$$
\begin{aligned}
&\tilde{F}_i{}^j{}_k = F_i{}^j{}_k - U_i{}^j{}_k, \quad \tilde{C}_i{}^j{}_k = C_i{}^j{}_k + \tau_i{}^j{}_k \\
(3.1) \qquad &U_i{}^j{}_k := \lambda^j l_i l_k - \lambda_i l^j l_k - \lambda_k l_i l^j - (C_i{}^j y_k + C_k{}^j y_i - C_{ki} y^j) + \\
&+ L^2 (C_k{}^r C_r{}^j{}_i + C_i{}^r C_r{}^j{}_k - C_r{}^j C_k{}^r{}_i)
\end{aligned}
$$

where $\lambda_i := \partial_i \lambda \quad \lambda^j = g^{ji}\lambda_i \quad C_i{}^j := g^{rs} C_r{}^j{}_i \lambda_s$

$$\tau_i{}^j{}_k := \frac{e^{2\lambda} - 1}{L e^{2\lambda}} h_{ik} l^j.$$

Torsion and curvature tensors are obtained in the relation of $C\Gamma$ as given in [3].

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ON ALMOST FINSLER STRUCTURES

BY

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In the present paper we define a notion of an almost Finsler structure, and show that, for some Lie group G, the tangent bundle over a manifold admits the G-structure under some condition, if and only if the manifold is Finslerian. In this case, an almost symplectic structure defined on the tangent bundle actually plays an important role. The G-connections relative to the almost Finsler structure are also investigated.

§1. Almost Finsler Structures. Let M be an n-dimensional C^∞-manifold and $T(M)$ be its tangent bundle. As is well-known, $T(M)$ admits the standard tangent structure $\mathcal{F}_0$. Let Q be its structure tensor. The set $T(n) = \{\left(\begin{smallmatrix} A & 0 \\ B & A \end{smallmatrix}\right) | A \in GL(n, R),\ B \in gl(n, R)\}$ is a well-known Lie group. The structure $\mathcal{F}_0$ is an integrable $T(n)$-structure on $T(M)$. Let π be the natural projection $T(M) \to M$. Any coordinate neighbourhood $\{U, x^i\}$ in M induces a coordinate neighbourhood $\{\pi^{-1}(U), (x^i, y^i)\}$ in $T(M)$, which we call a canonical coordinate neighbourhood. For any $\pi^{-1}(U)$ and $\pi^{-1}(\bar{U})$, if $2n$-frames $\{Z_A\}$ in $\pi^{-1}(U)$ and $\{\bar{Z}_A\}$ in $\pi^{-1}(\bar{U})$ are both adapted to the $T(n)$-structure, they satisfy $\bar{Z}_A = P_A^B Z_B ((P_A^B) \in T(n))$ in $\pi^{-1}(U) \cap \pi^{-1}(\bar{U})$. Of course, the components a_j^i and b_j^i of $(P_A^B) = \begin{pmatrix} a_j^i & 0 \\ b_j^i & a_j^i \end{pmatrix}$ are functions of x^i and y^i. Now, we may consider the case where a_j^i are positively homogeneous of degree 0 with respect to y^i and b_j^i are positively homogeneous of degree 1 with respect to y^i. If we treat only the structure $\mathcal{F}_0$ under the restriction of the above homogeneity condition, we call it the *homogeneous standard tangent structure* and denote it by $\mathcal{F}_0^*$.

Now, let G be a Lie subgroup of $T(n)$. If $T(M)$ admits the G-structure $\mathcal{D}$ and if any frame adapted to the structure $\mathcal{D}$ is always adapted to $\mathcal{F}_0$, the structure $\mathcal{D}$ is called a *G-structure depending on $\mathcal{F}_0$*. Moreover, if the G-structure $\mathcal{D}$ satisfies the above mentioned homogeneity condition, the structure $\mathcal{D}$ is called *a homogeneous G-structure depending on* $\mathcal{F}_0^*$ and is denoted by $\mathcal{D}^*$.

Now let us consider the set $F(n) = \{\left(\begin{smallmatrix} A & 0 \\ SA & A \end{smallmatrix}\right) \mid A \in O(n),\ S \in \mathbf{S}\mathrm{ym}(n)\}$. It is apparent that $F(n)$ is a Lie subgroup of $T(n)$. Hereafter the Lie group $F(n)$ is called a *Finsler group*, an $F(n)$-structure depending on $\mathcal{F}_0$ is called an *almost Finsler structure* and is denoted by $\mathcal{F}_1$. Similarly, a homogeneous $F(n)$-structure depending on $\mathcal{F}_0^*$ is called a *homogeneous almost Finsler structure* and is denoted by $\mathcal{F}_1^*$. Concerning these structures, we can show.

