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University „Al. I. Cuza"
Computer Center
6600 Iași — Romania

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CONTEXTUAL SYSTEMS AND CONTEXTUAL TRANSDUCERS

BY

DOREL LUCANU and DAN CRISTEA

1. Introduction. The ideas presented in this paper evolved from practical problems related to natural language comprehention in well bounded universes of discourse.

Rather frequently, in natural language sentences appear groups of words whose meanings are different than those done by the sums of their components. This is the case with idiomatic expressions or technical jargon. Such expressions must be considered without trying to breake their meanings into elementary building bricks. Let us refer to these linguistical phenomena as *sintagmae*. In fact what is to be done, before any other process of interpretation, is to translate them directly to symbols that, conventially or not, incorporate their whole meanings. Moreover there is often the case that a sintagma has more than a single meaning, even in a perfectly bounded universe of discourse. Nevertheless a human listener can easily decide between a sense or another. He uses the context to desambiguate among more interpretations. What precisely is meant by „context" is certainly hard to establish. At a first glance it is something like a neighbourhood around the ambiguous sintagma. How large this neighbourhood must be considered is again an open question and we prefer to pay no atention to this question here. A more attentive insight reveals that not everything in this neighbourhood is important in establishing the correct meaning and that usually there is there included a string of words that only matters. This remark modifies a little our definition for sintagma. When we insist on meaning rather then sintax, a sintagma could be seen as a pair of strings of words which, appearing in text at a certain distance one of the other, precisely determine a sense to one of them. Every one of the two strings of words will be denoted by *context*.

The paper proposes two mathematical tools for sintagmae transformation. The first of them, called *n-contextual systems*, is a generalization of rewriting systems on strings [2]. The second, called *n-contextual transducers*, is a combination between the first and the generalized sequential machine [3, 4]. The formal definitions of the two mechanisms and some properties of them are presented.

2. Preliminaries. In the rest of the paper V will denote a numerable set of symbols. The free monoid generated by V will be denoted with V^* and the empty string of V^* with e. A *sintagma* is a tuple

$$<c_1, \ldots, c_n>,$$

where $c_i \in V^*$, $i = 1, \ldots, n$ and $n \in N$ (the set of positive integers). A string that appears into a sintagma will be called *context*. When it is not explicitely given, n is understood to be a positive integer.

3. **Contextual systems. 3.1. Definition.** *An n-contextual system S is a finite set of rules of the form*

$$<c_1, \ldots, c_n> \to <c_1', \ldots, c_n'> \quad c_i, c_i' \in V^*, \; i = 1, \ldots, n$$

that satisfy the following properties:

(i) *there exists i such that $c_i \neq e$*

(ii) *$c_i = e$ implies $c_i' = e$.*

(iii) *$<c_{i_1}, \ldots, c_{i_n}> \to <c_{i_1}', \ldots, c_{i_n}'> \in S$ where $(i_1, \ldots, i_n)$ is a permutation of the set $\{1, \ldots, n\}$.*

3.2. **Definition.** *Let S be an n-contextual system, u, $v \in V^*$. We say that u is directly rewritten (derived) in v according to S if there exists a rule $<c_1, \ldots, c_n> \to <c_1', \ldots, c_n'>$ such that*

$$u = u_1 c_1 \ldots u_n c_n u_{n+1} \quad \text{and} \quad v = u_1 c_1' \ldots u_n c_n' u_{n+1}.$$

We write $u \xrightarrow{S} v$ if u is directly rewritten in v. We denote by $\xrightarrow{S+}$ the tranzitive closure of $\xrightarrow{S}$ and by $\xrightarrow{S*}$ the tranzitive and reflexive closure of $\xrightarrow{S}$.

Notations. n-CS denotes the set of n-contextual systems.

$$CS = \bigcup_{n>0} n\text{-}CS.$$

$$S(u) = \{v \in V^* / u \xrightarrow{S*} v\} \text{ for } u \in V^* \text{ and } S \in CS.$$

3.3. Definition. *Let $S \in CS$ and u, $v \in V^*$.*

(i) *v is irreducible with respect to S if it does not exist*

$$w \in V^* \text{ such that } v \xrightarrow{S} w$$

(ii) *v is a normal form for u if $u \xrightarrow{S*} v$ and v is irreducible.*

Notations. $u \overset{S*}{\leftrightarrow} v$ *if* $u \xrightarrow{S*} v$ *or* $v \xrightarrow{S*} u$,

$\equiv_S$ *is the tranzitive closure of* $\overset{S*}{\leftrightarrow}$.

