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SUMAR

MAXIMUM-DEGREE MATCHINGS

BY

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1. **Introduction.** Let $G=(V, E)$ be a finite graph or multigraph without loops. If v is a vertex of G, then $d_G(v)$ is the degree of v in G, and $\Delta(G)$ is the maximum degree of (a vertex of) G. If $A\subseteq E$, then $[A]_G$ denotes the subgraph spanned by A in G.

Definition 1. *Let $G=(V, E)$ and $A\subseteq E$. The degree of A in G, denoted by $\delta_G(A)$, is*

$$\delta_G(A)=\sum_{v\in V} d_{[A]_G}(v)\cdot d_G(v).$$

Let us remark that $\delta_G(A)$ is, in fact, the sum of the degrees of the extremities of each edge of A.

A matching of G is a set $M\subseteq E$ such that $\Delta([M]_G)\leq 1$. Let $\mathcal{M}_G$ be the set of all matchings of G. If $M\in\mathcal{M}_G$ then $E(M)$ (the set of exposed vertices with respect to M) and $S(M)$ (the set of saturated vertices with respect to M) are

$$E(M)=\{v:\ v\in V,\ d_{[M]_G}(v)=0\}\,,$$

$$S(M)=\{v:\ v\in V,\ d_{[M]_G}(v)=1\}\,.$$

Definition 2. *A maximum-degree matching of G (for short, m. d. m.) is a matching M^* of G such that,*

$$\delta_G(M^*)=\max\ \{\delta_G(M);\ M\in\mathcal{M}_G\}\,.$$

Clearly, a maximum-degree matching of G is a matching which maximizes the sum of the degrees of its saturated vertices. On the other hand, by the above remark a m. d. m. of G is a maximum-weighted matching of G with respect to the weight function $w(e)=\delta_G(e)$, for each $e\in E$.

Therefore any maximum-weighted matching algorithm [3], [5] can be used to obtain in polynomial time a m. d. m. of a graph. However, as we shall see in the next section, it is possible to use the intimate relation between this particular weight function and the structure of G in order to obtain a more efficient algorithm.

2. **Some properties of a m. d. m.** We give in this section a characterization of a m. d. m. similar to Berge's characterization [1] of a maximum cardinality matching, from which some interesting properties can be deduced.

Definition 3. *Let M be a matching of G. A degree-alternating chain (for short, d. a. c.) with respect to M in G is an alternating path P with respect to M starting from an exposed vertex v_0 with respect to M, and ending in a vertex v_f which is*

i) either exposed with respect to M,

ii) or saturated with respect to M and $d_G(v_0) > d_G(v_f)$.

Theorem 1. *A matching M of G is a m. d. m. if and only if it admits no d. a. c.*

Proof. If M is a m. d. m. and P is a d. a. c. with respect to M, then $M' = (M \setminus P) \cup (P \setminus M)$ is a matching of G with

$$\delta_G(M') = \delta_G(M) + d_G(v_0) + d_G(v_f) > \delta_G(M),$$

if P satisfies condition i) in definition 3, or

$$\delta_G(M') = \delta_G(M) + d_G(v_0) - d_G(v_f) > \delta_G(M),$$

if condition ii) is fulfilled. In both cases, the choice of M is therefore contradicted.

Conversely, let $M \in \mathcal{M}_G$ such that M admits no d. a. c. and let M' such that $\delta_G(M') > \delta_G(M)$. The connected components of $[M \Delta M']_G$ are isolated vertices, paths or even cycles. If K is an even cycle component, then $S(M) \cap V(K) = S(M') \cap V(K) = V(K)$. Therefore, since $\delta_G(M') > \delta_G(M)$, there exists a path component P of $[M \Delta M']_G$ such that $\delta_G(M' \cap P) > \delta_G(M \cap P)$ which implies that P has either both extremities exposed with respect to M, or one exposed with respect to M and with the other one saturated, but with a smaller degree. In both cases P is a d. a. c. with respect to M, thus contradicting the hypothesis.

Corollary. *If M^* is a m. d. m. of G, then M^* is a matching of maximum cardinality.*

A more interesting property of a m. d. m. is given by the following theorem.

Theorem 2. *If M^* is a m. d. m. of G, then*

$$\min_{M \in \mathcal{M}_G} \max_{v \in E(M)} d_G(v) = \max_{v \in E(M^*)} d_G(v).$$

Proof. Let $M^0 \in \mathcal{M}_G$ such that $\min\limits_{M \in \mathcal{M}_G} \max\limits_{v \in E(M)} d_G(v) = \max\limits_{v \in E(M^0)} d_G(v) = p$, and let us suppose that there exists $v_0 \in E(M^*)$ such that $d_G(v_0)$ is greater than p. By the choice of M^0 there exists $e_1 = v_0v_1 \in M^0$; since M^* is a m. d. m. of G, there exists $e_2 = v_1v_2 \in M^*$ and $d_G(v_2) \geq d_G(v_0) > p$, (otherwise v_0, e_1, v_1 or v_0, e_1, v_1, e_2, v_2 is a d. a. c. with respect to M^*).