Theorem 1. *If a tangent bundle $T(M)$ admits an almost Finsler structure, $T(M)$ also admits an almost symplectic structure. And $T(n)\cap S_p(n)=F(n)$ holds good.*

Let $\Omega=\omega_{AB}dx^A\wedge dx^B$ be the 2-form associated with the almost symplectic structure. Next, we shall define a pseudo inner product $<,>$ of rank n by $<Z_i, Z_j>=\delta_{ij}$, $<Z_i, Z_{\bar j}>=0$, $<Z_{\bar i}, Z_j>=0$, $<Z_{\bar i}, Z_{\bar j}>=0$ in each $\pi^{-1}(U)$. It is easy to see that the above pseudo inner product can be extended to the whole of $T(M)$ globally. Thus we obtain a pseudo Riemann metric G' of rank n on $T(M)$. In each $\pi^{-1}(U)$, if we put $\partial/\partial x^A=\beta^B_A Z_B$, then we see $(\beta^B_A)\in T(n)$ and we have $G'_{AB}=<\partial/\partial x^A, \partial/\partial x^B>=\sum_{a=1}^{n}\beta^a_A\beta^a_B$. On putting $\sum_{a=1}^{n}\beta^a_i\beta^a_j=g_{ij}$ and $\sum_{a=1}^{n}(\beta^{\bar a}_i\beta^a_j-\beta^a_i\beta^{\bar a}_j)=\alpha_{ij}$, we obtain

$$(G'_{AB})=\begin{pmatrix} g_{ij} & 0\\ 0 & 0\end{pmatrix},\quad (\omega_{AB})=\begin{pmatrix}\alpha_{ij} & -g_{ij}\\ g_{ij} & 0\end{pmatrix},\quad \Omega=\alpha_{ij}dx^i\wedge dx^j-2g_{ij}dx^i\wedge dy^j.$$

Let ∇ be a G-connection relative to the structure $\mathcal{F}_1$. Using the fact that the Lie algebra $f(n)$ of the Lie group $F(n)$ is given by $f(n)=\left\{\begin{pmatrix} a & 0\\ b & a\end{pmatrix}\middle| a\in\iota(n),\ b\in\mathfrak{S}\mathrm{ym}(n)\right\}$, we can show

Theorem 2. *Let $T(M)$ be a tangent bundle admitting an almost Finsler structure $\mathcal{F}_1$. In order that a linear connection ∇ on $T(M)$ be a G-connection relative to the structure $\mathcal{F}_1$, it is necessary and sufficient that $\nabla\omega=0$ and $\nabla Q=0$ hold good. With respect to the connection ∇, $\nabla G'=0$ is also true.*

§2. Finsler Structures. Concerning the almost Finsler structure $\mathcal{F}_1$, we can prove.

Theorem 3. *Let $T(M)$ be a tangent bundle admitting an almost Finsler structure $\mathcal{F}_1$. In order that the almost symplectic form Ω associated with $\mathcal{F}_1$ be closed, it is necessary and sufficient that (1) $\dot\partial_k g_{ij}=\dot\partial_j g_{ik}$, (2) $\dot\partial_k\alpha_{ij}=\partial_i g_{jk}-\partial_j g_{ki}$, (3) $\partial_k\alpha_{ij}+\partial_i\alpha_{jk}+\partial_j\alpha_{ki}=0$ hold.*

Theorem 4. *Suppose that the tangent bundle over a manifold M admits a homogeneous almost Finsler structure $\mathcal{F}^*_1$. Then, the almost symplectic form Ω associated with $\mathcal{F}^*_1$ is closed if and only if g_{ij} is a Finsler metric on M and $\alpha_{ij}=y^m(\dot\partial_i g_{jm}-\dot\partial_j g_{im})$ holds. In this case, $\Omega=d(2y^m g_{mj}dx^j)$ holds true.*

Theorem 5. *In order that a manifold M admits a Finsler metric, it is necessary and sufficient that the tangent bundle $T(M)$ admits a homogeneous $F(n)$-structure depending on $\mathcal{F}^*_0$ whose associated almost symplectic form is closed.*

On the tangent bundle over a Finsler manifold, the 2-form $\Omega=d(2y^m g_{mj}dx^j)$ is the well-known symplectic form, which we call the *symplectic form associated with a Finsler metric* and we denote by Ω^* hereafter.

