta}+\overset{*}{H}{}^{h}_{ia}D^{a}_{\beta}\right)+D^{\alpha}_{a}\left(\frac{\delta D^{a}_{\beta}}{\delta x^{i}}+\overset{\circ}{F}{}^{a}_{ib}D^{b}_{\beta}+\overset{\circ}{H}{}^{a}_{ij}B^{j}_{\beta}\right),$$
$$C^{\alpha}_{i\beta}=B^{\alpha}_{h}\left(\frac{\partial B^{h}_{\beta}}{\partial y^{i}}+\overset{\circ}{C}{}^{h}_{ik}B^{k}_{\beta}+\overset{*}{L}{}^{h}_{ia}D^{a}_{\beta}\right)+D^{\alpha}_{a}\left(\frac{\partial D^{a}_{\beta}}{\partial y^{i}}+\overset{\circ}{C}{}^{a}_{ib}D^{b}_{\beta}+\overset{\circ}{L}{}^{a}_{ij}B^{j}_{\beta}\right).$$

§4. Paths and autoparallel curves. Let $c=c(t)$ be a differentiable curve on TM, which, in $\widetilde{U}$, is given by $x^{i}=x^{i}(t)$, $y^{i}=y^{i}(t)$. For $\hat{Z}\in\mathrm{Sect}(FJ)$ we have, along c,

(18)
$$D^{FJ}_{\dot{c}}\hat{Z}=\left(\frac{d\hat{Z}^{\alpha}}{dt}+F^{\alpha}_{i\beta}\hat{Z}^{\beta}\frac{dx^{i}}{dt}+C^{\alpha}_{i\beta}\hat{Z}^{\beta}\frac{dy^{i}}{dt}\right)E_{\alpha}.$$

If c is a horizontal curve with respect to Γ^{T}, then

(19)
$$\frac{dy^{i}}{dt}+N^{i}_{j}\frac{dx^{j}}{dt}=0$$

For a horizontal curve c^{h}, $\dot{c}^{h}(t)=\frac{dx^{i}}{dt}\frac{\delta}{\delta x^{i}}$ and

(20)
$$D^{FJ}_{\dot{c}^{h}}\hat{Z}=\left(\frac{d\hat{Z}^{\alpha}}{dt}+F^{\alpha}_{i\beta}\hat{Z}^{\beta}\frac{dx^{i}}{dt}\right)E_{\alpha}.$$

A vertical curve c^{v} in TM has the form $x^{i}=x^{i}_{0}$, $y^{i}=y^{i}(t)$ and then

(21)
$$D^{FJ}_{\dot{c}^{v}}\hat{Z}=\left(\frac{d\hat{Z}^{\alpha}}{dt}+C^{\alpha}_{i\beta}\hat{Z}^{\beta}\frac{dy^{i}}{dt}\right)E_{\alpha}.$$

Let $c^{h}=c^{h}(t)$ be a horizontal curve in TM and $\hat{Z}\in\mathrm{Sect}(FJ)$ such that, along c^{h},

(22)
$$\hat{Z}=\left(\frac{dx^{i}}{dt}B^{\alpha}_{i}+v^{a}D^{\alpha}_{a}\right)E_{\alpha}.$$

The curve c^{h} is called *h-autoparallel curve* if there is a cross-section $\hat{Z}$ given by (22), such that

(23)
$$D^{FJ}_{\dot{c}^{h}}\hat{Z}=A(c^{h})\hat{Z}.$$

It follows that the equations of the h-autoparallel curves are given by:

(24)
$$\frac{dy^{i}}{dt}+N^{i}_{j}\frac{dx^{j}}{dt}=0,$$
$$\frac{d^{2}x^{i}}{dt^{2}}+\overset{\circ}{F}{}^{i}_{jk}\frac{dx^{j}}{dt}\frac{dx^{k}}{dt}+\overset{*}{H}{}^{i}_{ja}\frac{dx^{j}}{dt}v^{a}=A(t)\frac{dx^{i}}{dt},$$
$$\frac{dv^{a}}{dt}+\overset{\circ}{F}{}^{a}_{jb}v^{b}\frac{dx^{j}}{dt}+\overset{\circ}{H}{}^{a}_{ji}\frac{dx^{j}}{dt}\frac{dx^{i}}{dt}=A(t)v^{a}.$$

Let $c^{v}=c^{v}(t)$ be a vertical curve in TM and $\hat{Z}=\mathrm{Sect}(FJ)$ such that

(25)
$$\hat{Z}=\left(\frac{dy^{i}}{dt}B^{\alpha}_{i}+w^{a}D^{\alpha}_{a}\right)E_{\alpha},$$

along c^{v}. c^{v} is called *v-autoparallel curve* if there is a cross-section $\hat{Z}$ given by (25), such that

(26)
$$D^{FJ}_{\dot{c}^{v}}\hat{Z}=B(c^{v})\hat{Z}$$

It follows that the equations of the v-autoparallel curves are given by:

(27)
$$\frac{d^{2}y^{i}}{dt^{2}}+\overset{\circ}{C}{}^{i}_{jk}\frac{dy^{j}}{dt}\frac{dy^{k}}{dt}+\overset{*}{L}{}^{i}_{ja}\frac{dy^{j}}{dt}w^{a}=B(t)\frac{dy^{i}}{dt},$$
$$\frac{dw^{a}}{dt}+\overset{\circ}{C}{}^{a}_{ib}w^{b}\frac{dy^{i}}{dt}+\overset{\circ}{L}{}^{a}_{ij}\frac{dy^{i}}{dt}\frac{dy^{j}}{dt}=B(t)w^{a}.$$

We call *h-path* a horizontal curve which satisfies the equations :