By the choice of M^0 there exists $e_3 = v_2v_3 \in M^0$; since M^* is a m. d. m. of G, there exists $e_4 = v_3v_4 \in M^*$ and $d_G(v_4) \geq d_G(v_0) > p$. By repeating the above argument, we shall construct an infinite path P in G (we can not return to a vertex on the path just constructed, because M^0 and M^* are matchings and v_0 is exposed with respect to M^*), contradiction which finishes the proof.

Corollary. *If M^* is a m. d. m. of G, then*

$$\Delta(G - M^*) = \min \{\Delta(G - M) : M \in \mathcal{M}_G\}.$$

The first corollary of this section gives an idea for an algorithm for obtaining a m. d. m.: determine firstly a maximum cardinality matching of G, and again destroy the even d. a. c., if any. It is an open problem if it is possible to use some greedy ideas in order to obtain a fast algorithm for determining a m. d. m. and hence a maximum cardinality matching.

The second corollary gives a simple method of testing if a given graph has a matching saturating all vertices of maximum degree: choose a m. d. m. M^* of G and the answer to the question is yes if and only if M^* has the desired property. We shall apply this idea in the next section.

3. A generalization of a result on bipartite graphs. A well-known result [2] states that any bipartite (multi)graph has a matching saturating all vertices of maximum degree. Its usual proofs needs a trick in order to apply Hall's theorem or can be deduced from König theorem on the chromatic index of a bipartite (multi)graph.

Using the property of the m. d. m.'s, we can give a direct proof to a more general theorem.

Theorem 3. *If the vertices of maximum degree of G induce in G a bipartite sub(multi)graph, then G has a matching saturating all these vertices.*

Proof. Let M^* be a m. d. m. of G and let us suppose that there exists $v_0 \in E(M^*)$ such that $d_G(v_0) = \Delta(G)$. Let $\mathcal{O}$ and $\mathcal{E}$ be the sets of vertices accesible from v_0 on an odd, or even alternating paths respectively, with respect to M^*. Note that $v_0 \in \mathcal{E}$, and $|\mathcal{O}| + 1 = |\mathcal{E}|$.
Since v_0 is exposed with respect to M^* and M^* is a m. d. m. of G, it follows that any vertex of $\mathcal{E}$ has a maximum degree. The set $N_G(\mathcal{E})$ of the vertices adjacent in G with the vertices of $\mathcal{E}$ satisfies $N_G(\mathcal{E}) \subseteq \mathcal{O}$.

Indeed, if $v \in N_G(\mathcal{E})$, then v is accesible from v_0 on an odd alternating path with respect to M^* obtained from the even alternating path from v_0 to the neighbour $w \in \mathcal{E}$ of v, the edge e and the vertex v.

We have $\mathcal{O} \cap \mathcal{E} = \varnothing$.
Indeed, if $v \in \mathcal{O}(\mathcal{E})$ then, if P and P' are the alternating paths from v_0 to v, one even and the other one odd, then $P \Delta P'$ is an odd circuit whose all vertices belong to $\mathcal{E}$ and hence have a maximum degree, contradicting the hypothesis that the vertices of maximum degree induce a bipartite submultigraph. In particular, $\mathcal{E}$ is a stable set.

Let $E_{\mathcal{E}}$ and $E_{\mathcal{O}}$ the sets of edges incidents with vertices from $\mathcal{E}$, and from $\mathcal{O}$, respectively. From the above considerations, we have $E_{\mathcal{E}} \subseteq E_{\mathcal{O}}$ and $|E_{\mathcal{E}}| = |\mathcal{E}| \cdot \Delta(G)$. Therefore

$$|\mathcal{E}| \cdot \Delta(G) = |E_{\mathcal{E}}| \leq |E_{\mathcal{O}}| \leq |\mathcal{O}| \cdot \Delta(G) \text{ hence}$$

$$|\mathcal{O}| + 1 = |\mathcal{E}| \leq |\mathcal{O}|,$$

a contradiction which proves the theorem.

Let us remark that this theorem can not be deduced (as the result for bipartite multigraphs) from a result in edge-colourings, because there are graphs with a bipartite structure of the maximum degree vertices and with chromatic index greater than $\Delta(G)$ (for example, the graph obtained from $K_{2,2}$ by inserting a new vertex into one of its edges).

4. Degree-critical graphs. In this section we shall consider only graphs (without multiple edges).

Definition 4. *A connected graph G is called degree-critical if $\Delta(G-M)=\Delta(G)$ for any $M\in \mathcal{M}_G$, but for each $e\in E$, there exists $M\in\mathcal{M}_G$ such that $\Delta(G-(M\cup e))=\Delta(G)-1$.*

We are interested in the structure of the degree-critical graphs for reasons which will appear in section 5. We have the following theorem.