§3. Generalized Metrics and Non-linear Connections. In this section we consider a Lie group defined by $D(O(n))=\left\{\begin{pmatrix} A & 0\\ 0 & A\end{pmatrix}\middle| A\in O(n)\right\}$. It is clear that the Lie algebra of $D(O(n))$ is given by $d(O(n))=\left\{\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}\middle| a\in o(n)\right\}$.

Now let us consider the case where $T(M)$ admits a homogeneous $D(O(n))$-structure depending on $\mathcal{F}^*_0$, which we denote by $\mathcal{F}^*_2$. Let $Z_A=\gamma^B_A\partial_B$ be a $2n$-frame adapted to $\mathcal{F}^*_2$ and put $(\Gamma^B_A)=\begin{pmatrix}\gamma & 0\\ \sigma & \gamma\end{pmatrix}=\Gamma$ and $K=\begin{pmatrix}E & 0\\ 0 & -E\end{pmatrix}$. Then, $F=\Gamma K\Gamma^{-1}$ becomes a globally defined (1,1)-tensor field on $T(M)$ and gives $T(M)$ an almost product structure. Moreover $N=-\sigma\gamma^{-1}$ gives M a non-linear connection. Let us put $X_i=\partial_i-N^m_i\dot\partial_m$ and $X_{\bar i}=Y_i=\dot\partial_i$, $\{X_A\}$ is a $2n$-frame, but is not adapted to $\mathcal{F}^*_2$. By virtue of the fact $D(O(n))\subset O(2n)$, $T(M)$ admits a Riemann metric G. The components of G with respect to the frame $\{X_A\}$ are written as $<X_i, X_j>=g_{ij}$, $<X_i, Y_j>=0$ and $<Y_i, Y_j>=g_{ij}$. The functions g_{ij} are positively homogeneous of degree 0 with respect to y^i and they give M a so-called generalized metric. Concerning these, we can verify

Theorem 6. *A necessary and sufficient condition for a tangent bundle $T(M)$ to admit a structure $\mathcal{F}^*_2$ is that the manifold M admits a generalized metric and a non-linear connection.*

Theorem 7. *If a tangent bundle $T(M)$ admits a structure $\mathcal{F}^*_2$, $T(M)$ admits an almost symplectic form $\Omega=-\{(g_{ir}N^r_j-g_{jr}N^r_i)\,dx^i\wedge dx^j+2g_{ij}dx^i\wedge dy^j\}$. Moreover Ω is closed if and only if g_{ij} is a Finsler metric and Ω is the symplectic form Ω^* associated with the Finsler metric.*

§4. G-Connections Relative to the Structure $\mathcal{F}^*_2$. First, we assume that a tangent bundle $T(M)$ admits a structure $\mathcal{F}_2$, that is, a $D(O(n))$-structure depending on $\mathcal{F}_0$, and $\{Z_A\}$ is the $2n$-frame adapted to $\mathcal{F}_2$. Now we shall treat G-connections relative to the structure $\mathcal{F}_2$. Let ∇ be such a connection. Then ∇ is a linear connection on $T(M)$ and satisfies

$$(*)\qquad \nabla_U Z_A=\bar{\bar\Gamma}^B_{AC}U^C Z_B,\quad \bar{\bar\Gamma}^B_{AC}U^C\in d(o(n))\ \text{for any}\ U=U^C Z_C.$$

It is easy to see that $\nabla G=0$ holds. Hence a G-connection relative to $\mathcal{F}_2$ is a metrical connection for the Riemann metric derived from $\mathcal{F}_2$. And the condition (*) leads us to

$$(**)\qquad \nabla_{Z_j}Z_i=\bar{\bar\Gamma}^m_{ij}Z_m,\ \nabla_{Z_j}Z_{\bar i}=\bar{\bar\Gamma}^{\bar m}_{ij}Z_{\bar m},\ \nabla_{Z_{\bar j}}Z_i=\bar{\bar\Gamma}^m_{i\bar j}Z_m,\ \nabla_{Z_{\bar j}}Z_{\bar i}=\bar{\bar\Gamma}^{\bar m}_{i\bar j}Z_{\bar m}.$$

