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SOME CHARACTERISTIC PROPERTIES OF INTERVAL GRAPHS

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1. Let $G=(V, E)$ be a simple undirected graph, where V is the set of vertices and E is the set of the edges of G. The graph G is an interval graph if there exists a family $\mathcal{J}$ of open intervals on the real line such that G and the intersection graph associated to $\mathcal{J}$ are isomorphic [3]. Characteristic properties of interval graphs and algorithms for solving the problem whether a given graph is an interval graph are described in [1—4], for example. In this note some new characteristic properties of interval graphs are given. It is shown that a simple undirected graph G is an interval graph if and only if a specified graph obtained from G by deleting at least an edge or a vertex is an interval graph.

2. First of all we shall specify some notations. In this note we shall consider only simple undirected graphs, such a graph will be named simply a graph. If $G=(V, E)$ is a graph and $v \in V$ then we denote by $I(v, G)$, or simply $I(v)$, the set of the vertices of G which are adjacent to v, i.e. $I(v, G)=\{x \in V;\ [x, v] \in E\}$, and for $u, v \in V$ one denotes by $I(u, v, G)$, or simply $I(u, v)$, the set of the vertices of $V \setminus \{v\}$ which are adjacent to u and nonadjacent to v, i.e. $I(u, v)=I(u) \setminus (I(v) \cup \{v\})$. For a graph $G=(V, E)$ one denotes by $N(G)$ the subset of E defined by

$$N(G)=\{[u, v] \in E;\ I(u, v) \neq \emptyset,\ I(v, u) \neq \emptyset\}.$$

As usual, if $G=(V, E)$ is a graph and $X \subset V$ then $[X]_G$ is the subgraph of G, having the set of vertices X and the set of the edges E', $E'=\{[u, v] \in E;\ u \in X,\ v \in X\}$; if $M \subset E$ then $[M]_G$ denotes the partial subgraph of G having the set of the edges M and the set of the vertices $Y=\{u \in V;\ \exists v \in V,\ [u, v] \in M\}$. When a, b, c, are three distinct vertices of a graph G such that $[\{a, b, c\}]_G$ is a clique, then $[\{a, b, c\}]_G$ will be named a triangle of G. Let $G=(V, E)$ be a graph and $u, v \in V$, $u \neq v$; by $S(u, v, G)$, or simply $S(u, v)$, we shall denote the set of the vertices of G defined by

$$S(u, v)=\{x \in V;\ x \in I(u, v) \text{ or } \exists y \in S(u, v),\ [x, y] \in E,\ [x, u] \notin E\}.$$

We remark that for a graph G we have $[u, v] \in N(G)$ if and only if $[u, v]$ is an edge of G and $S(u, v) \neq \emptyset$, $S(v, u) \neq \emptyset$.

Now let $G=(V, E)$ be a graph and $[u, v] \in E$; we say that the edge $[u, v]$ has the property (A) when $[u, v] \in N(G)$ and for all $x \in I(u) \cap I(v)$ we have $\{x\} \cup I(x) \supset I(u) \cap I(v)$ or $I(x) \setminus \{u, v\} \subset I(u) \cap I(v)$. We say that an edge

$[u, v]$ of a graph G has the property (B) when $[u, v] \in N(G)$ and $I(x) = I(u) \cap I(v)$ for all $x \in I(u) \cap I(v)$, i.e. $[I(u) \cap I(v)]_G$ is a clique of G. It is clear that every edge which has the property (B) has the property (A), too.

3. The following theorem gives the structure of a simple undirected graph G such that $N(G) = \varnothing$.

Theorem 1. *If $G = (V, E)$ is a connected graph and $N(G) = \varnothing$ then there exists $v \in V$ such that $|I(v)| = |V| - 1$.*

Proof. Let v be a vertex of G such that $|I(v)| \geq |I(x)|$ for all $x \in V$. Since G is connected it follows $V = \{v\}$ or $I(v) \neq \varnothing$: when $V = \{v\}$ we have nothing to prove, so that we can suppose $I(v) \neq \varnothing$. Since G is connected, if $1 \leq |I(v)| < |V| - 1$ then there exist $x, y \in V \setminus \{v\}$ such that $[v, x], [x, y] \in E$ and $[v, y] \notin E$, i.e. $I(x, v) \neq \varnothing$, and because of $N(G) = \varnothing$ it follows $I(v, x) = \varnothing$, that is $I(v) \subset \{x\} \cup I(x)$. By using $I(v) \subset \{x\} \cup I(x)$ and $y \in I(x, v)$ one obtains $|I(x)| \geq |I(v)| + 1$, contradiction. Hence we have to accept $|I(v)| = |V| - 1$ and the Theorem is proved. Following the proof of Theorem 1 it is easy to prove the next

Corollary 1. *Let $G = (V, E)$ be a connected graph and $v \in V$ such that $|I(v)| \geq |I(x)|$ for all $x \in V$. If $|I(v)| < |V| - 1$ then there exists $u \in I(v)$ such that $[u, v] \in N(G)$.*

By using Theorem 1 and the fact that a graph is an interval graph if and only if every connected component of G is an interval graph one obtains the following Corollary, which is equivalent to the Theorem 2 in [5].

