We call *h-path* a horizontal curve which satisfies the equations:

$$\frac{dy^i}{dt}+N^i_j\frac{dx^j}{dt}=0,\qquad \frac{d^2x^i}{dt^2}+\overset{\circ}{F}{}^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}=0, \tag{28}$$

with respect to the induced Finsler connection $FT\overset{\circ}{\Gamma}$. A vertical curve is called *v-path* if it satisfies:

$$\frac{d^2y^i}{dt^2}+\overset{\circ}{C}{}^i_{jk}(x_0, y(t))\frac{dy^j}{dt}\frac{dy^k}{dt}=0. \tag{29}$$

Let c^h be a h-path and $\hat{X}=\frac{dx^i}{dt}s_i=\frac{dx^i}{dt}B^\alpha_i E_\alpha$ its tangent vector field. It results

$$D^{FJ}_{\dot{c}^h}\hat{X}=\overset{\circ}{H}{}^a_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}D^\alpha_a E_\alpha. \tag{30}$$

Theorem 1. *The tangent vector field along a h-path is parallel to himself if and only if*

$$\overset{\circ}{H}{}^a_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}=0. \tag{31}$$

Let c^v be a v-path and $\hat{X}=\frac{dy^i}{dt}s_i=\frac{dy^i}{dt}B^\alpha_i E_\alpha$ its tangent vector field. It results

$$D^{FJ}_{\dot{c}^v}\hat{X}=\overset{\circ}{L}{}^a_{jk}\frac{dy^j}{dt}\frac{dy^k}{dt}D^\alpha_a E_\alpha. \tag{32}$$

Theorem 2. *The tangent vector field along a v-path is parallel to himself if and only if*

$$L^a_{jk}\frac{dy^j}{dt}\frac{dy^k}{dt}=0. \tag{33}$$

We say that the manifold M is *h-parallel* (resp. *v-parallel*) if FT is h-parallel (resp. v-parallel) with respect to $FJ\Gamma$.

Theorem 3. *M is h-parallel (resp. v-parallel) if and only if $\overset{\circ}{H}{}^a_{ij}=0$ (resp. $\overset{\circ}{L}{}^a_{ij}=0$).*

The manifold M is called *totally h-geodesic* (resp. *totally v-geodesic*) if every h-autoparallel (resp. v-autoparallel) curve with respect to $FJ\Gamma$ is an autoparallel curve with respect to $FT\overset{\circ}{\Gamma}$.

Theorem 4. *M is totally h-geodesic (resp. totally v-geodesic) if and only if $\overset{\circ}{H}{}^a_{ij}+\overset{\circ}{H}{}^a_{ji}=0$ (resp. $\overset{\circ}{L}{}^a_{ij}+\overset{\circ}{L}{}^a_{ji}=0$).*

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ON MIRON'S THEORY OF SUBSPACES IN FINSLER SPACES

BY

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In this paper an invariant description of Miron's non-standard theory of subspaces in Finsler spaces is given. For the content of this theory we refer to R. M i r o n [5] and R. M i r o n and A. B e j a n c u [6]. After some notations and preliminaries (see [2] for details) and a study of the pairs of nonlinear connections with the H-property, the induced and intrinsic metrical Finsler connections are treated. Equations of Gauss and Codazzi will be included in a forthcoming paper in which detalied proofs of the present results will be given.

1. Notations and preliminaries. Let M be a smooth paracompact manifold and let be $\pi: TM\to M$ its tangent bundle. Let us denote by V the total space of the vertical subbundle of $TTM\to TM$. A Riemannian metric g in the vertical subbundle is called a metrical Finsler structure on M. The pair (M, g) will be called a Finsler manifold. We suppose g satisfies the Miron's regularity conditions (M i r o n [4]). It follows that there exists a canonical nonlinear connection on M determined only by g i.e. there exists a subbundle of $TTM\to TM$ whose total space will be denoted by H, such that $TTM=H\oplus V$ (Whitney'sum). Equivalently, there exists a tensor field P of type (1,1) on TM having the properties $JP=J$ and $PJ=-J$, where J is the natural almost tangent structure on TM. We recall that $J^2=0$ and $\ker J=\operatorname{im} J=V$. Let $\mathcal{X}(TM)$ be the module of vector fields on TM. The properties of P imply that P is an almost product structure on TM such that $P(X)=-X$ iff X is vertical and conversely. Let v and h the vertical and horizontal projectors, respectively. Sometimes we shall put $hX=X^h$ and $vX=X^v$ so $X=X^h+X^v$ for $X\in\mathcal{X}(TM)$. The restriction of J to horizontals is an isomorphism which will be denoted by J, too. Its reciprocal will be denoted by Q. We define $\hat{Q}:\mathcal{X}(TM)\to\mathcal{X}(TM)$ by $\hat{Q}(X)=Q(X^v)+J(X^h)$, It follows $\hat{Q}^2=I$ and $P\hat{Q}+\hat{Q}P=0$. It is easy to see that $F=P\hat{Q}$ is an almost complex structure on TM.

