Theorem 10. *Let $T(M)$ be a tangent bundle admitting a structure $\mathcal{F}_2^*$. A linear connection ∇ on $T(M)$ is a linear connection of Finsler type if and only if $\nabla Q=0$ and $\nabla F=0$ hold good. Moreover the linear connection ∇ is a G-connection relative to the structure $\mathcal{F}_2^*$ if and only if $\nabla Q=0$, ($\nabla F=0$ and $\nabla\omega=0$ hold good.*

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ON FINSLER SPACES SATISFYING THE CONDITION $L^2C^2=f(x)$

BY

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M. Matsumoto and C. Shibata [6] and G. S. Asanov and E. G. Kirnasov [2] have treated non-Riemannian Finsler spaces with the vanishing T-tensor, which are said to satisfy the T-condition.

In the above paper [6], it has been shown that if a Finsler space M^n satisfies the T-condition, then the function L^2C^2 of M^n is reduced to a function of the position only (i.e. $L^2C^2=f(x)$), where L is the fundamental function and C^2 is the square of length of the torsion vector C_i: For example, if the fundamental tensor g_{ij} has such a special form as $g_{ij}=Q_{ij}^{rs}(x)l_r l_s$ as in [2], then the function L^2C^2 becomes zero (i.e. $L^2C^2=f(x)=0$), because in this case, the T-condition is satisfied automatically and $C_i=0$. Therefore, we have been taking interest in the problem whether the condition $L^2C^2=f(x)$ is equivalent to the T-condition or not and have been considering on the properties of those Finsler spaces.

In the present paper, in §1, we shall study the T-tensor of such a Finsler space with the condition $L^2C^2=f(x)$ and get some corollaries. In §2, we shall obtain the condition for such a Finsler space to be a Landsberg or Berwald space and then show that a Landsberg (resp. Berwald) space satisfying the condition $L^2C^2=f(x)$ reduces to a Berwald (resp. locally Minkowski) space.

The terminology and notations are refered to the Matsumoto's monograph [4].

§1. The Condition $L^2C^2=f(x)$. Let l_i, h_{ij} and C_{ijk} denote the unit vector (i.e. $l_i=y_i/L$), the angular metric tensor and the $(h)hv$-torsion tensor (the Cartan torsion tensor), respectively. The T-tensor T_{ijkl} is defined by $T_{ijkl}=LC_{ijk}|_l+C_{ijk}l_l+C_{ijl}l_k+C_{ikl}l_j+C_{jkl}l_i$ and the torsion vector C_i is given by $C_i=g^{jk}C_{ijk}$, where the symbol $|_l$ denotes the v-covariant differentiation and g^{jk} is the reciprocal tensor of g_{jk}.

Assume that the function L^2C^2 is a non-zero function of the position only, i.e. $L^2C^2=f(x)$. The differentiation of this equation by y^i yields

$$L^2C^2{}_{;i}+2C^2y_i=0, \tag{1}$$

where the symbol $_{;i}$ denotes the differentiation by y^i. Since $C^2=g^{rs}C_rC_s$ and $T_{ij}(=g^{kl}T_{ijkl})=LC_i|_j+C_il_j+l_iC_j$, the equation (1) is rewritten as $L^2C^2{}_{;i}=T_{ij}C^j-2C^2y_i$, which gives $T_{ij}C^j=0$. On the contrary, if the condition $T_{ij}C^j=0$ is assumed, the function L^2C^2 becomes a function of the position only (i.e. $L^2C^2=f(x)$). Consequently, we have.

Theorem 1. *For an n-dimensional Finsler space M^n, the function L^2C^2 is a function of the position only, if and only if $T_{ij}C^j=0$ holds good.*

Corollary 1. *For a two-dimensional Finsler space, the condition $L^2C^2=f(x)$ is equivalent to the T-condition.*

Corollary 2. *For a three-dimensional Finsler space, the condition $L^2C^2=f(x)$ is equivalent to the condition $T_{ijkl}=Tn_in_jn_kn_l$, where n_i is the third vector of the Moor frame (l_i, m_i, n_i).*

Corollary 3. *For an n-dimensional C^2-like Finsler space (i.e. $C_{ijk}=C_iC_jC_k/C^2$), the condition $L^2C^2=f(x)$ is equivalent to the T-condition.*

Corollary 4. *For an n-dimensional C-reducible Finsler space (i.e. $C_{ijk}=(h_{ij}C_k+h_{jk}C_i+h_{ki}C_j)/(n+1)$), the condition $L^2C^2=f(x)$ is equivalent to the T-condition.*

