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ON SOME GENERALIZED CONNECTIONS AND THEIR APPLICATIONS

BY

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A Finsler manifold $F^n=(M^n, L)$ is a manifold M^n equipped with a fundamental function $L(x, y)$ defined on a tangent bundle $T(M^n)$ where (x, y) denotes canonical coordinates. Let g_{ij} be a fundamental tensor derived from L. We denote Finsler connection on F^n by its connection coefficients as $C\Gamma(F_i{}^j{}_k, N^j{}_k, C_i{}^j{}_k)$. Prof. Matsumoto showed in [8] that Cartan connection on a Finsler manifold F^n is uniquely determined by the following conditions :

(C1) $\quad g_{ij|k}=0, \quad g_{ij}|_k=0,$

(C2) $\quad T_i{}^j{}_k :=F_i{}^j{}_k-F_k{}^j{}_i=0, \qquad S_i{}^j{}_k :=C_i{}^j{}_k-C_k{}^j{}_i=0,$

(C3) $\quad D^j{}_k :=y^iF_i{}^j{}_k-N^j{}_k=0, \qquad$ (equivalent to $y^j{}_{|k}=0$).

In this paper we generalize these conditions and get a socalled generalized Cartan connection (with vanishing torsion) $C^{(p)}\Gamma$ and apply it to some problems. Terminologies and notations are refered to Matsumoto [11].

§1. Generalized Cartan Connections $C^{(p)}\Gamma$. p denotes a real constant number ($p\neq 1$). We put

$$\Phi^{(p)}{}_{ij} :=L^{p-3}(g_{ij}+(p-2)l_il_j). \tag{1.1}$$

We abbreviate $\Phi^{(p)}$ as Φ if it gives no confusion, for instance, $\Phi_{ijk} :=\dot\partial_k\Phi^{(p)}_{ij}$ and assume (Φ_{ij}) to be regular [4], and put

$$C^{(p)}{}_i{}^j{}_k :=(1/2)\Phi_{isk}\Phi^{sj}=C_i{}^j{}_k+\sigma^{(p)}{}_i{}^j{}_k, \tag{1.2}$$

where (Φ^{ij}) denotes the inverse of (Φ_{ij}) and

$$\sigma^{(p)}{}_i{}^j{}_k :=\frac{p-2}{2L}\left(\delta_i^jl_k+\delta_k^jl_i+\frac{1}{p-1}h_{ik}l^j-l_il_kl^j\right). \tag{1.3}$$

Definition. *Generalized Cartan connection* $C^{(p)}\Gamma$ *is a connection determined by coefficients* $F_i{}^j{}_k$, $N^j{}_k$, $C^{(p)}{}_i{}^j{}_k$ *where* $F_i{}^j{}_k$ *and* $N^j{}_k$ *are the same coefficients of* $C\Gamma(F_i{}^j{}_k, N^j{}_k, C_i{}^j{}_k)$.

Proposition 1.1. $C^{(p)}\Gamma$ *is determined uniquely by following conditions where symbols* $_|$ *and* $|$ *denote h- and v-covariant differentiations with respect to* $C^{(p)}\Gamma$.

(GC1) (1) $\Phi^{(p)}{}_{ij\|k}=0$ *for* $p\neq 0$ *and if* $p=0$ *this must be replaced by* $g_{ij\|k}=0$,

(2) $\Phi^{(p)}{}_{ij}\|_k=0$,

(GC2) $T^{(p)}{}_i{}^j{}_k:=F^{(p)}{}_i{}^j{}_k-F^{(p)}{}_k{}^j{}_i=0,\quad S^{(p)}{}_i{}^j{}_k:=C^{(p)}{}_i{}^j{}_k-C^{(p)}{}_k{}^j{}_i=0,$

(GC3) $D^{(p)j}{}_k:=y^iF^{(p)}{}_i{}^j{}_k-N^{(p)j}{}_k=0.$

Proposition 1.2. *Torsion and curvature tensors of* $C^{(p)}\Gamma$ *can be expressed in terms of corresponding tensors of* $C\Gamma$.