In a joint paper with I. P o p o v i c i [7] we have proved that the triad (P, J, D), where D is a linear connection on TM such that $DP=0$ and $D\hat{Q}=0$, defines a Finsler connection in the classical meaning (see M a t s u m o t o [3]) and conversely. By using a nonlinear connection, g can be prolonged to a Riemannian metric G on TM given by $G(X, Y)=g(X^v, Y^v)+g(JX^h, JY^h)$. The Finsler connection (P, J, D) will be called metrical iff $DG=0$.

Theorem 1.1. *Let (M, g) be a Finsler manifold endowed with the nonlinear connection determined only by g. There exists an unique metrical Finsler connection (P, J, D) determined by g only such that $T(vX, vY)=0$ and $hT(hX, hY)=0$, where T denotes the torsion of D.*

2. Finsler subspaces and nonlinear connections. Let M^n be an n-dimensional differentiable manifold of class C^∞ and M^m an m-dimensional differentiable manifold of class C^∞ imbedded in M^n with imbedding map $i: M^m \to M^n$. We call $i(M^n)$ a submanifold of M^n and identify it with the manifold M^m. If the manifold M^n is covered by a system of coordinate neighbourhoods (U, x^i) and M^m is covered by a system of coordinate neighbourhood (U', u^α), then the imbedding i can be represented locally by

$$x^i = x^i(u^\alpha) \text{ with rank } (\partial x^i/\partial u^\alpha) = m. \tag{2.1}$$

Here and in the sequel the indices $i, j, \ldots$ run over the range $1, 2, \ldots, n$ the Greek indices run over the range $1, 2, \ldots, m$ and the indices $a, b, \ldots$, run over the range $m+1, m+2, \ldots, n$. The differential of the map i denoted by $b: TM^m \to TM^n$ will be represented locally by

$$x^i = x^i(u^\alpha), \quad y^i = \frac{\partial x^i}{\partial u^\alpha} v^\alpha \tag{2.2}$$

Here v^α is the supporting element on M^m. The matrix of the differential of b with respect to the natural basis is as follows:

$$B = \begin{bmatrix} B^i_\alpha & 0 \\ B^i_{0\alpha} & B^i_\alpha \end{bmatrix}, \text{ where } B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}, \quad B^i_{0\alpha} = \frac{\partial B^i_\alpha}{\partial u^\beta} v^\alpha. \tag{2.3}$$

Let $C(C')$ be the canonical field on $TM^m(TM^n)$ and $J(J')$ the almost tangent structure on $TM^m(TM^n)$. We have

Lemma 2.1. $BC = \hat{C}'$, $BJ = \hat{J}' B$ *where* $\hat{C}'$ *and* $\hat{J}'$ *are the restriction of* C' *and* J' *to* $b(TM^m)$.

This Lemma implies $Bv = \hat{v}' B$, where v and $\hat{v}'$ are the vertical projectors. Also, it implies that if U is a vertical vector field on TM^m then BU is a vertical vector field on $b(TM^m)$. Let us suppose that M^m and M^n are endowed with the nonlinear connections P and P', respectively.

Definition 2.1. *The pair (P, P') has the H-property if $BP = \hat{P}'B$, where $\hat{P}'$ is the restriction of P' to $b(TM^m)$.*

Theorem 2.1. *Let h and h' be the horizontal projectors of the nonlinear connections P and P', respectively. The following assertions are equivalent:*

(i) *The pair (P, P') has the H-property,* (ii) $Bh = \hat{h}'B$,
(iii) *If U is horizontal then BU is horizontal,*
(iv) $B^i_\alpha N^i_i - N^\beta_\alpha B^i_\beta + B^i_{0\alpha} = 0$, *where (N^β_α) and (N^i_j) are the local components of the nonlinear connections defined by P and P', respectively.*

Remark 2.1. The relation (iv) from the Theorem 2.1 is the local transcription of $BP = P'B$. It shows the equivalence of our definition with that of R. Miron [5].