Corollary 5. *For an n-dimensional semi-C-reducible Finsler space (i.e. $C_{ijk}=p(h_{ij}C_k+h_{jk}C_i+h_{ki}C_j)/(n+1)+qC_iC_jC_k/C^2$), the condition that the function $L^2C^2=f(x)$ and the characteristic scalar $p=p(x)(\neq(n+1)/n)$ is equivalent to the T-condition.*

§2. Landsberg, Berwald and Locally Minkowski Spaces Satisfying the Condition $L^2C^2=f(x)$. Hereafter, assume that a Finsler space M^n satisfies the condition $L^2C^2=f(x)$. From the equation (1), the important tensors which will be used later are given by

$$g_{ij}=-L^2C^2{}_{;i;j}/2C^2+4l_il_j, \tag{2}$$

$$C_{ijk}=-L^2C^2{}_{;i;j;k}/4C^2+2(h_{ij}l_k+h_{jk}l_i+h_{ki}l_j-3l_il_jl_k)/L, \tag{3}$$

$$\begin{aligned} C_{ijk|l}=-L^2C^2{}_{;i;j;k|l}/4C^2-C^2{}_{|l}(C_{ijk}-2(h_{ij}l_k+h_{jk}l_i+\\ +h_{ki}l_j-3l_il_jl_k)/L)/C^2, \end{aligned} \tag{4}$$

$$\begin{aligned} P_{ijk}=-L^2C^2{}_{;i;j;k|0}/4C^2-C^2{}_{|0}(C_{ijk}-2(h_{ij}l_k+\\ +h_{jk}l_i+h_{ki}l_j-3l_il_jl_k)/L)/C^2, \end{aligned} \tag{5}$$

$$\begin{aligned} T_{ijkl}=-L^2C^2{}_{;i;j;k|l}/4C^2+5C_{ijk}l_l+C_{jkl}l_i+C_{ilk}l_j+C_{ijl}l_k+\\ +2(h_{ij}h_{kl}+h_{jk}h_{il}+h_{ki}h_{jl}-5(h_{ij}l_kl_l+h_{jk}l_il_l+h_{ki}l_jl_l+\\ +h_{kl}l_il_j+h_{il}l_jl_k+h_{jl}l_kl_i-3l_il_jl_kl_l))/L, \end{aligned} \tag{6}$$

where P_{ijk} is the $(v)hv$-torsion tensor, the symbol $_{|l}$ denotes the h-covariant differentiation and the index 0 means the contraction by y^i.

The above equation (4) (resp. (5)) gives the result that the condition $C_{ijk|l}=0$ (resp. $P_{ijk}=0$) is equivalent to $C^2{}_{;i;j;k|l}=0$ (resp. $C^2{}_{;i;j;k|0}=0$). So, we have

Theorem 2. *If an n-dimensional Finsler space M^n satisfies the condition $L^2C^2=f(x)$, then the necessary and sufficient condition for M^n to be a Berwald space is that $C^2{}_{;i;j;k|l}=0$ holds good. In this case, the function L^2C^2 is constant.*

Theorem 3. *If an n-dimensional Finsler space M^n satisfies the condition $L^2C^2=f(x)$, then the necessary and sufficient condition for M^n to be a Landsberg space is that $C^2{}_{;i;j;k|0}=0$ holds good.*

Assume that a Landsberg space satisfies the T-condition and that L^2C^2 is constant, then the equation (6) yields $C^2{}_{;i;j;k|l|0}=0$. The Ricci identity: $C^2{}_{;i;j;k|h|l}-C^2{}_{;i;j;k|l|h}=-C^2{}_{;i;j;k|r}C_h{}^r{}_l$ and the above result $C^2{}_{;i;j;k|l|0}=0$ give $C^2{}_{;i;j;k|l}=0$. Therefore, by Theorem 3, we have

Theorem 4. *If an n-dimensional Landsberg space M^n satisfies the T-condition and the condition $L^2C^2=constant$, then M^n becomes a Berwald space.*

Consider a locally Minkowski space (i.e. $g_{ij;k}=0$ for some coordinate system (x^i)), where the symbol $_{;k}$ denotes the differentiation by x^k. If the Berwald space satisfies the condition $L^2C^2=f(x)$, then the above condition $g_{ij;k}=0$ is equivalent to the equation $C^2{}_{;k}=0$ by Theorem 3. So, we have

Theorem 5. *If an n-dimensional Berwald space M^n satisfies the condition $L^2C^2=f(x)$, then the necessary and sufficient condition for M^n to be a locally Minkowski space is that $C^2{}_{;i}=0$ holds good for some coordinate system (x^i).*

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