(1.4) $$R^{(p)j}{}_{kl}=R^j{}_{kl},\quad P^{(p)j}{}_{kl}=P^j{}_{kl},$$

(1.5) $$R^{(p)}{}_i{}^j{}_{kl}=R_i{}^j{}_{kl}+\sigma^{(p)}{}_i{}^j{}_s R^s_{kl},$$

(1.6) $$P^{(p)}{}_i{}^j{}_{kl}=P_i{}^j{}_{kl}+\sigma^{(p)}{}_i{}^j{}_s P^s{}_{kl},$$

(1.7) $$S^{(p)}{}_i{}^j{}_{kl}=S_i{}^j{}_{kl}-\frac{(p-2)^2}{4(p-1)L^2}(h_{ik}h^j{}_l-h_{il}h^j_k).$$

§2. **Applications of** $C^{(p)}\Gamma$. The T-tensor of F^n was first introduced in [10] and studied in [2], [3], [7] especially W a t a n a b e [16] proved that if $T_{ijkl}=0$, F^n must be riemannian under an assumption that all indicatrices are closed. When closedness is not assumed, a problem to determine a fundamental function is not completely solved. (A s a n o v and K i r n a s o v [2])

Proposition 2.1.

(1) $$C_i{}^j{}_k|_l=0 \quad \textit{iff } F^n \textit{ is riemannian},$$

(2) $$C^{(0)}{}_i{}^j{}_k\|_l=0 \quad \textit{iff } T_{ijkl}=0$$

Thus examples of spaces which satisfy T-condition, given in [2] as (3.8) and (3.9) are all easily verified to satisfy T- condition, because we can treat the object as a sum of riemannian terms. Another more interesting application of $C^{(p)}\Gamma$ is a study of $S3$-like Finsler spaces.

Definition. *A Finsler space* F^n *is called S3-like if v-curvature tensor* S_{ijkl} *has a form*

(2.1) $L^2S_{ijkl}=S(h_{ik}h_{jl}-h_{il}h_{jk})$ *where* S *depends on* (x) *only.*

Referenced papers [1], [4], [5], [9], [13], [14] are mainly concerned with a problem to investigate $S3$-like Finsler spaces. We want to determine a fundamental function L of a $S3$-like F^n with constant S, (for more general cases, the problem was studied in a tangent Minkowskian space of F^n in [4] and [14]). Under an assumption of algebraic decomposition at a point (x_0, y_0) we have following results.

Theorem 2.1. *Let* S *be a constant number such that,* $S\geqq 0$ *or* $S<-1$. *Iff* F^n *has curvature tensors of a form*

(2.2) $$R_i{}^j{}_{kl}=0,\quad P_i{}^j{}_{kl}=0 \text{ and } L^2S_i{}^j{}_{kl}=S(h_{ik}h^j{}_l-h_{il}h^j_k)$$

L is of a special character such that

(2.3) $$L^p=\Sigma_{b=1}^n a_b(\eta^b)^p,$$

where (1) p *is a real constant satisfying* $(p-2)^2/4(p-1)=S$.
(2) η^b *are* (1) *p-homogeneous independent functions and satisfy*

(2.4) $$\Sigma_b a_b(\eta^b)^{p-1}\eta^b{}_{ij}=0,\quad (\eta^b{}_i=\dot\partial_i\eta^b,\ \eta^b{}_{ij}=\dot\partial_i\dot\partial_j\eta^b),$$

(2.5) $$\eta^b{}_{i|j}-\eta^c_{i|j}\eta^k{}_c\eta^b{}_{kl}=\eta^b{}_{j|i}-\eta^c_{j|i}\eta^k{}_c\eta^b{}_{kj}.$$

Theorem 2.2. *Iff* F^n *has curvature tensors of a form*

(2.6) $$R_i{}^j{}_{kl}=0,\quad P_i{}^j{}_{kl}=0 \text{ and } L^2S_i{}^j{}_{kl}=-(h_{ik}h^j{}_{l}-h_ih^j{}_k)$$

L has a form

(2.7) $$L=(\eta^1)^{a_1}(\eta^2)^{a_2}\ldots(\eta^n)^{a_n}\quad(\Sigma_b a_b=1).$$

N.B. Conditions (3.56) and (3.57) in [5] are weaker and they should be corrected as above.

