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MINIMAL GRAPHS WITH FOUR NEGATIVE EIGENVALUES

BY

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Results. Let $P(n)$ $(n \geq 1)$ be the set of all connected graphs with exactly n negative eigenvalues (including their multiplicities). A graph $G \in P(n)$ is said to be minimal if none of its induced subgraphs belongs to the same class. Denote by $P'(n)$ the set of all non-isomorphic minimal graphs from the class $P(n)$. Then in [3] is proved that class $P'(n)$ is finite for any $n \geq 1$.

The importance of the class $P'(n)$ is given by the following theorem proved also in [3].

Theorem A. *A connected graph G has at least n negative eigenvalues if and only if it has a graph $F \in P'(n)$ as its induced subgraph.*

A connected graph G belongs to the class $P(n)$ if and only if it has a graph $F \in P'(n)$ as its induced subgraph, and has no graph $H \in P'(n+1)$ as its induced subgraph.

It is known that class $P'(2)$ contains exactly two graphs (K_3 and P_4), where K_m and P_m are the complete graph and a path on m vertices, respectively $(m \geq 2)$. It is proved in [2] that class $P'(3)$ contains exactly 14 graphs which are presented bellow in the upper-diagonal form

$$a_{12}a_{13}a_{23} \ldots a_{1s} \ldots a_{s-1,s}$$

of their adjacency matrices.

We notice that the first number in each line denotes the order number of a graph, while the second denotes the number of vertices of this graph.

Table 1 *Minimal graphs with 3 negative eigenvalues*

(01.4)	1	11	111			(08.6)	1	01	001	0001	10001
(02.5)	1	11	001	0001		(09.6)	1	01	101	0010	00010
(03.5)	1	11	011	0001		(10.6)	1	11	100	0100	00100
(04.5)	1	11	001	0011		(11.6)	1	01	100	0011	00001
(05.5)	1	10	110	1011		(12.6)	1	10	011	0010	00011
(06.6)	1	01	001	0001	00100	(13.6)	1	01	101	0011	00100
(07.6)	1	01	001	0001	00001	(14.6)	1	01	100	0101	00111

We see that orders of all graphs from the class $P'(3)$ are just 4, 5 and 6. First graph is the graph K_4, while the seventh one is the graph P_6.

In [3] and [1], a method to generate the class $P'(n+1)$ from the class $P'(n)$ is also given. We will repeat this method in short.

Let the graph G run over the class $P'(n)$ and S run over the all non-empty subsets from the set $V(G)$. Form a graph Gx by adding a new vertex x

to G and by connecting it just with vertices from S. If the graph Gx has just $n+1$ negative eigenvalues, define $G_S = Gx$. If Gx has n negative eigenvalues, define $G_S = Gxy$ to be the graph generated from Gx by adding a new vertex y adjacent only to x. By a result of [3] we then have

$$P'(n+1)=\{G_S : G\in P'(n),\ S\subseteq V(G)\setminus\{\emptyset\}\}.$$

Using this general result, the set $P'(3)$, and the help of Computer with the subroutine „Eigen", one can generate the set $P'(4)$. We find that this set contains exactly 265 graphs, and their orders are respectively 5, 6, 7 and 8. The list of these graphs is presented in the following table.

The first number in each line is the order number of a graph, while the second one is the number of its edges.

Table 2. *Minimal graphs with 4 negative eigenvalues*

1. Graphs with 5 vertices

(001.10)	1	11	111	1111

2. Graphs with 6 vertices

(002.07)	1	11	001	0001	00011
(003.08)	1	11	111	1000	00001
(004.08)	1	11	011	0001	00011
(005.09)	1	10	110	1011	01010
(006.09)	1	11	111	0101	00001
(007.09)	1	11	001	0011	00111
(008.10)	1	11	001	0011	11101
(009.10)	1	11	111	1110	00001
(010.10)	1	11	011	0001	01111
(011.11)	1	11	011	0001	11111
(012.11)	1	11	001	0011	11111
(013.12)	1	10	110	1011	11111

3. Graphs with 7 vertices

(014.07)	1	01	001	0001	00001	001100
(015.07)	1	11	001	0001	00001	000001
(016.07)	1	01	001	0001	00001	100001
(017.07)	1	01	001	0001	00100	000110
(018.07)	1	11	001	0001	00100	000001
(019.08)	1	01	001	0001	00001	100011
(020.08)	1	11	001	0001	01010	000001
(021.08)	1	01	001	0001	00001	010011
(022.08)	1	11	001	0001	01100	000001
(023.08)	1	01	001	0001	00001	000111
(024.08)	1	11	001	0011	00100	000001
(025.08)	1	11	100	0100	00100	010010
(026.08)	1	11	011	0001	00010	000001
(027.08)	1	11	001	0011	00001	000001
(028.08)	1	01	101	0010	00010	001010
(029.08)	1	01	001	0001	00001	011100
(030.09)	1	10	011	0010	00011	010100
(031.09)	1	11	001	0011	10010	000001
(032.09)	1	01	101	0010	00010	100101
(033.09)	1	01	001	0001	00001	011110
(034.09)	1	01	100	0011	00001	111000
(035.09)	1	01	001	0001	00001	111001
(036.09)	1	01	001	0001	10001	001110
(037.09)	1	01	001	0001	00100	001111
(038.09)	1	01	101	0010	00010	001101
(039.09)	1	11	001	0001	01011	000001
(040.09)	1	01	100	0011	00001	100110
(041.09)	1	10	110	1011	00100	000001
(042.09)	1	11	001	0011	00101	000001
(043.09)	1	11	011	0001	01100	000001
(044.09)	1	11	100	0100	00100	101001
(045.09)	1	01	001	0001	00001	110011
(046.09)	1	01	101	0010	00010	110100
(047.09)	1	01	101	0011	00100	001001
(048.09)	1	10	110	1011	10000	000001
(049.10)	1	01	100	0011	00001	001111
(050.10)	1	01	001	0001	10001	111010
(051.10)	1	01	101	0011	00100	110001
(052.10)	1	11	011	0001	10110	000001
(053.10)	1	01	101	0011	00100	110100
(054.10)	1	11	001	0011	01011	000001
(055.10)	1	11	001	0011	10110	000001
(056.10)	1	11	100	0100	00100	111100
(057.10)	1	01	100	0011	00001	100111
(058.10)	1	01	001	0001	10001	111100
(059.10)	1	01	101	0011	00100	100110
(060.10)	1	10	110	1011	00011	000001
(061.10)	1	01	100	0011	00001	111100
(062.10)	1	10	110	1011	01001	000001
(063.10)	1	10	110	1011	10001	000001
(064.10)	1	01	101	0011	00100	001110
(065.10)	1	11	001	0001	11011	000001

(066.10)	1	01	101	0011	00100	011001
(067.10)	1	01	101	0010	00010	011110
(068.10)	1	01	101	0010	00010	001111
(069.10)	1	10	110	1011	10100	000001
(070.10)	1	10	011	0010	00011	001011
(071.10)	1	01	101	0011	00100	001011
(072.11)	1	10	011	0010	00011	101110
(073.11)	1	10	110	1011	01110	000001
(074.11)	1	01	100	0101	00111	110100
(075.11)	1	11	001	0011	11011	000001
(076.11)	1	01	100	0101	00111	111000
(077.11)	1	11	011	0001	11011	000001
(078.11)	1	10	011	0010	00011	111100
(079.11)	1	01	100	0011	00001	111110
(080.11)	1	10	110	1011	10110	000001
(081.11)	1	01	001	0001	10001	011111
(082.11)	1	10	011	0010	00011	111010
(083.11)	1	01	101	0011	00100	011011
(084.11)	1	10	110	1011	01011	000001
(085.11)	1	01	001	0001	00100	111111
(086.11)	1	01	101	0011	00100	001111
(087.11)	1	01	001	0001	00001	111111
(088.11)	1	01	101	0010	00010	011111
(089.11)	1	11	011	0001	11101	000001
(090.11)	1	11	100	0100	00100	111011
(091.11)	1	01	101	0011	00100	111001
(092.11)	1	01	100	0101	00111	000111
(093.12)	1	10	110	1011	01111	000001
(094.12)	1	01	100	0011	00001	111111
(095.12)	1	10	110	1011	11101	000001
(096.12)	1	01	100	0101	00111	011011
(097.12)	1	01	001	0001	10001	111111
(098.12)	1	01	100	0101	00111	100111
(099.12)	1	10	011	0010	00011	011111
(100.12)	1	01	101	0010	00010	111111
(101.12)	1	01	101	0011	00100	111101
(102.12)	1	01	101	0011	00100	011111
(103.12)	1	01	101	0011	00100	111011
(104.12)	1	01	100	0101	00111	111010
(105.13)	1	01	101	0011	00100	111111
(106.13)	1	01	100	0101	00111	111110
(107.13)	1	01	100	0101	00111	011111
(108.13)	1	10	011	0010	00011	111111
(109.14)	1	01	100	0101	00111	111111

4. Graphs with 8 vertices

(110.07)	1	01	001	0001	00001	000001	0000001
(111.07)	1	01	001	0001	00100	000010	0000001
(112.07)	1	01	001	0001	00100	001000	0000001
(113.07)	1	01	001	0001	00100	000001	0000001
(114.07)	1	01	001	0001	00100	000100	0000001
(115.08)	1	01	100	0011	00001	000100	0000001
(116.08)	1	01	001	0001	00100	001100	0000001
(117.08)	1	01	001	0001	00100	010100	0000001
(118.08)	1	01	100	0011	00001	100000	0000001
(119.08)	1	01	001	0001	10001	000010	0000001
(120.08)	1	01	100	0011	00001	000001	0000001
(121.08)	1	01	001	0001	00001	010100	0000001
(122.08)	1	01	100	0011	00001	000010	0000001
(123.08)	1	01	001	0001	00001	011000	0000001
(124.08)	1	01	001	0001	00001	101000	0000001
(125.08)	1	01	001	0001	00100	101000	0000001
(126.08)	1	01	001	0001	00001	100010	0000001
(127.08)	1	01	001	0001	00001	010010	0000001
(128.08)	1	01	001	0001	00100	100010	0000001
(129.09)	1	01	101	0011	01100	000010	0000001
(130.09)	1	01	001	0001	00001	101010	0000001
(131.09)	1	01	100	0011	00001	001100	0000001
(132.09)	1	11	100	0100	00100	001010	0000001
(133.09)	1	01	101	0011	00100	000001	0000001
(134.09)	1	01	001	0001	00100	010011	0000001
(135.09)	1	01	101	0010	00010	100010	0000001
(136.09)	1	01	101	0011	00100	100000	0000001
(137.09)	1	01	101	0011	00100	010000	0000001
(138.09)	1	11	100	0100	00100	000110	0000001
(139.09)	1	01	100	0011	00001	000011	0000001
(140.09)	1	01	101	0011	00100	001000	0000001
(141.09)	1	01	001	0001	00100	011010	0000001
(142.09)	1	01	101	0011	00100	000100	0000001
(143.09)	1	01	001	0001	00100	101010	0000001
(144.09)	1	01	100	0011	00001	010001	0000001
(145.09)	1	01	001	0001	00100	100011	0000001
(146.09)	1	01	001	0001	00001	100110	0000001
(147.09)	1	10	011	0010	00011	000001	0000001
(148.09)	1	01	100	0011	00001	001001	0000001
(149.09)	1	01	001	0001	00001	100101	0000001
(150.09)	1	01	101	0010	00010	010100	0000001
(151.09)	1	11	100	0100	00100	011000	0000001
(152.09)	1	01	001	0001	10001	010001	0000001
(153.09)	1	01	101	0010	00010	100001	0000001
(154.09)	1	01	001	0001	10001	001001	0000001
(155.09)	1	10	011	0010	00011	001000	0000001
(156.09)	1	01	101	0010	00010	000011	0000001
(157.10)	1	01	100	0011	00001	101010	0000001
(158.10)	1	10	011	0010	00011	010010	0000001
(159.10)	1	01	100	0011	00001	010101	0000001
(160.10)	1	10	011	0010	00011	100010	0000001
(161.10)	1	01	100	0011	00001	001101	0000001
(162.10)	1	01	101	0010	00010	010101	0000001
(163.10)	1	01	101	0011	00100	000011	0000001
(164.10)	1	01	001	0001	10001	110010	0000001
(165.10)	1	01	100	0101	00111	000001	0000001
(166.10)	1	01	101	0010	00010	010110	0000001
(167.10)	1	01	100	0011	00001	011001	0000001
(168.10)	1	01	100	0101	00111	100000	0000001
(169.10)	1	10	011	0010	00011	000110	0000001
(170.10)	1	10	011	0010	00011	001100	0000001
(171.10)	1	01	101	0010	00010	100011	0000001
(172.10)	1	01	100	0101	00111	010000	0000001
(173.10)	1	01	100	0011	00001	001110	0000001
(174.10)	1	01	100	0101	00111	000010	0000001
(175.10)	1	01	001	0001	10001	101010	0000001
(176.10)	1	01	100	0011	00001	110001	0000001
(177.10)	1	01	100	0011	00001	110010	0000001
(178.10)	1	01	100	0011	00001	100011	0000001
(179.10)	1	01	101	0011	00100	000101	0000001
(180.10)	1	01	101	0010	00010	110010	0000001
(181.10)	1	01	100	0011	00001	101100	0000001

(182.10)	1	01	101	0011	00100	100001	0000001
(183.10)	1	01	001	0001	00100	101101	0000001
(184.10)	1	10	011	0010	00011	110000	0000001
(185.10)	1	11	100	0100	00100	000111	0000001
(186.10)	1	01	001	0001	00100	100111	0000001
(187.10)	1	11	100	0100	00100	001011	0000001
(188.10)	1	01	001	0001	00100	110101	0000001
(189.10)	1	11	100	0100	00100	111000	0000001
(190.10)	1	01	001	0001	00100	101011	0000001
(191.10)	1	01	001	0001	00001	110110	0000001
(192.10)	1	01	001	0001	00001	101110	0000001
(193.10)	1	01	001	0001	00001	101101	0000001
(194.10)	1	11	100	0100	00100	110001	0000001
(195.11)	1	01	100	0011	00001	101011	0000001
(196.11)	1	01	101	0011	00100	010110	0000001
(197.11)	1	01	001	0001	10001	011011	0000001
(198.11)	1	01	100	0011	00001	110011	0000001
(199.11)	1	01	101	0011	00100	011010	0000001
(200.11)	1	01	100	0011	00001	101101	0000001
(201.11)	1	01	101	0011	00100	101010	0000001
(202.11)	1	01	101	0011	00100	110010	0000001
(203.11)	1	01	100	0101	00111	000110	0000001
(204.11)	1	11	100	0100	00100	101110	0000001
(205.11)	1	01	101	0011	00100	011100	0000001
(206.11)	1	01	100	0101	00111	010001	0000001
(207.11)	1	01	100	0101	00111	001001	0000001
(208.11)	1	01	100	0101	00111	000101	0000001
(209.11)	1	01	001	0001	00100	101111	0000001
(210.11)	1	01	100	0101	00111	100001	0000001
(211.11)	1	01	101	0011	00100	101001	0000001
(212.11)	1	01	100	0011	00001	111001	0000001
(213.11)	1	01	001	0001	01100	110111	0000001
(214.11)	1	01	101	0011	00100	100101	0000001
(215.11)	1	01	101	0011	00100	010101	0000001
(216.11)	1	01	101	0011	00100	001101	0000001
(217.11)	1	10	011	0010	00011	010011	0000001
(218.11)	1	11	100	0100	00100	010111	0000001
(219.11)	1	01	101	0011	00100	100011	0000001
(220.11)	1	01	101	0011	00100	010011	0000001
(221.11)	1	01	101	0010	00010	010111	0000001
(222.11)	1	10	011	0010	00011	001101	0000001
(223.11)	1	01	101	0011	00100	000111	0000001
(224.11)	1	10	011	0010	00011	100101	0000001
(225.11)	1	01	101	0010	00010	011011	0000001
(226.11)	1	10	011	0010	00011	011001	0000001
(227.11)	1	01	101	0010	00010	110011	0000001
(228.11)	1	01	100	0101	00111	101000	0000001
(229.11)	1	01	100	0101	00111	100010	0000001
(230.11)	1	01	100	0101	00111	010010	0000001
(231.12)	1	01	100	0011	00001	111101	0000001
(232.12)	1	01	100	0101	00111	101010	0000001
(233.12)	1	01	100	0011	00001	110111	0000001
(234.12)	1	01	100	0101	00111	011010	0000001
(235.12)	1	01	100	0101	00111	011100	0000001
(236.12)	1	01	100	0101	00111	101100	0000001
(237.12)	1	01	100	0101	00111	010110	0000001
(238.12)	1	10	011	0010	00011	110110	0000001
(239.12)	1	01	100	0101	00111	001110	0000001
(240.12)	1	10	011	0010	00011	011110	0000001
(241.12)	1	01	101	0011	00100	010111	0000001
(242.12)	1	01	101	0010	00010	111011	0000001
(243.12)	1	01	101	0011	00100	100111	0000001
(244.12)	1	01	101	0011	00100	101011	0000001
(245.12)	1	11	100	0100	00100	101111	0000001
(246.12)	1	01	101	0011	00100	110011	0000001
(247.12)	1	01	100	0101	00111	100101	0000001
(248.12)	1	01	100	0101	00111	010101	0000001
(249.12)	1	10	011	0010	00011	110011	0000001
(250.12)	1	01	101	0011	00100	101101	0000001
(251.12)	1	01	101	0011	1000x	110101	0000001
(252.12)	1	01	101	0011	00100	011110	0000001
(253.12)	1	01	101	0011	00100	111010	0000001
(254.13)	1	01	100	0101	00111	101011	0000001
(255.13)	1	10	011	0010	00011	110111	0000001
(256.13)	1	01	100	0101	00111	110011	0000001
(257.13)	1	01	100	0101	00111	010111	0000001
(258.13)	1	01	100	0101	00111	011101	0000001
(259.13)	1	01	100	0101	00111	101101	0000001
(260.13)	1	11	100	0100	00100	111111	0000001
(261.13)	1	01	100	0101	00111	111001	0000001
(262.13)	1	01	101	0011	00100	110111	0000001
(263.13)	1	01	101	0011	00100	101111	0000001
(264.14)	1	01	100	0101	00111	101111	0000001
(265.14)	1	01	100	0101	00111	111101	0000001

By a reviewing of the fourth negative eigenvalues λ_4 of graphs from the Table 2, we find that the 259-th graph G_{259} has the maximal such value

$$\lambda_4(G_{259}) = A_4 \approx -0.09088.$$

By the well-known interlacing theorem, this immediately implies the following result:

Corollary 1. *For any connected graph G with at least 4 negative eigenvalues, its fourth negative eigenvalue $\lambda_4(G)$ satisfies the inequality*

(1) $$\lambda_4(G) \leq A_4.$$

This inequality is the best possible.

Hence, the real number A_4 is the best upper bound of eigenvalues $\lambda_4(G)$ of all such graphs.

The last inequality can be of some importance, since many of methods related to the technique of forbidden subgraphs need to know if a particular eigenvalue of a graph is negative, positive or zero. But, as is also known, enough small real numbers are by computers very often confused with zero.

Remark. Using a similar method and the class P' (4), this author also intends to give in a subsequent note, a best upper bound of the fifth negative eigenvalues of all connected graphs with at least 5 negative eigenvalues.

REFERENCES

1. Cvetković D., Doob M., Sachs H. — *Spectra of Graphs — Theory and Application*, VEB Deutscher Verlag der Wiss., Berlin, 1982; Academic Press, New York, 1982.
2. Torgašev A. — *Graphs with exactly two negative eigenvalues*, Math. Nachr. 122 (1985), 135–140.
3. Torgašev A. — *On graphs with a fixed number of negative eigenvalues*, Discrete Math. 57 (1985), 311–317.
4. Torgašev A. — *On graphs with exactly three negative eigenvalues*, Graph Theory, Proc. VI Yugoslav Sem. Graph Theory, April 18–19, 1985, (Dubrovnik), to appear.

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