Conversely, if the above conditions (**) and $\nabla G=0$ are all satisfied, the condition (*) is fulfilled. Thus we obtain

Theorem 8. *In order that a linear connection ∇ on a tangent bundle admitting a structure $\mathcal{F}_2$ be a G-connection relative to $\mathcal{F}_2$, it is necessary and sufficient that the conditions (**) and $\nabla G=0$ are satisfied where G is the Riemann metric derived from $\mathcal{F}_2$.*

Now, we add the homogeneity condition to the structure $\mathcal{F}_2$, that is, we consider the structure $\mathcal{F}^*_2$. Then, there exist a non-linear connection N and a generalized metric g on the manifold M. On this case, we can show

Theorem 9. *On a tangent bundle admitting a structure $\mathcal{F}^*_2$, a linear connection ∇ is a G-connection relative to $\mathcal{F}^*_2$ if and only if ∇ is a linear connection of Finsler type and the Finsler connection derived from ∇ and N is a metric connection of the generalized metric.*

Theorem 10. *Let $T(M)$ be a tangent bundle admitting a structure $\mathcal{F}_2^*$. A linear connection ∇ on $T(M)$ is a linear connection of Finsler type if and only if $\nabla Q=0$ and $\nabla F=0$ hold good. Moreover the linear connection ∇ is a G-connection relative to the structure $\mathcal{F}_2^*$ if and only if $\nabla Q=0$, ($\nabla F=0$ and $\nabla\omega=0$ hold good.*

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ON FINSLER SPACES SATISFYING THE CONDITION $L^2C^2=f(x)$

BY

FUMIO IKEDA

M. Matsumoto and C. Shibata [6] and G. S. Asanov and E. G. Kirnasov [2] have treated non-Riemannian Finsler spaces with the vanishing T-tensor, which are said to satisfy the T-condition.

In the above paper [6], it has been shown that if a Finsler space M^n satisfies the T-condition, then the function L^2C^2 of M^n is reduced to a function of the position only (i.e. $L^2C^2=f(x)$), where L is the fundamental function and C^2 is the square of length of the torsion vector C_i : For example, if the fundamental tensor g_{ij} has such a special form as $g_{ij}=Q^{rs}_{ij}(x)l_r l_s$ as in [2], then the function L^2C^2 becomes zero (i.e. $L^2C^2=f(x)=0$), because in this case, the T-condition is satisfied automatically and $C_i=0$. Therefore, we have been taking interest in the problem whether the condition $L^2C^2=f(x)$ is equivalent to the T-condition or not and have been considering on the properties of those Finsler spaces.

In the present paper, in §1, we shall study the T-tensor of such a Finsler space with the condition $L^2C^2=f(x)$ and get some corollaries. In §2, we shall obtain the condition for such a Finsler space to be a Landsberg or Berwald space and then show that a Landsberg (resp. Berwald) space satisfying the condition $L^2C^2=f(x)$ reduces to a Berwald (resp. locally Minkowski) space.

The terminology and notations are refered to the Matsumoto's monograph [4].

§1. The Condition $L^2C^2=f(x)$. Let l_i, h_{ij} and C_{ijk} denote the unit vector (i.e. $l_i=y_i/L$), the angular metric tensor and the $(h)hv$-torsion tensor (the Cartan torsion tensor), respectively. The T-tensor T_{ijkl} is defined by $T_{ijkl}=LC_{ijk}|_l+C_{ijk}l_l+C_{ijl}l_i+C_{ikl}l_j+C_{ijl}l_k$ and the torsion vector C_i is given by $C_i=g^{jk}C_{ijk}$, where the symbol $|_l$ denotes the v-covariant differentiation and g^{jk} is the reciprocal tensor of g_{jk}.

Assume that the function L^2C^2 is a non-zero function of the position only, i.e. $L^2C^2=f(x)$. The differentiation of this equation by y^i yields

$$L^2C^2{}_{;i}+2C^2y_i=0, \tag{1}$$

where the symbol $_{;i}$ denotes the differentiation by y^i. Since $C^2=g^{rs}C_rC_s$ and $T_{ij}(=g^{kl}T_{ijkl})=LC_i|_j+C_il_j+l_iC_j$, the equation (1) is rewritten as $L^2C^2{}_{;i}=T_{ij}C^j-2C^2y_i$, which gives $T_{ij}C^j=0$. On the contrary, if the condi-