These theorems were proved by constructing a $(p/2)$-homogeneous complete family $f^a(x, y)$ $(a=1, 2, \ldots, n)$ with respect to $C^{(p)}\Gamma$. Consider a system of equations

(2.8) $$f|_i=0,\quad f\|_i=f_i,\quad f_i|_j=0,\quad f_i\|_j=0.$$

The complete integrability condition of this sistem is

(2.9) $$R^{(p)j}{}_{kl}=P^{(p)j}{}_{kl}=S^{(p)j}{}_{kl}=0,$$
$$R^{(p)}{}_i{}^j{}_{kl}=P^{(p)}{}_i{}^j{}_{kl}=S^{(p)}{}_i{}^j{}_{kl}=0,$$

and iff these vanishing conditions hold we have $f^a(x, y)$ such that $f^a, f^a_i:=\dot\partial_i f^a$ and $f^a_{ij}:=\dot\partial_i\dot\partial_j f^a$ satisfy :

(1) equations (2.8) hold

(2) rank $(f^a{}_i)=n$, i.e. $(f^a{}_i)$ has the inverse $(f^i{}_a)$

(2.10) (3) $f^a{}_iy^i=(p/2)f^a+e(p)$, $\quad e(p)=0$ if $p\neq 0$ and $p(0)=1$

(2.11) (4) $f^a{}_{i|j}-f^b{}_{ij}f^k{}_bf^a{}_{ik}=f^a_{j|i}-f^b{}_{ji}f^k{}_bf^a{}_{jk}.$

With these functions, the above theorems was proved in [5].

3. Distorted Conformal Change of Cartan Connection. We consider a generalized metric which depends on a function $\lambda(x)$ such that

$$\psi_{ij}(x, y)=g_{ij}+(e^{2\lambda}-1)l_il_j,$$

and define a connection $C(\lambda)\Gamma$ by following conditions :

(A1) $$\psi_{ij|k}=0,\quad \psi_{ij}\|_k=0,$$

(A2) $$\bar S_i{}^j{}_k:=\bar C_i{}^j{}_k-\bar C_k{}^j{}_i=0,\quad \bar T_i{}^j{}_k:=\bar F_i{}^j{}_k-\bar F_k{}^j{}_i=0,$$

(A3) $$\bar D^j{}_k:=y^i\bar F_i{}^j{}_k-\bar N^j{}_k=0.$$

As $\psi_{ij}y^iy^j=e^{2\lambda}L$, the connection has a very similar feature to that defined in H a s h i g u c h i [3], and connection coefficients are uniquely determined as in [3].

$$\tilde{F}_i{}^j{}_k = F_i{}^j{}_k - U_i{}^j{}_k, \quad \tilde{C}_i{}^j{}_k = C_i{}^j{}_k + \tau_i{}^j{}_k$$

$$(3.1) \qquad U_i{}^j{}_k := \lambda^j l_i l_k - \lambda_i l^j l_k - \lambda_k l_i l^j - (C_i{}^j y_k + C_k{}^j y_i - C_{ki} y^j) +$$

$$+ L^2 (C_k{}^r C_r{}^j{}_i + C_i{}^r C_r{}^j{}_k - C_r{}^j C_k{}^r{}_i)$$

where $\lambda_i := \partial_i \lambda \quad \lambda^j = g^{ji} \lambda_i \quad C_i^j := g^{rs} C_r{}^j{}_i \lambda_s$

$$\tau_i{}^j{}_k := \frac{e^{2\lambda} - 1}{L e^{2\lambda}} h_{ik} l^j.$$

Torsion and curvature tensors are obtained in the relation of $C\Gamma$ as given in [3].

