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ON SOME PROPERTIES OF SUBMODULAR FUNCTIONS

BY

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1. **Introduction.** Let E be a finite set. A real valued set function f: $f:\mathcal{P}(E)\to R$ is called *submodular* on E if it satisfies

$$f(A\cup B)+f(A\cap B)\leqslant f(A)+f(B)\qquad \forall A,\ B\subseteq E. \tag{1}$$

If in (1) the converse inequality is imposed, then f is called *supermodular* and a function which is both submodular and supermodular is a *modular* function. If f is a modular function on E then, obviously f satisfies

$$f(A)=f(\varnothing)+\sum_{a\in A} f(a)\quad \forall A\subseteq E, \tag{2}$$

(where, $f(a)$ is used instead of $f(\{a\})$; such simplified notations, which can be easily recognized, will be used throughout this paper). Conversely, if $f(\varnothing)=0$ and $f(a)$ $(a\in E)$ are given, then (2) defines the modular extension of the function $f:E\to R$.

Submodular functions play an important role in combinatorial optimization, matroid and graph theory [4, 5, 6], [8].

The verification of the submodularity of a specified set function can be done using the following remark:

A function $f:\mathcal{P}(E)\to R$ is submodular if and only if for every $a\in E$, the function $f_a:\ \mathcal{P}(E\setminus a)\to R$ given by $f_a(A)=f(A\cup a)-f(A)$ is monotone increasing.

However, the recognition of the submodularity in a combinatorial context, is perhaps a more difficult task. An example of a submodular function is given in section 2. In sections 3 and 4 of this paper we give an extension of R a d o's principle from transversal theory to submodular functions obtaining many results fundamental to combinatorial optimization.

2. **An example of a submodular function.** Let $f:\mathcal{P}(E)\to R$ be a fixed submodular function and $w:E\to R$ an arbitrary weight function on E. Let $E=\{e_1,\ e_2,\ \dots,\ e_n\}$ be ordered such that $i<j$ implies $w(e_i)\geqslant w(e_j)$. Define $W_f:\mathcal{P}(E)\to R$ by $W_f(\varnothing)=0$ and for every $A\subseteq E$ $W_f(A)=W_f(A\setminus e_t)+(f(A)-f(A\setminus e_t))\,w(e_t)$ where $t=\max\{j:\ e_j\in A\}$.

Theorem 1. *If f is submodular then, for every $w:E\to R_+$, W_f is submodular.*

Proof. We shall prove by induction on $n=|E|$ that

$$W_f(A\cup x)-W_f(A)\geq W_f(A\cup x\cup y)-W_f(A\cup y)\quad \forall A\subseteq E,\ x,\ y\in E\setminus A.$$

In the inductive step, we can suppose $w(x)\leqslant w(y)$, and let us define

$$w' : E \to R_+ \text{ by } w'(e) = w(e) - w(e_n) \quad \forall e \in E.$$

Now, the order induced by w' on E is the same as that induced by w and a simple computation gives

$$W'_f(A) = W_f(A) - (f(A) - f(\emptyset))\, w(e_n) \quad \forall A \subseteq E.$$

Using the inductive hypothesis for W'_f on $E \setminus e_n$, the obvious equality $W'_f(A \setminus e_n) = W'_f(A)$, and the above equation, we obtain

$$W_f(A \cup x) - W_f(A) \geq W'_f(A \cup x \cup y) - W'_f(A \cup y) + (f(A \cup x) - f(A) - f(A \cup x \cup y) +$$
$$+ f(A \cup y))\, w(e_n) \geq W_f(A \cup x \cup y) - W_f(A \cup y),$$

where the last inequality is a consequence of the submodularity of f and nonnegativity of $w(e_n)$.

An interesting consequence of this result will be made evident in what follows:

Let $IS = (E, \mathcal{F})$ an independence system, $w : E \to R_+$ and let $w(A) = 0 + \sum_{a \in A} w(a)$ the modular extension of w to $P(E)$. Many combinatorial optimization problems are concerned with the determination of the function $W : \mathcal{P}(E) \to R$ given by

$$W(A) = \max \{w(F) : F \in \mathcal{F},\ F \subseteq A\} \quad \forall A \subseteq E.$$

Theorem 2. *W is submodular for any $w \in R^E$ if and only if IS is a matroid.*

Proof. The necessity of the theorem is straightforward, considering $w \in \{0, 1\}^E$ and the rank's caracterization of matroids.