IRR(S) *denotes the set of irreductible words with respect to S.*

3.4. **Definition.** *Let $S \in CS$.*

(i) *S is confluent if for any u, v, $w \in V^*$ there exists $z \in V^*$ such that $u \xrightarrow{S*} v$, $u \xrightarrow{S*} w$ imply $v \xrightarrow{S*} z$, $w \xrightarrow{S*} z$.*

(ii) *S is locally confluent if for any u, v, $w \in V^*$ there exists $z \in V^*$ such that $u \xrightarrow{S} v$, $u \xrightarrow{S} w$ imply $v \xrightarrow{S*} z$, $w \xrightarrow{S*} z$.*

(iii) *S is Noetherian if for any $u \in V^*$ it does not exists an infinite sequence $u \xrightarrow{S} u_1 \xrightarrow{S} u_2 \xrightarrow{S} \ldots$*

3.5. **Proposition.** *Let S be a confluent n-contextual system, u, $v \in V^*$. Then v is a normal form of u iff $u \equiv_S v$ and v is irreducible.*

Proof. If v is a normal form of u *then* $u \equiv_S v$ by the definition of $\equiv_S$. Reciprocally, *if* $u \equiv_S v$, $u \neq v$ and v is irreducible then there exist $u_0, \ldots, u_n$ such that

$$u = u_0, \ldots, u_{n-2} \xleftarrow{S*} u_{n-1}, \; u_{n-1} \xrightarrow{S+} u_n = v.$$

We will show that $u \xrightarrow{S+} v$ using the induction on n. For $n = 1$ we have $u = u_0 \xrightarrow{S+} u_1 = v$. For $n > 1$, because S is confluent it follows that there exists $u_{n-1}' \in V^*$ such that $u_{n-2} \xrightarrow{S*} u_{n-1}'$, $u_n \xrightarrow{S*} u_{n-1}'$. But u_n is irreducible and hence $u_{n-1}' = u_n = v$. Now we have

$$u = u_0, \ldots, u_{n-2} \xrightarrow{S+} u_n = v$$

and by the inductive hypothesis it follows that $u \xrightarrow{S+} v$ #

3.6. **Theorem** [2]. *Let $S \in CS$. If S is Noetherian and confluent then for any $u \in V^*$ the set $[u] = \{v \in V^* / u \equiv_S v\}$ has an unique normal form.*

Theorem 3.6 shows that the processes of translation that use Noetherian and confluent n-contextual systems are nonambiguous. When natural language analysis is approached with contextual parsing, the property of non-ambiguity of the contextual system employed must not be thought of as also characterizing the computer „understanding" of the text. Some inerent ambiguity of an utterance could not be resolved by a set of nonambiguous rules. Vis-à-vis of the natural language parsing using contextual systems two needs could be expressed: (i) — one would like that one sentence that has a unique meaning to have only one normal form (this is obtained when Noetherian and confluent n-contextual systems are used); (ii) — one would like that two sentences that have the same meaning to be translated to the same normal form or to equivalent normal forms. This second need motivates the following.

3.7. **Definition.** *Consider $S \in CS$ and $\sim$ an equivalence relation on V^*.*

(i) *S has the Church-Rosser property with respect to $\sim$ if for any u, $v \in V^*$ there exists $w \in V^*$ such that $u \sim v$ implies $u \xrightarrow{S*} w$, $v \xrightarrow{S*} w$.*

(ii) *S is $\sim$-preserving if for any u, u', $v \in V^*$, there exists $v' \in V^*$ such that $u \xrightarrow{S*} u'$ and $u \sim v$ imply $u' \sim v'$ and $v \xrightarrow{S*} v'$.*

3.8. **Theorem** [1]. *Let S be a Noetherian n-contextual system. Then S is confluent iff it has the Church-Rosser property with respect to $\equiv_S$.*