Theorem 2.2. *Given a nonlinear connection P on M^m there exists a nonlinear connection P' on $i(M^m)$ such that the pair (P, P') has the H-property.*

Hypothesis. *There exists a prolongation of P' given by the Theorem 2.2 to the whole M^n. This prolongation will be denoted by P', too.*

3. Induced and intrinsic metrical Finsler connections. Let us suppose that P is a fixed nonlinear connection on M^m and P' is a fixed nonlinear connection on M^n such that the pair (P, P') has the H-property. If (M^n, g') is a Finsler manifold, the metrical Finsler structure g' induces a metrical Finsler structure g on M^m in an obvious manner. By using P, the metric g can be prolonged to a metric on TM^m which will be denoted by G.

Lemma 3.1. *The Riemannian metric G is induced by G'.*

Let us put $L'(v) = g'(v)(C', C')$ and $L(u) = g(u)(C, C)$.

Lemma 3.2. $L(u) = L'(b(u))$ *for every* $u \in TM^m$.

The existence of the pair (P, P') with the H-property assures the decomposition

$$T_{b(u)}TM^n = B(H_u) \oplus B(V_u) \oplus H^\perp_{b(u)} \oplus V^\perp_{b(u)}, \tag{3.1}$$

where $\perp$ denotes orthocomplements. The subspaces of this decomposition are mutual orthogonal with respect to G'.

Remark 3.1. The dimension of $T^\perp_{b(u)}TM^m = H^\perp_{b(u)} \oplus V^\perp_{b(u)}$ is $2p$, where $p = n - m$. By using the Gramm-Schmidt method we can find an orthonormal basis $(n_1, \ldots, n_p, \nu_1, \ldots, \nu_p)$ of $T^\perp_{b(u)}TM^m$ such that $n_1, \ldots, n_p$ are horizontal vectors and $\nu_1, \ldots, \nu_p$ are vertical vectors. So, at least locally we have $2p$ vector fields normal to $b(TM^m)$, mutual orthogonal and of length 1, the first p being horizontal and the last p being vertical.

Let (P', J', D') be the unique metrical Finsler connection having P' as nonlinear connection and satisfying $T'(v'X', v'Y') = 0$ and $h'T'(h'X', h'Y') = 0$.

Theorem 3.1. *Let BX and BY be two vector fields tangent to $b(TM^m)$. Then $D'_{X'}Y'|b(TM^m)$, where X' and Y' are extension of BX and BY, respectively, does not depend on the extension. Denoting this by $D'_{BX}BY$ we get*

$$D'_{BX}BY = BD_XY + H^a(X, Y)n_a + K^a(X, Y)\nu_a, \tag{3.2}$$

where D is a linear connection on TM^m and H^a and K^a are $2p$ tensor fields of type (0,2) on TM^m.

Lemma 3.3. *Let X and Y be two vector fields on TM^m and X' and Y' extension of BX and BY, respectively. Then $[X', Y']|_{b(TM^m)}$ is independent of the extension and denoting it by $[BX, BY]$ we get $[BX, BY] = B[X, Y]$.*

Theorem 3.2. *The triad (P, J, D), where D is given by (3.2), is a metrical Finsler connection satisfying $T(vX, vY) = 0$ and $hT(hX, hY) = 0$.*

The metrical Finsler connection (P, J, D) will be called the *induced metrical Finsler connection*. According to the Theorem 1.1. there exists on (M^m, g) an unique metrical Finsler connection satisfying $T(vX, vY) = 0$ and $hT(hX, hY) = 0$ wihch will be called the *intrinsic metrical Finsler connection*.

Corollary 3.1. *The intrinsic metrical Finsler connection coincide to the induced metrical Finsler connection.*

Lemma 3.4. *The following properties of H^a and K^a hold:*

(i) $H^a(X, PY) = H^a(X, Y)$, (ii) $K^a(X, PY) = -K^a(X, Y)$,
(iii) $H^a(X, QY) = (q^{-1})^a_b K^b(X, Y)$, (iv) $K^a(X, QY) = q^a_b H^b(X, Y)$,
(v) $H^a(X, vY) = 0$, (vi) $K^a(X, hY) = 0$,
(vii) $H^a(hX, hY) = H^a(hY, hX)$, (viii) $K^a(vX, vY) = K^a(vY, vX)$,

where $X, Y \in \mathcal{X}(TM^m)$ *and* $\hat{Q}(n_a) = q^a_b \nu_b$.