(28) $$\frac{dy^i}{dt}+N^i_j\frac{dx^j}{dt}=0,\qquad \frac{d^2x^i}{dt^2}+\overset{\circ}{F}{}^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}=0,$$

with respect to the induced Finsler connection $FT\overset{\circ}{\Gamma}$. A vertical curve is called *v-path* if it satisfies :

(29) $$\frac{d^2y^i}{dt^2}+\overset{\circ}{C}{}^i_{jk}(x_0, y(t))\frac{dy^j}{dt}\frac{dy^k}{dt}=0.$$

Let c^h be a h-path and $\hat{X}=\frac{dx^i}{dt}s_i=\frac{dx^i}{dt}B^\alpha_i E_\alpha$ its tangent vector field. It results

(30) $$D^{FJ}_{\dot{c}^h}\hat{X}=\overset{\circ}{H}{}^a_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}D^\alpha_a E_\alpha.$$

Theorem 1. *The tangent vector field along a h-path is parallel to himself if and only if*

(31) $$\overset{\circ}{H}{}^a_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}=0.$$

Let c^v be a v-path and $\hat{X}=\frac{dy^i}{dt}s_i=\frac{dy^i}{dt}B^\alpha_i E_\alpha$ its tangent vector field. It results

(32) $$D^{FJ}_{\dot{c}^v}\hat{X}=\overset{\circ}{L}{}^a_{jk}\frac{dy^j}{dt}\frac{dy^k}{dt}D^\alpha_a E_\alpha.$$

Theorem 2. *The tangent vector field along a v-path is parallel to himself if and only if*

(33) $$L^a_{jk}\frac{dy^j}{dt}\frac{dy^k}{dt}=0.$$

We say that the manifold M is *h-parallel* (resp. *v-parallel*) if FT is h-parallel (resp. v-parallel) with respect to $FJ\Gamma$.

Theorem 3. *M is h-parallel (resp. v-parallel) if and only if* $\overset{\circ}{H}{}^a_{ij}=0$ (*resp.* $\overset{\circ}{L}{}^a_{ij}=0$).

The manifold M is called *totally h-geodesic* (resp. *totally v-geodesic*) if every h-autoparallel (resp. v-autoparallel) curve with respect to $FJ\Gamma$ is an autoparallel curve with respect to $FT\overset{\circ}{\Gamma}$.

Theorem 4. *M is totally h-geodesic (resp. totally v-geodesic) if and only if* $\overset{\circ}{H}{}^a_{ij}+\overset{\circ}{H}{}^a_{ji}=0$ (*resp.* $\overset{\circ}{L}{}^a_{ij}+\overset{\circ}{L}{}^a_{ji}=0$).

BIBLIOGRAPHY

1. O p r i ş, D. — *Fibrés vectoriels de Finsler et connexions associées*, Proc. Naţ. Sem. on Finsler Spaces, Braşov, Febr. 1980, 185—193
2. Y a n o, K., I s h i h a r a, S. — *On hyperconnections*, Bull. Calcutta Math. Soc., vol. 56 (2, 3) 1964, 110—133

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ON MIRON'S THEORY OF SUBSPACES IN FINSLER SPACES

BY

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In this paper an invariant description of Miron's non-standard theory of subspaces in Finsler spaces is given. For the content of this theory we refer to R. M i r o n [5] and R. M i r o n and A. B e j a n c u [6]. After some notations and preliminaries (see [2] for details) and a study of the pairs of nonlinear connections with the H-property, the induced and intrinsic metrical Finsler connections are treated. Equations of Gauss and Codazzi will be included in a forthcoming paper in which detalied proofs of the present results will be given.

1. Notations and preliminaries. Let M be a smooth paracompact manifold and let be $\pi : TM\to M$ its tangent bundle. Let us denote by V the total space of the vertical subbundle of $TTM\to TM$. A Riemannian metric g in the vertical subbundle is called a metrical Finsler structure on M. The pair (M, g) will be called a Finsler manifold. We suppose g satisfies the Miron's regularity conditions (M i r o n [4]). It follows that there exists a canonical nonlinear connection on M determined only by g i.e. there exists a subbundle of $TTM\to TM$ whose total space will be denoted by H, such that $TTM=H\oplus V$ (Whitney'sum). Equivalently, there exists a tensor field P of type (1,1) on TM having the properties $JP=J$ and $PJ=-J$, where J is the natural almost tangent structure on TM. We recall that $J^2=0$ and ker $J=$im $J=V$. Let $\mathcal{X}(TM)$ be the module of vector fields on TM. The properties of P imply that P is an almost product structure on TM such that $P(X)=-X$ iff X is vertical and conversely. Let v and h the vertical and horizontal projectors, respectively. Sometimes we shall put $hX=X^h$ and $vX=X^v$ so $X=X^h+X^v$ for $X\in\mathcal{X}(TM)$. The restriction of J to horizontals is an isomorphism which will be denoted by J, too. Its reciprocal will be denoted by Q. We define $\hat{Q} : \mathcal{X}(TM)\to\mathcal{X}(TM)$ by $\hat{Q}(X)=Q(X^v)+J(X^h)$, It follows $\hat{Q}^2=I$ and $P\hat{Q}+\hat{Q}P=0$. It is easy to see that $F=P\hat{Q}$ is an almost complex structure on TM.

In a joint paper with I. P o p o v i c i [7] we have proved that the triad (P, J, D), where D is a linear connection on TM such that $DP=0$ and $D\hat{Q}=0$, defines a Finsler connection in the classical meaning (see M a t s u m o t o [3]) and conversely. By using a nonlinear connection, g can be prolonged to a Riemannian metric G on TM given by $G(X, Y)=g(X^v, Y^v)+g(JX^h, JY^h)$. The Finsler connection (P, J, D) will be called metrical iff $DG=0$.