3.9. **Definition.** *Consider S, $S' \in CS$.*

(i) *$S \equiv S'$ if for any $u \in V^*$, $S(u) = S'(u)$.*

(ii) *$S \simeq S'$ if IRR(S) = IRR(S').*

3.10. **Definition.** *Let K and K' be two subsets of CS.*

(i) *$K \leq K'$ if for any $S \in K$, there exists $S' \in K'$ such that $S \equiv S'$.*

(ii) *$K < K'$ if $K \leq K'$ and furthermore there exists $S' \in K'$, such that $S \equiv S'$ for none $S \in K$.*

(iii) *$K \leq_1 K'$ if for any $S \in K$ there exists $S' \in K'$, such that $S \simeq S'$,*

(iv) *$K <_1 K'$ if $K \leq_1 K'$ and furthermore there exists $S' \in K'$, such that $S \simeq S'$ for none $S \in K$.*

3.11. Proposition. *$\equiv$ and $\simeq$ are equivalence relations on CS and $\leqslant$, $\leqslant_1$ are preorders on the family of subsets of CS.*

3.12. Theorem. *For $n>0$, n-$CS<(n+1)$-CS.*

Proof. The relation n-$CS\leqslant(n+1)$-CS follows by replacing any rule $<c_1, \ldots, c_n>\to<c'_1, \ldots, c'_n>$ by a new rule $<c_1, \ldots, c_n, e>\to<c'_1, \ldots, c'_n, e>$. In order to show that the equality does not hold, we consider the $(n+1)$-contextual system

$$S'=\{<a_1, \ldots, a_{n+1}>\to<u_1, \ldots, u_{n+1}>\},$$

where $a_1, \ldots, a_{n+1}\in V$, $u_1, \ldots, u_{n+1}\in(V-\{a_1, \ldots, a_{n+1}\})^*$. Assume that there is $S\in n$-CS such that $S\equiv S'$. Because V is infinite and S is finite there exists $x\in V$ such that x does not belong to any context of a rule of S. For $u=a_1x\ldots a_nxa_{n+1}$ we have

$$S'(u)=\{v/v=u_1x\ldots u_nxu_{n+1}\}\cup\{u\}.$$

If a rule of S can be applied to u then it must have the form

$$<y_1, \ldots, y_n>\to<c_1, \ldots, c_n>,\ y_i\in\{a_1, \ldots, a_{n+1}\}\cup\{e\}.$$

But this implies $S(w)\neq\{w\}$ for $w=y_1x\ldots y_{n-1}xy_n$. Because $S\equiv S'$ it follows that $S'(w)\neq\{w\}$ and this is a contradiction. Therefore it does not exist $S\in n$-CS such that $S\equiv S'$ and hence n-$CS<(n+1)$-CS #

In the proof of the theorem 3.12 v and w are irreducible with respect to S' and the following theorem holds true.

3.13. Theorem. *For $n>0$, n-$CS <_1 (n+1)$-CS.*

The theorems 3.12 and 3.13 establish that the sets n-CS, $n>0$, form an infinite hierarchy in CS with respect to both preorders.

3.14. Proposition. *Consider S, $S'\in CS$ and $\sim$ an equivalence relation on V^*. If $S\equiv S'$ then*

(i) *$S\simeq S'$,*

(ii) *S is confluent iff S' is confluent,*

(iii) *S has the Church-Rosser property with respect to $\sim$ iff S' has the Church-Rosser property with respect to $\sim$*

(iv) *S is $\sim$-preserving iff S' is $\sim$-preserving.*

A proposition like 3.14 for $\simeq$ is not possible. Indeed, the systems $S=\{<a>\to<e>\}$, $S'=\{<ab>\to<aab>, <a>\to<e>\}$ satisfy $S\simeq S'$ because $\mathrm{IRR}(S)=\mathrm{IRR}(S')=\{u/a$ is not a subword of $u\}$. It is easy to see also that $S(ab)\neq S'(ab)$ and hence the two preorders are distinct.