Definition 3.1. *The Finsler tensor fields* $h^a(X, Y) = H^a(hX, hY)$ *and* $k^a(X, Y) = K^a(vX, vY)$ *will be called the h-and v-second fundamental form, respectively, of the imbedding i.*

We remark that h^a and k^a are symmetrical Finsler tensor fields.

Theorem 3.3 *Let BX be a vector field tangent to* $b(TM^m)$ *and n a vector field normal to* $b(TM^m)$. *Let X' and n' be extensions of BX and n, respectively. Then* $D'_{X'} n'|_{b(TM^m)}$ *does not depend on the extension and denoting this by* $D'_{BX} n$ *we get*

$$D'_{BX} n = -B(A_n(X)) + \nabla_{BX} n, \tag{3.3}$$

where A_n *is a tensor field of type (1.1) on* TM^m *and* ∇ *is a covariant derivative in the normal bundle. Moreover, we have :*

$$G(A_n(X), Y) = H^a(X, Y) G'(n_a, n) + K^a(X, Y) G'(\nu_a, n), \quad X, Y \in \mathcal{X}(TM^m) \tag{3.4}$$

and ∇ *defines a Finsler connection in the normal bundle.*

Summarizing up, we have obtained the following formulae of Gauss-Weingarten type :

$$\begin{cases} D'_{h'BX} BY = B(D_{hX} Y) + H^a(hX, hY) n_a + K^a(hX, vY) \nu_a \\ D'_{h'BX} n = -B(A_n(hX)) + \nabla_{h'BX} n \end{cases} \tag{3.5}$$

$$\begin{cases} D'_{v'BX} BY = B(D_{vX} Y) + H^a(vX, hY) n_a + K^a(vX, vY) \nu_a \\ D'_{v'BX} n = -B(A_n(vX)) + \nabla_{v'BX} n \end{cases} \tag{3.6}$$

By virtue of Lemma 3.4 the normal components from the first rows of these formulas depend only by h^a and k^a, respectively.

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STRUCTURES FINSLER PRESQUE HORSYMPLECTIQUES

PAR

GHEORGHE ATANASIU

Les variétés différentiables douées de structure presque horsymplectiques ont été définies et approfondies par C. M. de Barros [7], V. Oproiu [2], Gh. Atanasiu [1], etc.

La note présente, qui emploie la terminologie de M. Matsumoto [8], aborde le cas Finsler de la structure presque horsymplectique, détermine les connexions Finsler compatibles avec cette structure Finsler et le groupe de transformations entre ces connexions.

La structure Finsler presque horsymplectique appartient à la famille des structures Finsler dégénerées [2]—[5] et elle peut être étudiée comme celles-ci, par recours aux méthodes de R. Miron et M. Hashiguchi, [9], [10], etc.

La portée d'une telle étude est prouvée par la note [6].

1. Une structure Finsler presque horsymplectique sur une C^∞-variété M est donnée par un champ de tenseurs Finsler du type (0, 2), $a_{ij}(x, y)$ (y étant l'élément de support au point x), antisymétrique $[a_{ij}(x, y) = -a_{ji}(x, y)]$ et s champs de co-vecteurs Finsler $\overset{\alpha}{\eta}_k(x, y)$ de manière que

$$\text{rang}\ \| a_{ij}(x, y) \| = \dim M - s = 2p, \quad \text{dét} \begin{Vmatrix} 0 & -\overset{\alpha}{\eta}_i \\ \overset{\alpha}{\eta}_j & a_{ij} \end{Vmatrix} \neq 0, \tag{1.1}$$

$(\alpha = 2p+1, \ldots, \dim M\ ;\ i, j = 1, 2, \ldots, \dim M)$.

De (1.1.) il résulte que la matrice $\begin{Vmatrix} 0 & -\overset{\alpha}{\eta}_i \\ \overset{\alpha}{\eta}_j & a_{ij} \end{Vmatrix}$ admet une inverse, notamment $\begin{Vmatrix} 0 & \underset{\beta}{\xi}^i \\ -\underset{\beta}{\xi}^j & a^{ij} \end{Vmatrix}$.

On note

$$l^i_j = \overset{\alpha}{\eta}_j \underset{\alpha}{\xi}^i, \quad m^i_j = \delta^i_j - \overset{\alpha}{\eta}_j \underset{\alpha}{\xi}^i \ \text{(on somme selon } \alpha). \tag{1.2}$$

On constate que

$$l^i_j l^k_i = l^k_j, \quad l^i_j m^k_i = 0, \quad m^i_j l^k_i = 0, \quad m^i_j m^k_i = m^k_j, \quad m^i_j + l^i_j = \delta^i_j,$$