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ON ALMOST FINSLER STRUCTURES

BY

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In the present paper we define a notion of an almost Finsler structure, and show that, for some Lie group G, the tangent bundle over a manifold admits the G-structure under some condition, if and only if the manifold is Finslerian. In this case, an almost symplectic structure defined on the tangent bundle actually plays an important role. The G-connections relative to the almost Finsler structure are also investigated.

§1. Almost Finsler Structures. Let M be an n-dimensional C^∞-manifold and $T(M)$ be its tangent bundle. As is well-known, $T(M)$ admits the standard tangent structure $\mathcal{F}_0$. Let Q be its structure tensor. The set $T(n) = \{\left(\begin{smallmatrix} A & 0 \\ B & A \end{smallmatrix}\right) | A \in GL(n, R),\ B \in gl(n, R)\}$ is a well-known Lie group. The structure $\mathcal{F}_0$ is an integrable $T(n)$-structure on $T(M)$. Let π be the natural projection $T(M) \to M$. Any coordinate neighbourhood $\{U, x^i\}$ in M induces a coordinate neighbourhood $\{\pi^{-1}(U), (x^i, y^i)\}$ in $T(M)$, which we call a canonical coordinate neighbourhood. For any $\pi^{-1}(U)$ and $\pi^{-1}(\bar{U})$, if $2n$-frames $\{Z_A\}$ in $\pi^{-1}(U)$ and $\{\bar{Z}_A\}$ in $\pi^{-1}(\bar{U})$ are both adapted to the $T(n)$-structure, they satisfy $\bar{Z}_A = P_A^B Z_B ((P_A^B) \in T(n))$ in $\pi^{-1}(U) \cap \pi^{-1}(\bar{U})$. Of course, the components a_j^i and b_j^i of $(P_A^B) = \begin{pmatrix} a_j^i & 0 \\ b_j^i & a_j^i \end{pmatrix}$ are functions of x^i and y^i. Now, we may consider the case where a_j^i are positively homogeneous of degree 0 with respect to y^i and b_j^i are positively homogeneous of degree 1 with respect to y^i. If we treat only the structure $\mathcal{F}_0$ under the restriction of the above homogeneity condition, we call it the *homogeneous standard tangent structure* and denote it by $\mathcal{F}_0^*$.

Now, let G be a Lie subgroup of $T(n)$. If $T(M)$ admits the G-structure $\mathcal{D}$ and if any frame adapted to the structure $\mathcal{D}$ is always adapted to $\mathcal{F}_0$, the structure $\mathcal{D}$ is called a *G-structure depending on* $\mathcal{F}_0$. Moreover, if the G-structure $\mathcal{D}$ satisfies the above mentioned homogeneity condition, the structure $\mathcal{D}$ is called *a homogeneous G-structure depending on* $\mathcal{F}_0^*$ and is denoted by $\mathcal{D}^*$.

Now let us consider the set $F(n) = \{\left(\begin{smallmatrix} A & 0 \\ SA & A \end{smallmatrix}\right) | A \in O(n),\ S \in \mathrm{Sym}(n)\}$. It is apparent that $F(n)$ is a Lie subgroup of $T(n)$. Hereafter the Lie group $F(n)$ is called a *Finsler group*, an $F(n)$-structure depending on $\mathcal{F}_0$ is called an *almost Finsler structure* and is denoted by $\mathcal{F}_1$. Similarly, a homogeneous $F(n)$-structure depending on $\mathcal{F}_0^*$ is called a *homogeneous almost Finsler structure* and is denoted by $\mathcal{F}_1^*$. Concerning these structures, we can show