3.15. Definition. *Consider $W\subseteq V$, K, $K'\subseteq CS$.*

(i) *for S, $S'\in CS$ we have $S\equiv_W S'$ if $S(u)\cap W^*=S'(u)\cap W^*$ for any $u\in W^*$,*

(ii) *$K\leqslant_W K'$ if for any $S\in K$ there exists $S'\in K'$ such that $S\equiv_W S'$*

(iii) *$K<_W K'$ if $K\leqslant_W K'$ and furthermore there exists $S'\in K'$ such that $S\equiv_W S'$ for none $S\in K$.*

(iv) *$K\equiv_W K'$ if $K\leqslant_W K'$ and $K'\leqslant_W K$.*

3.16. Proposition. *Consider K, $K'\subseteq CS$. $K\leqslant_1 K'$ iff $K\leqslant K'$.*

3.17. Theorem. *Let $W\subseteq V$ be such that $V-W$ is infinite. Then 2-$CS\equiv_W n$-CS for any $n>2$*

Proof. The relation 2-$CS\leqslant_W n$-CS follows immediately. In order to show n-$CS\leqslant_W 2$-CS we consider $S'\in n$-CS. Initially we set S to $\varnothing$. For any rule

$$<c_1, \ldots, c_n>\to<c'_1, \ldots, c'_n>\in S'$$

we consider $A_1, \ldots, A_{n-1}, A'_2, \ldots, A'_{n-1}\in V-W$. They are not implicated in the definitions of the rules of $S\cup S'$, and we add to S the set of rules

$$\begin{aligned}
&<c_1, c_2>\to<A_1, A_2>,\\
&<A_2, c_3>\to<A'_2, A_3>,\\
&\ldots\\
(1)\qquad &<A_{n-1}, c_n>\to<A'_{n-1}, c'_n>,\\
&<A'_{n-1}, A'_{n-2}>\to<c'_{n-1}, A'_{n-2}>,\\
&\ldots\\
&<A'_2, A_1>\to<c'_2, c'_1>.
\end{aligned}$$

Let X be the set of all new symbols added. We show that $S(u)\cap(V-X)^*=S'(u)\cap(V-X)^*$, $u\in(V-X)^*$. In order to show that $S'(u)\cap(V-X)^*\subseteq S(u)\cap(V-X)^*$ we use the induction on k, the length of $u\xrightarrow{S'*}v$, $v\in S'(u)\cap(V-X)^*$. For $k=0$ it follows immediately. For $k>0$ there exist u' and a rule

$$<c_1, \ldots, c_n>\to<c'_1, \ldots, c'_n>\in S'$$

such that $u=u_1c_1\ldots u_nc_nu_{n+1}$, $u'=u_1c'_1\ldots u_nc'_nu_{n+1}$ and $u'\xrightarrow{S'*}v$.

By the definition of S there exists in S a set of rules of the form (1) that defines

$$\begin{aligned}
&u\xrightarrow{S}u_1A_1\ldots u_nc_nu_{n+1}\xrightarrow{S}\ldots\\
&\xrightarrow{S}u_1A_1\ldots A_{n-1}u_nc'_nu_{n+1}\xrightarrow{S}\ldots\\
&\xrightarrow{S}u_1c'_1\ldots u_nc'_nu_{n+1}=u'.
\end{aligned}$$

Because $u\in(V-X)^*$ and in the definition of S' are not implicated symbols from X it follows that $u'\in(V-X)^*$. By inductive hypothesis $v\in S(u')\cap\cap(V-X)^*$ and hence $u\xrightarrow{S*}u'\xrightarrow{S*}v$, i.e. $v\in S(u)$. In order to show the inverse inclusion we use the induction on k, the length of $u\xrightarrow{S*}v$, $v\in(V-X)^*$. For $k=0$ it follows immediately. Consider $k>0$. The fact u, $v\in(V-X)^*$ implies that there exists u' such that u is rewritten in u' by the application of a set of rules of the form (1). By the definition of S there is a rule in S' that defines $u\xrightarrow{S'}u'$ and by inductive hypothesis it follows $u'\xrightarrow{S'*}v$ and hence $u\xrightarrow{S'*}v$. Now, the conclusion of the theorem is a consequence of the fact that $W \subset (V - X)$

3.18. **Theorem.** *If W is an infinite subset of V then $1\text{-}CS <_W 2\text{-}CS$*

The proof is similar to that of theorem 3.12. if we consider.

$$S' = \{\langle a_1, a_2 \rangle \to \langle u_1, u_2 \rangle\}.$$

where $a_1, a_2 \in W$, $u_1, u_2 \in (W - \{a_1, a_2\})^*$.