To prove the sufficiency, we note that $W(A) = W_r(A)$, as a byproduct of the greedy algorithm, where r is the rank function of the matroid. Using theorem 1 we deduce the submodularity of W, since r is a submodular function.

We note that the proof of sufficiency is given in [3] and is more involved.

3. **An extension of Rado's principle.** The simplest proof of H a l l 's theorem, which applies for independent transversals also, is that of R a d o [7], which constructs a transversal by iterating a dichotomycal deleting of the elements from the members of a finite family of subsets of a finite set, family which fulfills a Rado-Hall condition. In this section we shall extend this method to submodular functions.

Theorem 3. *Let $f, g : \mathcal{P}(E) \to Z_+$ be two submodular functions on E, such that $f(\emptyset) = g(\emptyset) = 0$. If*

$$f(A) + g(A) \geq |A| \quad \forall A \subseteq E \tag{3}$$

then there exists a partition (E_1, E_2) of E such that

$$f(A) \geq |A| \quad \forall A \subseteq E_1 \tag{4}$$

$$g(A) \geq |A| \quad \forall A \subseteq E_2. \tag{5}$$

Proof. We shall prove firstly that if f, g satisfy the hypotheses of the theorem and e is an arbitary element of E, then

$$f(A \setminus e) + g(A) \geq |A| \quad \forall A \subseteq E, \tag{6}$$

or

$$f(A) + g(A \setminus e) \geq |A| \quad \forall A \subseteq E. \tag{7}$$

Suppose this claim is false. Let $A_1 \cup e$, $A_2 \cup e$ ($A_1, A_2 \subseteq E \setminus e$) violating respectively (6) and (7). We have $f(A_1) + g(A_1 \cup e) \leq |A_1|$ and $f(A_2 \cup e) + g(A_2) \leq \leq |A_2|$ and using the submodularity of f and g, together with condition (3) we obtain

$$|A_1| + |A_2| \geq f(A_1 \cup A_2 \cup e) + f(A_1 \cap A_2) + g(A_1 \cup A_2 \cup e) + g(A_1 \cap A_2) \geq$$
$$\geq |A_1 \cup A_2| + 1 + |A_1 \cap A_2| = |A_1| + |A_2| + 1,$$

a contradiction which proves our claim.

Now, let us define for a function $h : \mathcal{P}(E) \to R$ and $X \subseteq E$ the function $h^X : \mathcal{P}(E) \to R$ given by $h^X(A) = h(A \setminus X)$, $A \subseteq E$. Clearly, if h is submodular then h^X is submodular. Furthermore, $h^X(\emptyset) = h(\emptyset)$. With this remark, our claim can be stated: if f anf g satisfy the hypothesis of the theorem then either $f^{\{e\}}$ and g or f and $g^{\{e\}}$ satisfy the hypothesis, too.

In order to prove the theorem, let us consider the following algorithm:

1. $X := \emptyset$; $Y := \emptyset$;
2. *while* $X \cup Y \neq E$ *do begin*
   let $e \in E \setminus X \cup Y$;
   if $f^{X \cup e}$, g *satisfy* (3) *then* $X := X \cup e$ *else* $Y := Y \cup e$;
   end;
3. $E_1 := Y$; $E_2 := X$.

After the execution of the algorithm, we obtain a partition E_1, E_2 of E such that $f^{E_2}(A) + g^{E_1}(A) \geq |A|$ $\forall A \subseteq E$.

This implies for $A \subseteq E_1$: $\quad f(A) + g(\emptyset) \geq |A|$ and for $A \subseteq E_2$ $f(\emptyset) + g(A) \geq |A|$, which proves the theorem.

Theorem 4. *If f and g are submodular functions on E, nondecreasing, integer valued, with $f(\emptyset) = g(\emptyset) = 0$, then*

$$f(A) + g(A) \geq |A| \quad \forall A \subseteq E$$

if and only if there exists a partition E_1, E_2 of E such that

$$f(A) \geq |A| \quad \forall A \subseteq E_1 \text{ and } g(A) \geq |A| \quad \forall A \subseteq E_2.$$

Proof. The sufficiency is an obvious consequence of the monotony, and the necessity is given by the theorem 3.

We next present two applications of this theorem.

1. **Matroid intersection theorem.** We shall derive the famous „matroid intersection theorem" of E d m o n d s (see [8]).

Let $M_1=(E, r_1)$, $M_2=(E, r_2)$ two matroids on E, where r_1, r_2 are their rank functions. Let $k\in Z_+$, $k\leq r_1(E)$, $r_2(E)$. Let us consider $f(A)=r_1(A)$ and $g(A)=\min(r_2(E\setminus A), k)+|A|-k$. If

$$r_1(A)+r_2(E\setminus A)\geq k \quad \forall A\subseteq E. \tag{8}$$

then all the hypotheses of the theorem 4 are satisfied (as can be easily seen), and therefore (8) is equivalent with the existence of the partition E_1, E_2 of E such that $r_1(A)\geq |A|$ $\forall A\subseteq E_1$ and $g(A)\geq |A|$ $\forall A\subseteq E_2$.

Hence we have obtained that E_1 is independent in M_1, and since $g(E_2)=r_2(E_1)+|E_2|-k$, we have $r_2(E_1)\geq k$, that is, there exists a common k-independent set of M_1 and M_2, contained in E.

II. A submodular generalization of „defect" versions of Rado-Hall theorems

Let $\mathcal{A}: E\to\mathcal{P}(S)$ be a finite family (indexed by E) of subsets of the finite set S, λ and μ nondecreasing integer valued submodular functions on S, respectively E.

A function $r: E\to S$ such that $r(e)\in\mathcal{A}(e)$ is called a *representation function* for $\mathcal{A}$. By theorem 4, we have $\lambda(\mathcal{A}(J))+\mu(J)\geq |J|$ $\forall J\subseteq E$ if and only if there exists a partition E_1, E_2 of E such that

$$\lambda(\mathcal{A}(J))\geq |J| \quad \forall J\subseteq E_1 \text{ and } \mu(J)\geq |J| \quad \forall J\subseteq E_2.$$

Using a theorem of W e l s h [8], the first of the above relations is equivalent to the existence of representation function $r_1: E\to S$ such that $\lambda(r(J))\geq$ $\geq |J|$ $\forall J\subseteq E_1$. Hence we have obtained the following generalization of Rado-Hall's theorems:

Theorem 5. *Let $A=(A_e : e\in E)$ be a finite family of subsets of a finite set S, and let λ and μ be nondecreasing integer-valued submodular functions on S and E respectively. There exist a partition E_1, E_2 of E and a representation function $r: E_1\to S$ for A such that*

$$\lambda(r(J))\geq |J| \quad \forall J\subseteq E_1 \text{ and } \mu(J)\geq |J| \quad \forall J\subseteq E_2$$

if and only if

$$\lambda(A(J))+\mu(J)\geq |J| \quad \forall J\subseteq E.$$

Particular choices for λ and μ in theorem give all the known transversality theorems, e.g. B r u a l d i's „symmetrized version of Rado's theorem" [1].

4. A proof of the „sandwich" theorem. We shall extend the method used in setion 2 in order to give a simple proof of a theorem of F r a n k [4].

Let $f: P(E)\to R$ be a submodular function on E and, for every $w: E\to Z_+$, let us consider $f^w: P(E)\to R$ given by

$$f^w(A)=\min\{f(A),\ f(A\setminus \sup(w))+\sum_{a\in A\cap\sup(w)}(f(a)-w(a))\} \quad \forall A\subseteq E,$$

where, $\sup(w)=\{x\in E: w(x)>0\}$. It is straightforward to verify that if f is integer-valued, submodular and with $f(\emptyset)=0$, then f^w has all these properties too. In fact, it is sufficient to consider weight functions $w_a: E\to\{0, 1\}$ given by $w_a(x)=0$ if $x\neq a$ and $w_a(x)=1$ if $x=a$ (where a is a fixed element

of E) and to repeat the proof with appropriate choosings of the element a. We have the following analogue of theorem 3:

Theorem 6. *If f and g are integer valued submodular functions on E such that $f(A)+g(A)\geq 0$ $\forall A\subseteq E$, then there exist w_1, $w_2: E\to Z$ such that*

$$f^{w_1}(A)+g^{w_2}(A)=0 \quad \forall A\subseteq E$$

Proof. We may suppose without loss of generality that $f(\emptyset)=g(\emptyset)=0$. If $f(A)+g(A)=0$ $\forall A\subseteq E$, then $w_1(a)=w_2(a)=0$ $\forall a\in E$. If there exists $A\subseteq E$ such that $f(A)+g(A)>0$, then there is $a\in E$ such that $f(a)+g(a)>0$ (otherwise, using submodularity, we derive $f(A)+g(A)=0$). Furthermore, we have

$$f^{w_a}(A)+g(A)\geq 0 \quad \forall A\subseteq E \tag{9}$$

or

$$f(A)+g^{w_a}(A)\geq 0 \quad A\subseteq E. \tag{10}$$

If not, there exist A_1, $A_2\in E\setminus a$ such that

$$f(A_1)+f(a)-1+g(A_2\cup a)<0 \text{ and}$$
$$f(A_2\cup a)+g(A_2)+g(a)-1<0.$$

By adding these two inequalities and using submodularity and the hypothesis of the theorem we obtain:

$0>f(a)+g(a)+f(A_1\cup A_2\cup a)+g(A_1\cup A_2\cup a)+f(A_1\cap A_2)+g(A_1\cap A_2)\geq f(a)+g(a)>0$
a contradiction.

Repeating the above argument to the pair f^{w_a} and g, or f and g^{w_a}, in a finite number of steps, we obtain the conclusion of the theorem.

Let us note that, since f^{w_1} and g^{w_2} are submodular functions, and $f^{w_1}=-g^{w_2}$, we obtain that f^{w_1} and g^{w_2} are modular functions and furthermore, by construction $f\geq f^{w_1}$ and $-g^{w_2}\geq -g$. Hence we can derive from theorem 6 the following theorem of F r a n k [4].

If f, g are integer valued functions on E, f submodular, g supermodular such that $f\geq g$, then there exists a modular integer valued function h such that $f\geq h\geq g$.

Note. The method described in sections 2 and 3 can be extended, in a nontrivial sense, to three (and more, as was proved by J. Moscovici) submodular functions [2]. An open problem is how to apply these results to the problem of intersection of three matroids.

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SYSTOLIC ARRAYS FOR SOLVING NONLINEAR EQUATIONS BY A NONITERATIVE METHOD

BY

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1. **Introduction.** Ioakimidis presents in [7] a new method for the solution of a single real nonlinear equation $f(x)=0$ with a simple root in the interval (a, b). The noniterative approximate formula has very good performances and uses just a simple knowledge of the interval where the root lies and no approximate knowledge of its value.

These qualities and the regular form of the approximate formula make it suitable for a systolic implementation.

This paper presents a systolic implementation of this method. In Section 2 we present an outline of the method and specify the real functions whose roots finding will be systolized. In Section 3 we describe the components of the systolic network, and analyse their performances. The case of algebraic equations is specified in Section 4, where we give a new solution to the systolization of polynomial roots finding approchead in [1] by using the Lin-Bairstow method. Some remarks are given in the last section.

2. **Outline of the method and functions specification.** Our notation is taken from [7]. If c is the simple root of the nonlinear equation $f(x)=0$ in the interval (a, b), then [7] states that

$$c \approx S1/S2 \tag{1}$$

with

$$S1 = SI1 + \Sigma_{i=1}^{n} T1_i, \qquad S2 = SI2 + \Sigma_{i=1}^{n} T2_i, \tag{2.1}$$

$$SI1 = b/f(b) - a/f(a), \quad SI2 = 1/f(b) - 1/f(a), \tag{2.2}$$

$$T1_i = A_{in}(x_{in} f'(x_{in})/f^2(x_{in}) - 1/f(x_{in})), \tag{3.1}$$

$$T2_i = A_{in} f'(x_{in})/f^2(x_{in}), \tag{3.2}$$

where

$$x_{in} = (b + a + (b - a)\, t_{in})/2, \tag{4}$$

$$A_{in} = (b - a)\, w_{in}/2, \tag{5}$$

$i=1, \ldots, n$. The nodes t_{in} and the weights w_{in} of the Gauss quadrature rule can be found in [5, p. 608]. The convergence of this formula is stated by Theorem 1 in [7].