Theorem 4. *If G is a degree-critical graph, then G has an odd number of vertices.*

Proof. Let G be degree-critical and let M^* be a m. d. m. of G. Let $v_0\in E(M^*)$ with $d_G(v_0)=\Delta(G)$. If v_1 is any vertex adjacent with v_0 then there exists v_2 such that $v_1v_2\in M^*$ and $d_G(v_2)=\Delta(G)$. Let $M'=M^*\cup\{v_0v_1\}\setminus\{v_1v_2\}$. Clearly M' is a matching of $G'=G-\{v_1v_2\}$. Furthermore M' is a m. d. m. of G'. Indeed, since $|M^*|=|M'|$, there is no odd d. a. c. with respect to M' in G'. Let P an even d. a. c. with respect to M' in G' with initial extremity x and final extremity y.

We have two posibilities

i) $x\neq v_2$. If P does not use the edge v_0v_1 then P is a d. a. c. in G with respect to M^*, contradicting the choice of M^*.
Hence P uses the edge v_0v_1. If v_0 is before v_1 on the path P, then the initial part of P relating x and v_0 is an odd d. a. c. with respect to M^* in G, contradicting the choice of M^*. On the other hand, if v_1 is before v_0 on the path P, then the final part of P relating v_0 and y is an even d. a. c. with respect to M^* in G, since $d_G(v_0)=\Delta(G)\geqslant d_G(x)>d_G(y)$, and hence contradicting the choice of M^*. In all cases we have obtained contradictions which show that the first possibility can not occur.

ii) $x=v_2$. If P does not use the edge v_0v_1 then v_0v_1, v_1, v_1v_2, P is an even d. a. c. with respect to M^* in G (because $d_G(v_0)=d_G(v_2)=d_{G'}(v_2)+1>d_{G'}(y)+1>d_G(y)$); contradiction. Hence P uses the edge v_0v_1. If v_1 is before v_0 on the path P, then as above the final part of P is an even d. a. c. with respect to M^* in G, contradiction. If v_0 is before v_1 on the path P, then $v_1\neq y$ (since $d_{G'}(v_2)=d_{G'}(v_1)$) and then the path from v_0 to v_2 on P, the edge v_2v_1 and the final part of P relating v_1 to y gives a d. a. c. with respect to M^* in G, contradiction. Hence the second possibility can not occur, either.

We have therefore proved that M' is a m. d. m. of G' and from the definition of a degree-critical graph and the corollary of theorem 2, we obtain that v_0 is the only exposed vertex with respect to M^* in G which has the degree $\Delta(G)$.

Now, we show that $E(M^*)=\{v_0\}$. Let $w\in E(M^*)\setminus\{v_0\}$. We have, therefore, $d_G(w)<\Delta(G)$. Let w_1, w_2 such that $ww_1\in E$, $w_1w_2\in M^*$.
Since G is degree-critical and v_0 is exposed with respect to M^* it follows that M^* is not a m. d. m. of $G'=G-ww_1$. Hence there exists a d. a. c. P with respect to M^* in G'. Clearly P starts from v_0 and can finish in w_1 or w_2. In both cases we obtain a d. a. c. with respect to M^* in G, contradicting the choice of M^*.

The theorem results from the obvious relation

$$|V(G)|=2\cdot|M^*|+|E(M^*)|.$$

5. Applications to edge-colourings of graphs. An edge-colouring of $G=(V,E)$ is a partition of E such that each class of this partition is a matching of G. The chromatic index of G is the minimum number of matchings which partioned E, and is denoted by $\chi'(G)$.

Let us device the following greedy-colour algorithm for the colouring of G.

Step 0. $k:=0$, $G_0:=G$.

Step 1. If $E(G_k):=\varnothing$ STOP: greedy-colour number $:=k$; $M_1,\ldots,M_k$ is a greedy edge-colouring of G.
Otherwise go to step 2.

Step 2. Determine a m. d. m. M_{k+1} of G_k;
Put $G_{k+1}:=G_k-M_{k+1}$, $k:=k+1$;
Go to step 1.

From theorem 3 we obtain that if the greedy-colour algorithm is applied to a bipartite multigraph then the greedy-colour number determined is $\Delta(G)$, therefore we obtain König's theorem [4].

If the greedy-colour algorithm is applied to an arbitrary multigraph, then we can apply theorem 3 in order to obtain that any three consecutive iterations of step 2 decrease the maximum degree of the current multigraph with at least two, hence the greedy-colour number is at most $3\,\Delta(G)/2$ and therefore we obtain the well-known result of Shannon [6].

The above greedy-colour algorithm does not can be used in order to obtain Vizing's theorem [7].
However, using the theorem of Vizing and the corollary of our theorem 2, we obtain the following interesting property of a m. d. m.:

If G is a graph and M a m. d. m. of G, then

$$\Delta(G-M)+\chi'(G-M)\leqslant\Delta(G)+\chi'(G)-1.$$

Theorem 4 and the above property of a m. d. m. give a progress in the critical graph conjecture due to Jacobsen:

If G is a (edge-colour) critical graph without a matching saturating all the maximum degree vertices, then G has an odd number of vertices. In this direction we give the following conjecture (which implies that of Jacobsen):

If G is a (edge-colour) critical graph which has a matching saturating all the maximum degree vertices, then G has a matching M such that $G-M$ is critical.

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