A consequence of theorems 3.17 and 3.18 is the following

3.19. **Theorem.** *If W is an infinite subset of V such that $V - W$ is infinite then $1\text{-}CS <_W 2\text{-}CS$ and $2\text{-}CS \equiv_W n\text{-}CS$ for any $n > 2$.*

4. **Contextual transducers.** A disadvantage of contextual systems is that they do not permit to replace a sintagma a limited number of times. However this need can be satisfied by introducing a mechanism able to check the application of rules. Contextual systems equiped with this mechanism become automata.

4.1. **Definition.** *An n-contextual transducer is a triple $T = (Q, q_0, P)$ where Q is a finite set of states, $q_0 \in Q$ is the initial state and P is a finite set of rules of the form*

$$(q, \langle c_1, \ldots, c_n \rangle) \to (q', \langle c'_1, \ldots, c'_n \rangle)$$

and furthermore every rule satisfies the following conditions :

(i) *there exists i such that $c_i \neq e$.*

(ii) *$c_i = e$ implies $c'_i = e$.*

(iii) *$(q, \langle c_{i_1}, \ldots, c_{i_n} \rangle) \to (q', \langle c'_{i_1}, \ldots, c'_{i_n} \rangle) \in T$ where $(i_1, \ldots, i_n)$ is a permutation of the set $\{1, \ldots n\}$, $q, q' \in Q$.*

4.2. **Definition.** *Consider T an n-contextual transducer and $u, v \in V^*$. We say that u is directly rewritten in v according to T if there exists a rule $(q, \langle c_1, \ldots, c_n \rangle) \to (q', \langle c'_1, \ldots, c'_n \rangle)$ such that*

$$u = u_1 c_1 \ldots u_n c_n u_{n+1}, \quad v = u_1 c'_1 \ldots u_n c'_n u_{n+1}$$

for some $u_1, \ldots, u_{n+1} \in V^*$.

We denote this fact by $(q, u) \xrightarrow{T} (q', v)$, the tranzitive closure of $\xrightarrow{T}$ by $\xrightarrow{T+}$ and the tranzitive and reflexive closure of $\xrightarrow{T}$ by $\xrightarrow{T*}$.

Notations. $T(q, u) = \{v \in V^* / (q, u) \xrightarrow{T*} (q', v), q' \in Q\}$.

$T(u) = T(q_0, u)$.

$n\text{-}CT$ = *the set of the n-contextual transducers,*

$CT = \bigcup_{n>0} n\text{-}CT$

4.3. **Definition.** *Let $T \in n\text{-}CT$, $u, v \in V^*$. We say that v is a normal from for u if $v \in T(u)$ and $T(q, v) = \{v\}$ for any q with the property $(q_0, u) \xrightarrow{T*} (q, v)$.*

Notation. $NF(T) = \{v /$ *there is u such that v is the normal form for u*$\}$

4.4. **Definition.** *Consider T, $T' \in CT$.*

(i) *$T \equiv T'$ if $T(u) = T'(u)$ for any $u \in V^*$.*

(ii) *$T \simeq T'$ if $NF(T) = NF(T')$*

4.5. **Definition.** *Consider K, $K' \subseteq CT$.*

(i) *$K \leqslant K'$ if for any $T \in K$ there exists $T' \in K'$ such that $T \equiv T'$,*

(ii) *$K < K'$ if $K \leqslant K'$ and furthermore there exists $T' \in K'$ such that $T \equiv T'$ for none $T \in K$.*

(iii) *$K \leqslant_1 K'$ if for any $T \in K$ there exists $T' \in K'$ such that $T \simeq T'$.*

(iv) *$K <_1 K'$ if $K \leqslant_1 K'$ and furthermore there exists $T' \in K'$ such that $T \simeq T'$ for none $T \in K$.*

The next two examples show that the relations $\equiv$ and $\simeq$ are incomparable.

