Remarque. La connexion Finsler presque horsymplectique (2.6) s'obtient de (3.3) par $X=0$, $Y=0$, $Z=0$.

Si on note par $F\Gamma(N)$ les connexions Finsler qui possèdent une même connexion non-linéaire N, alors on a le

Théorème 5. *L'ensemble de toutes les connexions Finsler $F\Gamma(N)$ compatibles avec une structure Finsler presque horsymplectique est donné par*

$$(3.4)\qquad N^h_j=\overset{*}{N}{}^h_j,\quad F^h_{jk}=\overset{*}{F}{}^h_{jk}+\underset{1}{\Phi}{}^{ih}_{jr}Y^r_{ik},\quad C^h_{jk}=\overset{*}{C}{}^h_{jk}+\underset{1}{\Phi}{}^{ih}_{jr}Z^r_{ik}$$

où $F\overset{*}{\Gamma}=(\overset{*}{N},\overset{*}{F},\overset{*}{C})$ est la connexion Finsler presque horsymplectique donnée par (2.6) et Y^h_{jk}, Z^h_{jk} sont des champs de tenseurs Finsler arbitraires.

Théorème 6. *Les transformations entre les connexions Finsler presque horsymplectiques qui possèdent une même connexion non-linéaire N, $F\Gamma(N)\to\to F\overline{\Gamma}(N)$ sont données par*

$$(3.5)\qquad \overline{N}{}^h_j=N^h_j,\quad \overline{F}{}^h_{jk}=F^h_{jk}+\underset{1}{\Phi}{}^{ih}_{jr}Y^r_{ik},\quad \overline{C}{}^h_{jk}=C^h_{jk}+\underset{1}{\Phi}{}^{ih}_{jr}Z^r_{ik},$$

où Y^h_{jk}, Z^h_{jk} sont des champs de tenseurs Finsler arbitraires.

Les transformations (3.5) et le produit d'applications en tant que loi de composition, forment un groupe abélien $\mathcal{G}$. Ce groupe est isomorphe avec le groupe additif des couples de champs de tenseurs Finsler $(\underset{1}{\Phi}{}^{ih}_{jr}Y^r_{ik},\ \underset{1}{\Phi}{}^{ih}_{jr}Z^r_{ik})$.

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ON GENERAL PATH SPACES

BY

MASAKI FUKUI

Z. I. Szabo showed that a Finsler space is of scalar curvature if and only if the Weyl's projective curvature tensor $W_i{}^h{}_{jk}$ vanishes. And the present author and T. Yamada found a tensor $V_i{}^h{}_{jk}$ characterizing a Finsler space of constant curvature [3, 4]. These tensors can be defined in a so-called general path space, because they are both constructed by the h-curvature tensor $H_i{}^h{}_{jk}$ independently of a metric tensor.

The general path spaces were treated by J. Douglas, L. Berwald and so on [1, 2]. In this lecture we shall be concerned with such spaces. We shall define special general path spaces by means of the above mentioned tensors and investigate the relations between those spaces. The theory of general path spaces seems to be important to the projective theory of Finsler spaces.

§1. **General Path Spaces.** Let M^n be an n-dimensional differentiable manifold and (x^i, y^i) be a usual coordinate system of the tangent bundle. Assuming that for each local coordinate system there exist n functions $G^i(x, y)$ positively homogeneous of degree two in y^i, then M^n is called a general path space if the functions G^i satisfy the transformation law

$$(1.1)\qquad 2\overline{G}{}^i(\overline{x}, \overline{y})=2G^m(x, y)\frac{\partial\overline{x}^i}{\partial x^m}-\frac{\partial^2\overline{x}^i}{\partial x^j\partial x^k}y^jy^k,$$

under a coordinate transformation $\overline{x}^i=\overline{x}^i(x)$.

Now, differentiating G^i partially with respect to y^i, we get

$$(1.2)\qquad G^i_j=\partial G^i/\partial y^j,\quad G^i_{jk}=\partial G^i_j/\partial y^k.$$

It is easily shown from (1.1) that the G^i_{jk} satisfy the transformation law of the connection coefficients under a coordinate transformation. We can define the Berwald connection in a general path space M^n by means of the quantities G^i_j and G^i_{jk}. For example, the covariant derivatives of a contravariant vector X^i are given by

$$(1.3)\qquad X^i{}_{;j}=\partial X^i/\partial x^j-G^m_j(\partial X^i/\partial y^m)+G^i_{jm}X^m,\quad X^i{}_{.j}=\partial X^i/\partial y^j.$$

If a curve $x^i=x^i(t)$ satisfies a system of differential equations

$$(1.4)\qquad \frac{d^2x^i}{dt^2}+2G^i\left(x, \frac{dx}{dt}\right)=0$$

it is called a path in the general path space. The assumption (1.1) is a condition so that the equations (1.4) are invariant under any coordinate transformation. It is evident that a path is nothing but an auto-parallel curve with respect to the Berwald connection.