Corollary 2. *If G is a graph and $N(G) = \varnothing$ then G is an interval graph.*

We can obtain a better result by using the characterisation of interval graphs in terms of prohibitted subgraphs. [3].

Theorem 2. *If G is a graph and every connected component of $[N(G)]_G$ is a path, that is $[N(G)]_G$ has no subgraphs isomorphic to K_{13} and no subgraphs isomorphic to a cycle, then G is an interval graph.*

Proof. It is easy to prove that when $[N(G)]_G$ has no subgraphs isomorphic to K_{13} or to a cycle, then G cannot contain any prohibitted subgraph described in [3]. We remark that another proof of Theorem 2 can be obtained by using the characteristic properties of interval graphs given in this note.

Lemma 1. *Let $G = (V, E)$ be an interval graph, $[a, b] \in N(G)$, $\mathcal{J}$ a family of open intervals on the real line and $f : V \to \mathcal{J}$ a bijection such that $f(x) \cap f(y) \neq \varnothing$ if and only if $x = y$ or $[x, y] \in E$, that is $(\mathcal{J}, f)$ represents the graph G. If we denote $f(x) = (\alpha(x), \beta(x))$ then for all $x \in I(a, b)$ and for all $y \in I(b, a)$ either*

$$\alpha(a) < \beta(x) \leq \alpha(b) < \beta(a) \leq \alpha(y) < \beta(b), \text{ or}$$

$$\alpha(b) < \beta(y) \leq \alpha(a) < \beta(b) \leq \alpha(x) < \beta(a).$$

Proof. Obvious.

The next Corollary is a direct consequence of the previous Lemma.

Corollary 3. *If G is an interval graph and $[a, b] \in N(G)$ then $S(a, b, G) \cap I(b, a) = \varnothing$.*

Lemma 2. *If $[\{a, b, c\}]_G$ is a triangle of an interval graph G then at least one of the sets $I(a, c) \cap I(b, c)$, $I(b, a) \cap I(c, a)$, $I(a, b) \cap I(c, b)$ is empty and at least one of the sets $I(a, b) \cap I(a, c)$, $I(b, a) \cap I(b, c)$, $I(c, a) \cap I(c, b)$ is an empty set.*

Proof. If $I(a, c) \cap I(b, c) \neq \varnothing$, $I(b, a) \cap I(c, a) \neq \varnothing$, $I(a, b) \cap I(c, b) \neq \varnothing$ then $[a, b], [a, c], [b, c] \in N(G)$ and, by using Lemma 1, one obtains a contradiction. So we have to accept that at least one of these three sets is an empty set. Similarly one proves that at least one of the sets $I(a, b) \cap I(a, c)$, $I(b, a) \cap I(b, c)$, $I(c, a) \cap I(c, b)$ is an empty set.

Lemma 3. *Let $G = (V, E)$ be an interval graph and $[a, b] \in N(G)$. If $x, y \in I(a, b)$, $x \neq y$, and $[a, x], [a, y] \in N(G)$ then $[x, y] \in E$.*

Proof. By using Lemma 1 for the edges $[a, b]$, $[a, x]$, $[a, y]$ it results that for every representation (J, f) of the graph G we must have $f(x) \cap f(y) \neq \varnothing$ and consequently $[x, y] \in E$.

Lemma 4. *Let $G = (V, E)$ be an interval graph, $[a, b] \in N(G)$ and $x, y \in I(a) \cap I(b)$, $x \neq y$. If $[a, x] \in N(G)$ and $[x, y] \notin E$ then either $I(a, b) \subset I(a, x)$ or $I(b, a) \subset I(b, x)$.*

Proof. Suppose $I(a, b) \not\subset I(a, x)$. In this case it is clear that $I(a, b) \cap I(x, b) \neq \Phi$ and because of $y \in I(a) \cap I(b)$, $[x, y] \notin E$ it follows $y \in I(a, x) \cap I(b, x)$. By using Lemma 2 for the triangle $[\{a, b, x\}]_G$ one obtains $I(b, a) \cap I(x, a) = \varnothing$ and therefore $I(b, a) \subset I(b, x)$. On the other hand, if we accept $I(a, b) \subset I(a, x)$ and $I(b, a) \subset I(b, x)$ then $I(x, a) \subset I(x, b)$ and because of $I(x, a) \neq \varnothing$ it follows $I(a, b) \cap I(a, x) \neq \varnothing$, $I(b, a) \cap I(b, x) \neq \varnothing$ $I(x, a) \cap I(x, b) \neq \varnothing$, which contradicts the Lemma 2. The lemma is proved.