Example 1. Consider

$$T = (\{q_0, q_1, q_2\}, q_0, \{(q_0, \langle a \rangle) \to (q_1, \langle b \rangle), (q_1, \langle b \rangle) \to (q_2, \langle c \rangle)$$
$$(q_0, \langle a, b \rangle) \to (q_0, \langle b, a \rangle)\}).$$

$$T' = (\{q_0, q_1, q_2\}, q_0, \{(q_0, \langle a \rangle) \to (q_1, \langle c \rangle), (q_1, \langle c \rangle) \to (q_2, \langle b \rangle)$$
$$(q_0, \langle a, c \rangle) \to (q_0, \langle c, a \rangle)\}).$$

Let $u \in V^*$. If a is not a subword of u then $T(u) = T'(u) = \{u\}$. Otherwise, denote by n_1 the number of occurrences of a in u, by n_2 the number of occurrences of b in u and by n_3 the number of occurrences of c in u. If $v \in T(u)$ then v contains $n_1 - 1$ occurrences of a, $n_2 + 1$ occurrences of b and n_3 occurrences of c or $n_1 - 1$ occurrences of a, n_2 occurrences of b and $n_3 + 1$ occurrences of c. In every one of these two cases we have $v \in T'(u)$. The inclusion $T'(u) \subseteq T(u)$ follows by symetry. Hence $T(u) = T'(u)$. If $v \in NF(T)$ then v contains $n_1 - 1$ occurrences of a, n_2 occurrences of b and $n_3 + 1$ occurrences of c and if $w \in NF(T')$ then w contains $n_1 - 1$ occurrences of a, $n_2 + 1$ occurences of b and n_3 occurrences of c. This implies $NF(T) \neq NF(T')$.

Example 2. Let $T = (\{q_0\}, q_0, \{(q_0, \langle a, b \rangle) \to (q_0, \langle c, x \rangle), (q_0, \langle a, c \rangle) \to (q_0, \langle c, y \rangle)\})$ and $T' = (\{q_0\}, q_0, \{(q_0, \langle a, b \rangle) \to (q_0, \langle c, y \rangle), (q_0, \langle a, c \rangle) \to (q_0, \langle c, x \rangle)\})$.

where $x \neq y$, and a, b and c are not subwords of x or y. Let us show that $T \simeq T'$. If $w \in NF(T)$ then there is u such that $w \in T(u)$ and w is rewritten from u by replacing m times the pair (a, b) with x and n times the pair (a, c) with y. Let also u' be the word obtained from u by the simultaneous replacement of all b with c and of all c with b. This makes $w \in T'(u')$ and of course w is also irreducible with respect to T', so $w \in NF(T')$. By simmetry we can also conclude $NF(T') \subseteq NF(T)$ and hence $NF(T) = NF(T')$. But we cannot state that $T \equiv T'$ because for $m, n \in N$ and $u = a^{m+n} b^m c^n$ we have $T(u) = \{x^m y^n\} \neq \{y^m x^n\} = T'(u)$.

4.6. **Theorem.** *For any $n > 0$ we have*

(i) *$n\text{-}CT < (n+1)\text{-}CT$,*

(ii) *$n\text{-}CT <_1 (n+1)\text{-}CT$.*

The proof is similar to that of theorem 3.12. if we consider

$$T = (\{q_0\}, q_0, \{(q_0, \langle a_1, \ldots, a_{n+1} \rangle) \to (q_0, \langle u_1, \ldots, u_{n+1} \rangle)\}),$$

where $a_1, \ldots, a_{n+1} \in V$, $u_1, \ldots, u_{n+1} \in (V - \{a_1, \ldots, a_{n+1}\})^*$ #

4.7. **Definition.** *Consider $W \subseteq V$, K, $K' \subseteq CT$.*

(i) *if T, $T' \in CT$ then $T \equiv_W T'$ if $T(u) \cap W^* = T'(u) \cap W^*$ for any $u \in W^*$.*

(ii) *$K \leqslant_W K'$ if for any $T \in K$ there exists $T' \in K'$ such that $T \equiv_W T'$,*

(iii) $K \equiv_u K'$ *if* $K \leqslant_u K'$ *and* $K' \leqslant_u K$.

For n-contextual transducers the theorem 3.17 is slightly modified.

4.8. **Theorem.** *Let* W *be a finite subset of* V. *Then* $1\text{-}CT \equiv_u n\text{-}CT$ *for any* $n > 1$.