In a general path space, Ricci formulas and Bianchi identities are the very same as ones of Berwald connection in a Finsler space. In the following, we denote the torsion tensor by R^h_{jk}, h- and hv-curvature tensor by $H_i{}^h{}_{jk}$ and $G_i{}^h{}_{jk}$.

A general path space M^n is called a space with an affine connection when there exists a coordinate system around any point of M^n in which all the G^i are quadratic forms of y^i, that is, G^i_{jk} are functions of position only. This property is also represented by $G_i{}^h{}_{jk}=0$. More especially, M^n is called a space with a flat affine connection when there exists a coordinate system in which all the G^i vanish. About such spaces we know.

Theorem 1. *Let M^n be a general path space. Then the following conditions are equivalent each other :*

(1) *M^n is a space with a flat affine connection.*

(2) *$G_i{}^h{}_{jk}=0$ and $H_i{}^h{}_{jk}=0$.*

(3) *There exists a flat Riemannian metric such that any path of M^n is a geodesic with respect to the metric.*

From the above theorem we can say

Corollary. *In a locally Minkowskian Finsler space, there exists a flat Riemannian metric.*

§ 2. **Projective Mappings.** A diffeomorphism of a general path space $\{M^n, G^i\}$ to another general path space $\{\overline{M}^n, \overline{G}^i\}$ is called a projective mapping if it maps an arbitrary path of M^n onto a path of $\overline{M}^n$. A diffeomorphism is a projective mapping, if there exists a scalar field $p(x, y)$ satisfying

$$\overline{G}^i=G^i+py^i. \tag{2.1}$$

It is fundamental that the Douglas tensor $D_i{}^h{}_{jk}$ and the Weyl tensor $W_i{}^h{}_{jk}$ are invariant under any projective mapping. They are defined by

$$D_i{}^h{}_{jk}=G_i{}^h{}_{jk}-y^hG_{ij.k}/(n+1)-C_{(ijk)}\{\delta^h_iG_{jk}\}/(n+1), \tag{2.2}$$

$$W_i{}^h{}_{jk}=H_i{}^h{}_{jk}+\mathfrak{A}_{(jk)}\{y^hH_{jk.i}+\delta^h_iH_{jk}+\delta^h_jH_{k.i}\}/(n+1), \tag{2.3}$$

where $G_{ij}=G_i{}^m{}_{jm}$, $H_{ij}=H_i{}^m{}_{jm}$, $H_i=(nH_{mi}+H_{im})y^m/(n-1)$. The importance of these tensors is illustrated by

Theorem 2. [1, 5] *A general path space may be mapped by means of a projective mapping onto a space with a flat affine connection if and only if the equations*

$$D_i{}^h{}_{jk}=0,\ W_i{}^h{}_{jk}=0 \tag{2.4}$$

are satisfied.

For the later use we notice that

$$\overline{H}_{ij}-\overline{H}_{ji}=H_{ij}-H_{ji}-(n+1)(p_{i;j}-p_{j;i}), \tag{2.5}$$

$$\overline{G}_{ij}=G_{ij}+(n+1)p_{ij}, \tag{2.6}$$

where $p_i=p_{.i}$ and $p_{ij}=p_{i;j}$.

§ 3. **General Path Spaces of Scalar or Constant Curvature.** In the theory of Finsler spaces, a Finsler space of scalar curvature is characterized by vanishing of the Weyl tensor [7]. And a space of constant curvature is characterized by $V_i{}^h{}_{jk}=0$[4], where

$$V_i{}^h{}_{jk}=H_i{}^h{}_{jk}+\mathfrak{A}_{(jk)}\{\delta^h_jH_{k.i}\}/(n+1). \tag{3.1}$$

A Finsler space of constant curvature is mapped onto a space of constant curvature under a projective mapping satisfying $p_{i;j}-p_{j;i}=0$. The tensor $V_i{}^h{}_{jk}$ is invariant under such a mapping.

Returning to the theory of general path spaces, we can give the following :

Definition 1. *A general path space is called a space of scalar curvature, if the Weyl curvature tensor vanishes.*

Definition 2. *A general path space is called a space of constant curvature if the tensor $V_i{}^h{}_{jk}$ defined in (3.1) vanishes.*

Without making use of a metric tensor, we can prove the same lemma as in the metric case.

Lemma. *In order that a general path space of scalar curvature turns into a space of constant curvature, it is necessary and sufficient that*

$$H_{ij}-H_{ji}=0. \tag{3.2}$$

By means of this lemma and (2.5), we have

Theorem 3. *A general path space of constant curvature is mapped onto a space of constant curvature under a projective mapping satisfying*

$$p_{i;j}-p_{j;i}=0. \tag{3.3}$$

The following theorem has been proved in the metric case [4] :

Theorem 4. *A general path space is of constant curvature if and only if*

$$H_i{}^h{}_{jk}=(H_{ij}\delta^h_k-H_{ik}\delta^h_j)/(n-1). \tag{3.4}$$

As to spaces with affine connections, we have

Theorem 5. (Douglas, Matsumoto) *In order that a general path space might be projectively mapped onto a space with an affine connection, it is necessary and sufficient that the space has vanishing Douglas tensor.*

Theorem 6. *A general path space with an affine connection is mapped onto a space with an affine connection under a projective mapping satisfying*

$$p_{ij}=0. \tag{3.5}$$

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ON SOME GENERALIZED CONNECTIONS AND THEIR APPLICATIONS

BY

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A Finsler manifold $F^n=(M^n, L)$ is a manifold M^n equipped with a fundamental function $L(x, y)$ defined on a tangent bundle $T(M^n)$ where (x, y) denotes canonical coordinates. Let g_{ij} be a fundamental tensor derived from L. We denote Finsler connection on F^n by its connection coefficients as $C\Gamma(F_i{}^j{}_k, N^j{}_k, C_i{}^j{}_k)$. Prof. M a t s u m o t o showed in [8] that Cartan connection on a Finsler manifold F^n is uniquely determined by the following conditions :

(C1) $g_{ij|k}=0, \quad g_{ij}|_k=0,$

(C2) $T_i{}^j{}_k : =F_i{}^j{}_k-F_k{}^j{}_i=0, \qquad S_i{}^j{}_k : =C_i{}^j{}_k-C_k{}^j{}_i=0,$

(C3) $D^j{}_k : =y^iF_i{}^j{}_k-N^j{}_k=0, \qquad$ (equivalent to $y^j{}_{|k}=0$).

In this paper we generalize these conditions and get a socalled generalized Cartan connection (with vanishing torsion) $C^{(p)}\Gamma$ and apply it to some problems. Terminologies and notations are refered to M a t s u m o t o [11].

§1. Generalized Cartan Connections $C^{(p)}\Gamma$. p denotes a real constant number ($p\neq 1$). We put

$$\Phi^{(p)}{}_{ij} : =L^{p-2}(g_{ij}+(p-2)l_il_j). \tag{1.1}$$

We abbreviate $\Phi^{(p)}$ as Φ if it gives no confusion, for instance, $\Phi_{ijk} : =\dot\partial_k\Phi^{(p)}_{ij}$ and assume (Φ_{ij}) to be regular [4], and put

$$C^{(p)}{}_i{}^j{}_k : =(1/2)\Phi_{isk}\Phi^{sj}=C_i{}^j{}_k+\sigma^{(p)}{}_i{}^j{}_k, \tag{1.2}$$

where (Φ^{ij}) denotes the inverse of (Φ_{ij}) and

$$\sigma^{(p)}{}_i{}^j{}_k : =\frac{p-2}{2L}\left(\delta_i^jl_k+\delta_k^jl_i+\frac{1}{p-1}\,h_{ik}l^j-l_il_kl^j\right). \tag{1.3}$$

Definition. *Generalized Cartan connection $C^{(p)}\Gamma$ is a connection determined by coefficients $F_i{}^j{}_k$, $N^j{}_k$, $C^{(p)}{}_i{}^j{}_k$ where $F_i{}^j{}_k$ and $N^j{}_k$ are the same coefficients of $C\Gamma(F_i{}^j{}_k, N^j{}_k, C_i{}^j{}_k)$.*

Proposition 1.1. *$C^{(p)}\Gamma$ is determined uniquely by following conditions where symbols $_{\|}$ and $\|$ denote h- and v-covariant differentiations with respect to $C^{(p)}\Gamma$.*