Lemma 5. *If $G = (V, E)$ is an interval graph, $[a, b] \in N(G)$, $x, y \in I(a) \cap I(b)$ $x \neq y$, $[a, x] \in N(G)$ and $[x, y] \notin E$ then $|I(a) \cap I(x)| < |I(a) \cap I(b)|$ or $[b, x] \in N(G)$ and $|I(b) \cap I(x)| < |I(a) \cap I(b)|$.*

Proof. By using Lemma 1, either $I(a, b) \subset I(a, x)$ or $I(b, a) \subset I(b, x)$. Suppose $I(a, b) \subset I(a, x)$ and consider $z \in I(a) \cap I(x)$. If $z \notin I(b) \cup \{b\}$ then $z \in I(a, b) \subset I(a, x)$, that is $z \notin I(x)$, impossible. So we have to accept $I(a) \cap I(x) \subset I(b) \cup \{b\}$ and consequently $I(a) \cap I(x) \subset (I(a) \cap I(b)) \cup \{b\}$. But $x, y \in I(a) \cap I(b)$ and $x, y \notin I(a) \cap I(x)$ hence $|I(a) \cap I(x)| \leq |I(a) \cap I(b)| - 1$. Similarly one proves that when $I(b, a) \subset I(b, x)$, we have $|I(b) \cap I(x)| \leq |I(a) \cap I(b)| - 1$. In order to complete the proof, we shall show that in the case $I(b, a) \subset I(b, x)$ the relation $[b, x] \in N(G)$ is true. Since $I(b, a) \neq \varnothing$ it is clear that $I(b, x) \neq \varnothing$. By using again Lemma 1 we obtain $I(a, b) \not\subset I(a, x)$ and because of $I(a, b) \neq \varnothing$ it follows that there exists $y \in I(a, b)$, $y \notin I(a, x)$, i.e. $y \in I(x, b) \neq \Phi$. Consequently $[b, x] \in N(G)$ and the Lemma is proved.

Theorem 3. *If $G = (V, E)$ is an interval graph then $N(G) = \Phi$ or there exists $[a, b] \in E$ such that $[a, b]$ has the property (A).*

Proof. Suppose $N(G) \neq \varnothing$ and let $[a, b]$ be an edge of G such that $[a, b] \in N(G)$ and

$$(1) \qquad |I(a) \cap I(b)| \leq |I(x) \cap I(y)| \quad \text{for all } [x, y] \in N(G).$$

We shall prove that $[a, b]$ has the property (A). If $[a, b]$ has not the property (A) then there exists $u \in I(a) \cap I(b)$ such that $\{u\} \cup I(u) \not\supset I(a) \cap I(b)$ and $I(u) \setminus \{a, b\} \not\subset I(a) \cap I(b)$. Consider $v \in I(a) \cap I(b)$ and $w \in I(u) \setminus \{a, b\}$ such that $v \notin I(u) \cup \{u\}$ and $w \notin I(a) \cap I(b)$. It is clear that we can suppose $w \notin I(a)$, that is $w \in I(u, a)$, $v \in I(a, u)$ which shows that $[a, u] \in N(G)$. Now by using the Lemma 5 one obtains $|I(a) \cap I(u)| < |I(a) \cap I(b)|$ or $[u, b] \in N(G)$ and $|I(u) \cap I(b)| < |I(a) \cap I(b)|$, which contradicts (1). Hence we have to accept that $[a, b]$ has the property (A) and the Theorem is proved.

We note that, by Theorem 3, when $N(G)\neq\emptyset$ and G has no edges with the property (A) then G is not an interval graph. On the other hand, the proofs of the Theorem 3 and Corollary 1 also as the Lemma 5 give some methods for finding an edge having the property (A), when such an edge exists.

Theorem 4. *Let $[a, b]$ be an edge having the property (A) in a graph $G=(V, E)$ and let V' be the set of vertices defined by*

$$V'=\{x\in I(a)\cap I(b):\ I(x)\setminus\{a, b\}\subset I(a)\cap I(b)\}.$$

The graph G is an interval graph if and only if the graph $G'=(V, E')$ where $E'=\{[x, y]\in E:\ x, y\in V'\ \text{or}\ x, y\in V\setminus V'\}$ is an interval graph. (The graph G' is obtained from G by deleting all the edges $[x, y]$ with $x\in V'$ and $y\in V\setminus V'$).

Proof. If G is an interval graph then it is clear that $[V']_G$ and $[V\setminus V']_G$ are interval graphs, and because of every edge of G' is either an edge of $[V']_G$ or an edge of $[V\setminus V']_G$ it follows that G' is an interval graph too.

Now suppose that G' is an interval graph and let $\mathcal{J}$ be a family of open intervals on the real line and $f: V\to\mathcal{J}$ a bijection such that $(\mathcal{J}, f)$ represents G'. If we denote $f(x)=(\alpha(x), \beta(x))\in\mathcal{J}$ then we can suppose $\inf\{\alpha(x):\ x\in V\}>-\infty$, $\sup\{\beta(x):\ x\in V\}<+\infty$, and because of $[a, b]\in N(G')$ we can suppose $\alpha(a)<\alpha(b)<\beta(a)<\beta(b)$. When $V'=\emptyset$ then $G'=G$ hence G is an interval graph. Consider $V'\neq\emptyset$ and let α_0, β_0 be the real numbers defined by

$$\alpha_0=\inf\{\alpha(x):\ x\in V'\},\quad \beta_0=\sup\{\beta(x):\ x\in V'\}.$$

Since $[x, y]\notin E'$ for all $x\in V'$ and all $y\in V\setminus V'$ we can consider the pair $(\mathcal{J}, f)$ such that $f(x)\cap(\alpha_0, \beta_0)=\emptyset$ for all $x\in V\setminus V'$. Now consider the set of open intervals $\mathcal{J}'=\{(\alpha'(x), \beta'(x)):\ x\in V'\}$ defined by

$$\alpha'(x)=\alpha(b)+(\alpha(x)-\alpha_0)(\beta(a)-\alpha(b))/(\beta_0-\alpha_0),$$

$$\beta'(x)=\alpha(b)+(\beta(x)-\alpha_0)(\beta(a)-\alpha(b))/(\beta_0-\alpha_0).$$

If $\mathcal{G}=(\mathcal{J}\setminus f(V'))\cup\mathcal{J}'$ and $g: V\to\mathcal{G}$ is defined by

$$g(x)=\begin{cases} f(x):\ x\in V\setminus V',\\ (\alpha'(x), \beta'(x)):\ x\in V',\end{cases}$$

it is straightforward to prove that $(\mathcal{G}, g)$ represents the graph G and the theorem is proved.

We remark that the edge $[a, b]$ has the property (B) in the graph G' defined in the Theorem 4, hence in order to decide whether a given graph is an interval graph it remains to establish how to do this for a graph G, for which all the edges having the property (A) have also the property (B).

Lemma 6. *Let G be an interval graph and $[a, b]\in N(G)$. If $c\in I(a)\cap I(b)$ then $I(c)\subset I(a)\cup I(b)$ or $I(c)\supset I(a, b)$ or $I(c)\supset I(b, a)$.*

Proof. Suppose $I(c)\not\subset I(a)\cup I(b)$ and $I(c)\not\supset I(a, b)$; hence $I(c, a)\cap I(c, b)\neq\emptyset$, $I(a, b)\cap I(a, c)\neq\emptyset$. By using Lemma 2 for the triangle $[\{a, b, c\}]_G$ it results $I(b, c)\cap I(b, a)=\emptyset$, which shows that $I(b, a)\subset I(c)$ and the Lemma is proved.

Theorem 5. *Let $G=(V, E)$ be an interval graph and $[a, b]\in N(G)$. If $M=I(a)\cap I(b)\neq\emptyset$ then at least one of the following three relations holds:*

$$\text{(2)}\qquad \begin{aligned} &\text{i)}\ \bigcap_{x\in M} I(x)\subset I(a)\cup I(b),\\ &\text{ii)}\ \bigcap_{x\in M} I(x)\supset I(a, b),\\ &\text{iii)}\ \bigcap_{x\in M} I(x)\supset I(b, a).\end{aligned}$$

Proof. By using the previous Lemma it results that for every $x\in M$ we have $I(x)\subset I(a)\cup I(b)$ or $I(x)\supset I(a, b)$ or $I(x)\supset I(b, a)$. If there exists $x\in M$ such that $I(x)\subset I(a)\cup I(b)$ then it is clear that (2i) holds. Now suppose that for every $x\in M$ we have $I(x)\supset I(a, b)$ or $I(x)\supset I(b, a)$ and consider $M_1=\{x\in M:\ I(x)\supset I(a, b)\}$, $M_2=\{x\in M:\ I(x)\supset I(b, a)\}$. If (2ii) and (2iii) are false then there exist $c\in M_1$ and $d\in M_2$ such that $I(c)\not\supset I(b, a)$ and $I(d)\not\supset I(a, b)$. Consider $\mathcal{J}$ a family of open intervals and $f: V\to\mathcal{J}$ such that $(\mathcal{J}, f)$ represents the graph G. If we denote $f(x)=(\alpha(x), \beta(x))\in\mathcal{J}$ then, taking account on $[a, b]\in N(G)$ we can suppose $\alpha(a)<\alpha(b)<\beta(a)<\beta(b)$ and because of $I(c)\supset I(a, b)$, $I(d)\supset I(b, a)$, $I(c)\not\supset I(b, a)$, $I(d)\not\supset I(a, b)$ it follws $\alpha(a)<\beta(c)<\beta(b)$ and $\alpha(a)<\alpha(d)<\beta(b)$. If $\beta(c)\leqslant\alpha(d)$ then for all $x\in I(c)\cap I(d)$ we have $\beta(c)\in f(x)\cap(f(a)\cup f(b))$ and, if $\beta(c)>\alpha(d)$ then for all $x\in I(c)\cap I(d)$ we must have $\emptyset\neq f(x)\cap(\alpha(d), \beta(c))\subset f(a)\cup f(b)$. Therefore, for every $x\in I(c)\cap I(d)$ we must have $f(x)\cap(f(a)\cup f(b))\neq\emptyset$, i.e. $I(c)\cap I(d)\subset I(a)\cup I(b)$ and consequently (2i) holds. The Theorem is proved.

Theorem 6. *Let $G=(V, E)$ be a graph and $[a, b]\in E$ such that $[a, b]$ has the property (B). If $M=I(a)\cap I(b)\neq\emptyset$ and $\bigcap_{x\in M} I(x)\subset I(a)\cup I(b)$ then G is an interval graph if and only if $S(a, b)\cap I(b, a)=\emptyset$ and the graph obtained from G by deleting the edge $[a, b]$ is an interval graph.*

Proof. Let us suppose that G is an interval graph. By using Corollary 3 it results $S(a, b)\cap I(b, a)=\emptyset$. Now let $\mathcal{J}$ be a family of open intervals on the real line and $f: V\to\mathcal{J}$ a bijection such that the pair $(\mathcal{J}, f)$ represents G. If $f(x)=(\alpha(x), \beta(x))\in\mathcal{J}$ then we can suppose $\alpha(a)<\alpha(b)<\beta(a)<\beta(b)$ and $\alpha(x)<\alpha(b)$ for all $x\in M$. Consider the family $\mathcal{G}$ and the function $g: V\to\mathcal{G}$ defined by

$$\mathcal{G}=(\mathcal{J}\setminus\{f(a)\})\cup\{(\alpha(a), \alpha(b))\},$$

$$g(x)=\begin{cases} f(x):\ x\neq a,\\ (\alpha(a), \alpha(b));\ x=a.\end{cases}$$

It is clear that g is a bijection and, because $[a, b]$ has the property (B), it is easy to prove that $g(x)\cap g(y)\neq\emptyset$ if and only if $x=y$ or $[x, y]\in E\setminus\{[a, b]\}$. Hence the graph obtained from G by deleting the edge $[a, b]$ is an interval graph.

Conversely, suppose that $G=(V, E')$, where $E'=E\setminus\{[a, b]\}$ is an interval graph and let $\mathcal{J}$ be a family of open intervals and $f: V\to\mathcal{J}$ a bijection such that $(\mathcal{J}, f)$, represents G'. Since $[a, b]\notin E'$ we have $f(a)\cap f(b)=\emptyset$ and because of $[a, x]$, $[b, x]\in E'$ for all $x\in I(a, G)\cap I(b, G)=I(a, G')\cap I(b, G')$ it results $f(x)\cap f(a)\neq\emptyset$, $f(x)\cap f(b)\neq\emptyset$ for all $x\in I(a, G)\cap I(b, G)$. If $f(x)=(\alpha(x), \beta(x))\in\mathcal{J}$ we can suppose $\beta(a)<\alpha(b)$ and consequently $f(x)\supset(\beta(a), \alpha(b))$ for all $x\in M$. Since $\bigcap_{x\in M} I(x, G')=\bigcap_{x\in M} I(x, G)\subset I(a, G)\cup I(b, G)=I(a, G')\cup I(b, G')\cup$

$\cup \{a, b\}$ it results that there exists no $y \in V$ such that $f(y) \subset (\beta(a), \alpha(b))$. Taking account on $S(a, b, G) \cap I(b, a, G) = \emptyset$ it follows that $f(x) \cap f(y) = \emptyset$ for every $x \in I(a, b, G) = I(a, b, G')$ and every $y \in I(b, a, G) = I(b, a, G')$. Therefore we can suppose the pair $(\mathcal{I}, f)$ such that $\beta(x) < \alpha(y)$ for all $x \in I(a, b, G)$ and all $y \in I(b, a, G)$. Consequently the family $\mathcal{I}$ and the function f can be chosen such that $f(x) \cap (\beta(a), \alpha(b)) = \emptyset$ for all $x \in V \setminus M$. Now consider the family of open intervals $\mathcal{J}$ and $g : V \to \mathcal{J}$ defined by

$$\mathcal{J} = (\mathcal{I} \setminus \{f(a), f(b)\} \cup \{(\alpha(a), \alpha(b)), (\beta(a), \beta(b))\}$$

$$g(x) = \begin{cases} f(x)\ ; & x \notin \{a, b\}, \\ (\alpha(a), \alpha(b))\ ; & x = a, \\ (\beta(a), \beta(b))\ ; & x = b. \end{cases}$$

We shall prove that $(\mathcal{J}, g)$ represents G. Since $g(x) \supset f(x)$ for all $x \in V$ it is clear that $g(x) \cap g(y) \neq \emptyset$ whenever $[x, y] \in E \setminus \{[a, b]\}$ and because of $g(a) \cap g(b) = (\beta(a), \alpha(b)) \neq \emptyset$ one obtains $g(x) \cap g(y) \neq \emptyset$ for all $[x, y] \in E$. Now suppose $x \neq y$ and $g(x) \cap g(y) \neq \emptyset$. If $x, y \in V \setminus \{a, b\}$ then $g(x) = f(x)$, $g(y) = f(y)$ and from $f(x) \cap f(y) \neq \emptyset$ one obtains $[x, y] \in E' \subset E$. When $x = a$, $y \in V \setminus \{a, b\}$ then $g(y) = f(y)$ and $g(x) = (\alpha(a), \alpha(b))$; hence $f(y) \cap (\alpha(a), \alpha(b)) \neq \emptyset$. Now, when $y \in M$ it is clear that $[a, y] \in E' \subset E$ and if $y \notin M$, taking account on $f(y) \cap (\beta(a), \alpha(b)) = \emptyset$ it results $f(y) \cap f(a) \neq \emptyset$ and therefore $[a, y] \in E' \subset E$, too. Similarly, if $g(b) \cap g(y) \neq \emptyset$, $y \in V \setminus \{a, b\}$ it results $[b, y] \in E' \subset E$. The theorem is proved.

For a graph $G(V, E)$ and $[a, b] \in E$ we shall denote by $G(a, b)$ the graph (V', E'), where $V' = V \setminus \{b\}$ and

$$E' = (E \setminus \{[b, x]\ ;\ x \in I(b, G)\}) \cup \{[a, x]\ :\ x \in I(b, a, G)\}.$$

that is $G(a, b)$ is the graph obtained from G by deleting the vertex b and by adding the edges $[a, x]$, $x \in I(b, a, G)$.

Lemma 7. *If $G = (V, E)$ is an interval graph and $[a, b] \in E$ then $G(a, b)$ is an interval graph, too.*

Proof. Suppose G is an interval graph and let $\mathcal{I}$ be a family of intervals and $f : V \to \mathcal{I}$ such that $(\mathcal{I}, f)$ represents the graph G. Consider the family $\mathcal{J}$ and $g : V \setminus \{b\} \to \mathcal{J}$ defined by

$$\mathcal{J} = (\mathcal{I} \setminus \{f(a), f(b)\}) \cup \{f(a) \cup f(b)\}.$$

$$g(x) = \begin{cases} f(x)\ ; & x \neq a, \\ f(a) \cup f(b)\ ; & x = a. \end{cases}$$

Since $[a, b] \in E$ it results that $f(a) \cup f(b)$ is an interval on the real line and it is not difficult to prove that $g(x) \cap g(y) \neq \emptyset$ if and only if $x = y$ or $[x, y]$ is an edge of $G(a, b)$, i.e. $G(a, b)$ is an interval graph. The Lemma is proved.

Theorem 7. *Let $G = (V, E)$ be a graph and $[a, b] \in E$ such that $[a, b]$ has the property (B) in G. If $M = I(a) \cap I(b) = \emptyset$ or $\bigcap_{x \in M} I(x) \supset I(a, b) \cup I(b, a)$ then the graph G is an interval graph if and only if*

(3)

i) *$S(a, b) \cap I(b, a) = \emptyset$.*

ii) *$[\{x \in I(b, a) : [b, x] \in N(G)\}]_G$ and*

$[\{x \in I(a, b) : [a, x] \in N(G)\}]_G$ are cliques.

iii) *$G(a, b)$ is an interval graph.*

Proof. Suppose that G is an interval graph. By using the Lemmas 3,7 and Corollary 3 it results that (3) are fulfilled. Conversely, suppose that the conditions (3) are fulfilled and let $\mathcal{I}$ be a set of intervals and $f : V \setminus \{b\} \to \mathcal{I}$ such that $(\mathcal{I}, f)$ represents $G(a, b)$. Let us denote $f(x) = (\alpha(x), \beta(x)) \in \mathcal{I}$ and $C_1 = I(a, b, G) \subset I(a, G(a, b))$, $C_2 = I(b, a, G) \subset I(a, G(a, b))$. Taking account on the way the graph $G(a, b)$ is defined and on (3i) it follows

$$[C_1]_G = [C_1]_{G(a, b)},\quad [C_2]_G = [C_2]_{G(a, b)},\quad [C_1 \cup C_2]_G = [C_1]_G \cup [C_2]_G.$$

On the other hand, by using (3i), $M = \Phi$ or $\bigcap_{x \in M} I(x, G) \supset I(b, a, G) \cup I(a, b, G)$ it follows $[a, x] \in N(G(a, b))$ if and only if either $x \neq b$ and $[a, x] \in N(G)$ or $x \in I(b, a, G)$ and $[b, x] \in N(G)$. Now, by using (3ii) it follows that we can define $[\bar{C}_1]_G$, $[\bar{C}_2]_G$ the connected components of $[C_1]_G$, $[C_2]_G$ respectively such that $\bar{C}_1 \supset \{x \in I(a, b, G)\ ;\ [a, x] \in N(G)\}$, $\bar{C}_2 \supset \{x \in I(b, a, G) : [b, x] \in N(G)\}$. Since $C_1 \neq \emptyset$, $C_2 \neq \emptyset$ and $[a, x]$ is an edge of $G(a, b)$ if and only if $x \neq b$ and $[a, x] \in E$ or $[b, x] \in E$ it follows that for all $x \in C_1 \cup C_2$ we have $f(x) \cap f(a) \neq \emptyset$, $f(x) \not\supset f(a)$ and more, because of $S(a, b, G) \cap I(b, a, G) = \emptyset$ it results $f(x) \cap f(y) = \emptyset$ for every $x \in \bar{C}_1$ and every $y \in \bar{C}_2$. Hence we can consider the family $\mathcal{I}$ and the bijection f such that

(4)

i) $\beta_0 = \sup \{\beta(x) : x \in \bar{C}_1\} < \inf \{\alpha(y) : y \in \bar{C}_2\} = \alpha_0$

ii) $\alpha(a) \leqslant \beta_0 < \alpha_0 \leqslant \beta(a)$.

When $\bar{C}_1 = \emptyset$ or $\bar{C}_2 = \emptyset$, we shall consider $\beta_0 = \alpha(a)$ and $\alpha_0 = \beta(a)$, respectively, the relations (4ii) being also true. Now we can suppose that $\mathcal{I}$ and f are such that $f(x) \subset (\beta_0, \alpha_0)$ for all $x \in (C_1 \setminus \bar{C}_1) \cup (C_2 \setminus \bar{C}_2)$ and $f(y) \supset (\alpha(a), \beta(a))$ for all $y \in M$. By using again $f(x) \cap f(y) = \emptyset$ for all $x \in C_1$ and all $y \in C_2$ it results that we can suppose $\mathcal{I}$ and f such that

$$\beta_1 = \sup \{\beta(x) : x \in C_1\} < \inf \{\alpha(x) : x \in C_2\} = \alpha_1$$

$$\alpha(a) \leqslant \beta_0 \leqslant \beta_1 < \alpha_1 \leqslant \alpha_0 \leqslant \beta(a).$$

Now consider the family of intervals $\mathcal{J}$ and $g : V \to \mathcal{J}$ defined by

$$\mathcal{J} = (\mathcal{I} \setminus \{f(a)\}) \cup \{(\alpha(a), \alpha_1), (\beta_1, \beta(a)).$$

$$g(x) = \begin{cases} f(x) : & x \notin \{a, b\}, \\ (\alpha(a), \alpha_1)\ ; & x = a, \\ (\beta_1, \beta(a))\ ; & x = b. \end{cases}$$

Taking into account that $[a, b]$ has the property (B) and $M = \emptyset$ or $\bigcap_{x \in M} I(x, G) \supset I(a, b, G) \cup I(b, a, G)$ it is not difficult to prove that $g(x) \cap g(y) \neq \emptyset$ if and only if $x=y$ or $[x, y]\in E$. Hence G is an interval graph and the Theorem is proved.

Theorem 8. *Let $G=(V, E)$ be a graph and $[a, b]$ an edge of G such that $[a, b]$ has the property* (B) *If $M=I(a)\cap I(b)\neq\emptyset$ and $\bigcap_{x\in M} I(x)\not\subset I(a)\cup I(b)$, $\bigcap_{x\in M} I(x)\neq I(a, b)$, $\bigcap_{x\in M} I(x)\supset I(b, a)$ then consider $c\in M$ such that $I(c)\not\supset I(a, b)$. The graph G is an interval graph if and only if*

$$\text{(5)}\qquad \begin{aligned} &\text{i) } S(a, b)\cap I(c, a)=\emptyset \text{ and}\\ &\text{ii) } G(a, b) \text{ is an interval graph.}\end{aligned}$$

Proof. Suppose G is an interval graph. By using Lemma 7 it results that $G(a, b)$ is an interval graph. In order to prove (5i) let us consider the graph $G=(V, E')$, where $E'=E\setminus\{[c, x]\,;\, x\in I(c, G)\cap I(a, b, G)\}$. If $(\mathcal{I}, f)$ represents G and for $x\in V$ we denote $f(x)=(\alpha(x), \beta(x))\in\mathcal{I}$ then, because of $[a, b]$ has the property (B) and $I(c, G)\not\supset I(a, b, G)$, $I(c, G)\supset I(b, a, G)$, we can suppose $\alpha(a)<\alpha(c)\leqslant\alpha(b)<\beta(a)<\beta(b)<\beta(c)$. Consider the family of open intervals $\mathcal{J}$ and $g: V\to\mathcal{J}$ defined by

$$\mathcal{J}=(\mathcal{I}\setminus\{f(c)\})\cup\{(\alpha(b), \beta(c))\}.$$

$$g(x)=\begin{cases} f(x)\,;\ x\neq c,\\ (\alpha(b), \beta(c))\,;\ x=c.\end{cases}$$

It is easy to prove that $(\mathcal{J}, g)$ represents G', hence G' is an interval graph. By using Corollary 3 it results $S(a, c, G')\cap I(c, a, G')=\emptyset$ and taking account on $I(a, c, G')=I(a, b, G)$, $I(c, a, G')=I(c, a, G)$ one obstain $S(a, b, G)\cap I(c, a, G)=\emptyset$.

Conversely, suppose that $G(a, b)$ is an interval graph and $S(a, b, G)\cap I(c, a, G)=\emptyset$. Let $\mathcal{I}$ be a family of open intervals and $f: V\setminus\{b\}\to\mathcal{I}$ such that $(\mathcal{I}, f)$ represents $G(a, b)$. Taking account on the way the graph $G(a, b)$ is defined it follows $f(x)\cap f(a)\neq\emptyset$ for all $x\in(I(a, G)\cup I(b, G))\setminus\{b\}$. If $f(x)=(\alpha(x), \beta(x))\in\mathcal{I}$, $x\in V\setminus\{b\}$, we shall prove that $\mathcal{I}$ and f can be chosen such that $\beta(x)<\alpha(y)$ for all $x\in I(a, b, G)$ and all $y\in I(b, a, G)$.

Since $I(c, G(a, b))=I(c, G)\setminus\{b\}\not\supset I(a, b, G)$, $I(a, G(a, b))=(I(a, G)\setminus\{b\})\cup I(b, a, G)$, $I(c, G)\supset I(b, a, G)$, $I(c, G)\not\subset I(a, G)\cup I(b, G)$ it follows $[a, c]\in N(G(a, b))$; hence we can suppose $\alpha(a)<\alpha(c)<\beta(a)<\beta(c)$. Let us denote $C_1=I(a, b, G)$, $C_2=I(b, a, G)$. If $x\in C_1$, $y\in C_2$ then $y\notin I(x, G(a, b))$ because of $I(c, G(a, b))=I(c, G)\setminus\{b\}$ and $S(a, b, G)\cap I(c, a, G)=\emptyset$; hence $f(x)\cap f(y)=\emptyset$ for all $x\in C_1$ and all $y\in C_2$. When $x\in C_1$, $[a, x]\in N(G)$ then $[a, x]\in N(G(a, b))$, because of $I(x, G)=I(x, G(a, b))$ and $I(a, G(a, b))=C_2\cup I(a, G)\setminus\{b\}$, $I(x, G)\cap C_2=\Phi$. Now, if $x\in C_1$, $[a, x]\in N(G)$ and $\alpha(a)<\alpha(x)<\beta(a)<\beta(x)$ it follows that there exists $y\in I(x, a, G(a, b))\cap I(c, a, G(a, b))=I(x, a, G)\cap I(c, a, G)\cap I(c, b, G)$, that is $S(a, b, G)\cap I(c, a, G)\neq\Phi$, impossible. Hence for every $x\in C_1$, $[a, x]\in N(G)$ we have $\alpha(x)<\alpha(a)<\beta(x)<\beta(a)$. In the same manner as in the previous case one proves that for every $y\in C_2$, $[b, y]\in N(G)$ we have $[a, y]\in N(G(a, b))$ and $\alpha(a)<\alpha(y)<\beta(a)<\beta(y)$. Consequently, if $\{x\in C_1\,;\,[a, x]\in N(G)\}\neq\emptyset$ or $\{y\in C_2\,;\,[b, y]\in N(G)\}\neq\emptyset$, we can consider the connected components $[\bar{C}_1]_G=[\bar{C}_1]_{G(a, b)}$, $[\bar{C}_2]_G=[\bar{C}_2]_{G(a, b)}$ of $[C_1]_G=[C_1]_{G(a, b)}$ and $[C_2]_G=[C_2]_{G(a, b)}$, respectively, such that $C_1\supset\bar{C}_1\supset\{x\in C_1\,;\,[a, x]\in N(G)\}$, $C_2\supset\bar{C}_2\supset\{y\in C_2\,;\,[b, y]\in N(G)\}$. When $\{x\in C_1\,;\,[a, x]\in N(G)\}=\emptyset$ or $\{y\in C_2\,;\,[b, y]\in N(G)\}=\emptyset$ then we shall consider $\bar{C}_1=\emptyset$ and $\bar{C}_2=\emptyset$, respectively.

Now, taking account on $S(a, b, G)\cap I(b, a, G)\subset S(a, b, G)\cap I(c, a, G)=\emptyset$ it follows that we can consider $(\mathcal{I}, f)$ such that

$$\text{(6)}\qquad \alpha(a)\leqslant\alpha_0=\sup\{\beta(x)\,;\, x\in\bar{C}_1\}<\inf\{\alpha(y)\,;\, y\in\bar{C}_2\}=\beta_0\leqslant\beta(a).$$

If $\bar{C}_1=\Phi$ or $\bar{C}_2=\emptyset$ then we shall consider $\alpha_0=\alpha(a)$ and $\beta_0=\beta(a)$, respectively. By using again $S(a, b, G)\cap I(c, a, G)=\emptyset$, $I(a, G(a, b))=C_2\cup(I(a, G)\setminus\{b\})$ $f(x)\cap f(y)=\emptyset$ for every $x\in C_1$ and every $y\in C_2$ it is not difficult to prove that $(\mathcal{I}, f)$ can be chosen such that

$$\text{(7)}\qquad \alpha_0\leqslant\alpha_1=\sup\{\beta(x)\,;\, x\in C_1\setminus\bar{C}_1\}<\inf\{\alpha(y)\,;\, y\in C_2\setminus\bar{C}_2\}=\beta_1\leqslant\beta_0.$$

On the other hand, by using $\bigcap_{x\in M} I(x, G)=\bigcap_{x\in M} I(x, G(a, b))\cup\{b\}\not\subset I(a, G)\cup I(b, G)=I(a, G(a, b))\cup\{a, b\}$, it is easy to prove that for all $x\in M$, $I(x, G)\not\supset I(a, b, G)$ we have $\alpha(a)<\alpha(x)<\beta(a)<\beta(x)$, and taking account on $I(x, G)\supset I(b, a, G)$ for all $x\in M$ it follows that we can consider $(\mathcal{I}, f)$ such that

$$\text{(8)}\qquad \alpha(x)<\beta_1<\beta(x) \text{ for all } x\in M.$$

Consequently we can consider a family $\mathcal{I}$ of open intervals and $f: V\setminus\{b\}\to\mathcal{I}$ such that $(\mathcal{I}, f)$ represents $G(a, b)$ and the relations (6–8) are fulfilled. Now consider the family $\mathcal{J}$ of open intervals and $g: V\to\mathcal{J}$ defined by $\mathcal{J}=(\mathcal{I}\setminus\{f(a)\})\cup\{(\alpha(a), \beta_1), (\alpha_1, \beta(a))\}$.

$$g(x)=\begin{cases} f(x)\,;\ x\notin\{a, b\},\\ (\alpha(a), \beta_1)\,;\ x=a\\ (\alpha_1, \beta(a))\,;\ x=b\end{cases}$$

Since $(\mathcal{I}, f)$ satisfies (6–8) and represents $G(a, b)$, taking account on the hypotheses of the Theorem, it is straightforward to prove that $(\mathcal{J}, g)$ is a representation of G and the Theorem is proved.

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