Proof. The relation $1\text{-}CT \leqslant_u n\text{-}CT$ follows immediately. In order to show that $n\text{-}CT \leqslant_u 1\text{-}CT$ we consider $T' = (Q', q'_0, P') \in n\text{-}CT$. We initially set $Q = Q'$, $P = \emptyset$. For any rule

$$(q, \langle c_1, \ldots, c_n \rangle) \to (q', \langle c'_1, \ldots, c'_n \rangle) \in P'$$

we add to Q $2n-2$ new states $q_1, \ldots, q_{n-1}, q'_1, \ldots, q'_{n-1}$ and to P the following set of rules:

$$(q, \langle c_1 \rangle) \to (q_1, \langle A_1 \rangle),$$

$$\ldots$$

$$(q_{n-1}, \langle c_n \rangle) \to (q'_{n-1}, \langle c'_n \rangle),$$

$$(q'_{n-1}, \langle A_{n-1} \rangle) \to (q'_{n-2}, \langle c'_{n-1} \rangle),$$

$$\ldots$$

$$(q'_1, \langle A_1 \rangle) \to (q', \langle c'_1 \rangle)$$

where $A_1, \ldots, A_{n-1}$ are new symbols that do not belong to W and that are not implicated in the definition of T'. The fact that $T = (Q, q'_0, P) \equiv_u T'$ is proved in the same manner as in 3.17 #

Since we can consider n-contextual systems as being n-contextual transducers with a single state the following theorem holds true.

4.9. **Theorem.** $CS < CT$.

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„Al. I. Cuza" University,
Computer Center,
16 Berthelot str.,
6600 Iaşi, România

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ABOUT RECURSIVE DOMAIN EQUATIONS

BY

ANDREI BARANGA

Introduction. The paper presents some facts about solving recursive domain equations, which lead us, sometimes, to obtain in a simpler way solutions of such equations.

In the first section we assume that there is a solution in a given category and we try to find a solution in a full subcategory of it.

In the second, a proof of the main theorem of [1] in a more generally case is given. The paper also contains examples of using the facts described above.

1. **On Solving Equations in Subcategories.** We start by recalling the recursive domain equation problem:

Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \mathcal{C}$ be a functor. We try to find an object X of the category $\mathcal{C}$ such that $F(X)$ is isomorphic to X.

We assume that there is an object X of the category $\mathcal{C}$ which is a solution of the problem. Let $\bar{\mathcal{C}} \subseteq \mathcal{C}$ a full subcategory of $\mathcal{C}$. Can we find another solution in $\bar{\mathcal{C}}$ of the problem, by using the solution in $\mathcal{C}$?

Notation. If P is an object of the category $\mathcal{C}$, we define a functor $\mathcal{X}_P : \bar{\mathcal{C}} \to \mathcal{E}ns$ if we set:

1) $\mathcal{X}_P(A) = \mathrm{Hom}_{\mathcal{C}}(P, A)$, for any object A of the category $\bar{\mathcal{C}}$.

2) $\mathcal{X}_P(f)(u) = fu$, for any morphism $f \in \mathrm{Hom}_{\bar{\mathcal{C}}}(A, B)$ $(= \mathrm{Hom}_{\mathcal{C}}(A, B))$ and $u \in \mathrm{Hom}_{\mathcal{C}}(P, A)$.

Definition. *An object* P *of* $\mathcal{C}$ *is said to have a reflection in* $\bar{\mathcal{C}}$ *if the functor* $\mathcal{X}_P$ *is representable (i.e. there is a functorial isomorphism between* $\mathcal{X}_P$ *and* $\mathcal{X}_{\bar{P}}$ *for some object* $\bar{P}$ *of* $\bar{\mathcal{C}}$*).*

Theorem. *An object* P *of* $\mathcal{C}$ *has a reflection in* $\bar{\mathcal{C}}$ *if and only if there is an object* $\bar{P}$ *of* $\bar{\mathcal{C}}$ *and a morphism* $i_P : P \to \bar{P}$ *in* $\mathcal{C}$ *such that for any object* A *of* $\bar{\mathcal{C}}$ *and any morphism* $f : P \to A$ *in* $\mathcal{C}$, *there is a unique morphism* $\bar{f} : \bar{P} \to A$ *such that the following diagram is commutative:*

$$\begin{tikzcd} P \arrow[rr, "i_P"] \arrow[dr, "f"'] & & \bar{P} \arrow[dl, "\bar{f}"] \\ & A & \end{tikzcd}$$

We say that the pair $(\bar{P}, i_P)$ (or briefly $\bar{P}$) reflects P in $\bar{\mathcal{C}}$.

We outline the main ideas of the proof: