7. Ioakimidis N. I. — *Application of the Gauss quadrature rule to the numerical solution of nonlinear equations*, Intern. J. Computer Math., vol. 18, 1986, 311—322.
8. Leiserson C. E., Rose F. M., Saxe, J. B. — *Optimizing synchronous circuitry by retiming*, Proc. of the Third Caltech Conference on VLSI, Randal Bryant, ed., Springer-Verlag, Berlin Heidelberg, 1983, 87—116.
9. Leiserson, C.E. — *Systolic and semisystolic design* , Proc. of Computer Design/VLSI in Computers, ICCD' 83, 1983.
10. Megson G. M., Evans D. J. — *Soft systolic pipelined matrix algorithms*, Parallel Computing 85, M. Feilmeier, G. Joubert, U. Schendel, eds. , Elsevier Science Publishers B. V. (North-Holland), 1986, 171—180.
11. Pam, M. S., Mostow J. — *A transformational model of VLSI systolic design*, Proc. IFIP 6th Intern. Symp. Comput. Hardware Descriptive Lang. Appl., Carnegie-Mellon Univ. Pittsburgh, May 1983.

Received 25.III.1987

Computer Centre
Polytechnic Institute
6600 — Iași, România

ANALELE ȘTIINȚIFICE ALE UNIVERSITĂȚII „AL. I. CUZA" DIN IAȘI
Tomul XXXIII, s. I a. Matematică, 1987, fasc. 4

ON THE DEFINITION OF M-FLOWCHARTS

BY

V. E. CĂZĂNESCU and Ș. GRAMA

This paper is a natural continuation of the following Căzănescu and Ungureanu result : the M-flowcharts over an algebraic theory with iterate T form a T-module with iterate freely generated by M.

The definition of M-flowcharts uses an artificial linear order on the set of internal vertices. We remove this deficency by a simple factorization. In this way we obtain a new free algebraic structure which differs from the previous one by a new axiom.

1. **Introduction.** Let us begin with some notation. Let T be an S-sorted algebraic theory. Let M be a monoid and let $r : M \to S^*$ be a monoid morphism. The letter m will denote an element of M. If $f \in N(n, p)$ is a morphism in the initial (one-sorted) algebraic theory and $m_1, m_2, \ldots, m_p$ are p elements of M then we denote by

$r(f : m_1, m_2 \ldots, m_p) \in T(r(m_{f(1)} m_{f(2)} \ldots m_{f(n)}), r(m_1 m_2 \ldots m_p))$ the unique morphism of T such that for each $i \in [n]$

$$(O_{r(m_{f(1)} \ldots m_{f(i-1)})} + 1_{r(m_{f(i)})} + O_{r(m_{f(i+1)} \ldots m_{f(n)})}) r(f : m_1, \ldots, m_p) =$$
$$= O_{r(m_1 m_2 \ldots m_{f(i)-1})} + 1_{r(m_{f(i)})} + O_{r(m_{f(i)+1} \ldots m_p)}.$$

Notice that

$$r(f : m_1, m_2, \ldots, m_p) = 1_{S^*}(f : r(m_1), r(m_2), \ldots, r(m_p)),$$

where 1_{S^*} is the identity morphism of S^*.

The following equalities are easy to prove :

$$(1.1) \qquad r(fg : m_1, m_2, \ldots, m_q) = r(f : m_{g(1)}, m_{g(2)}, \ldots, m_{g(p)})\, r(g ; m_1, \ldots, m_q)$$

for every $f \in N(n, p)$ and $g \in N(p, q)$.

$$(1.2) \qquad r(<f, g> : m_1, m_2, \ldots, m_p) = <r(f ; m_1, m_2, \ldots, m_p), r(g ; m_1, m_2, \ldots, m_p)>$$

for every $f \in N(n, p)$ and $g \in N(q, p)$.

$$(1.3) \qquad r(f + g : m_1, m_2, \ldots, m_p, m'_1, m'_2, \ldots, m'_q) = r(f ; m_1, \ldots, m_p) + r(g ; m'_1, \ldots, m'_q)$$

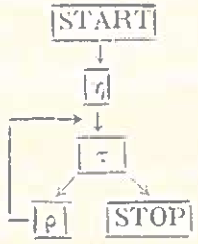
for every $f \in N(n, p)$ and $g \in N(n', q)$.

$$(1.4) \qquad r(1_n ; m_1, m_2, \ldots, m_n) = 1_{r(m_1 m_2 \ldots m_n)}.$$

If $f \in N(n, n)$ is an isomorphism then:
$r(f; m_1, m_2, \ldots, m_n)$ is an isomorphism too and

$$(1.5) \qquad r(f; m_1, m_2, \ldots, m_n)^{-1} = r(f^{-1}; m_{f(1)}, m_{f(2)}, \ldots, m_{f(n)}).$$

If $f \in N(2,2)$, $f(1)=2$ and $f(2)=1$ then $r(f; m, m') = S^{r(m)}_{r(m')}$.
Let as regard again the following example



The set of statements $\Sigma = \{\eta, \tau, \rho\}$ was naturally endowed with the ranking function $r' : \Sigma \to \omega$ defined by $r'(\eta) = r'(\rho) = 1$ and $r'(\tau) = 2$.

Using a linear order on the set of internal vertices reflected by the order of statements in the word $m = \eta\tau\rho \in \Sigma^*$, the above flowchart was represented by the triple (i, t, m) where the function $i : [1] \to [3+1]$ is defined by $i(1) = 1$ and the function $t : [1+2+1] \to [3+1]$ is defined by the table

j	1	2	3	4
$t(j)$	2	3	4	2

Now let us change the linear order on the set of internal vertices. Let us put for example $m' = \rho\eta\tau$. The corresponding input function $i' : [1] \to [3+1]$ is defined by $i'(1) = 2$. The corresponding function $t' : [1+1+2] \to [3+1]$ is defined by the next table

j	1	2	3	4
$t'(j)$	3	3	1	4

Hence we have in $Fl_{\Sigma,N}(1,1)$ six elements representing the same flowchart. The triples (i, t, m) and (i', t', m') are two of them. We have to do something to obtain a bijection between flowcharts and their abstract representations.

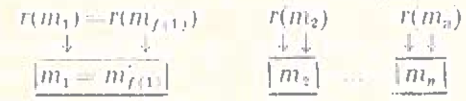
Coming back to our example, it is natural to introduce the unique isomorphism between the two above linear orders. This isomorphism $f : [3] \to [3]$ is defined by

j	1	2	3
$f(j)$	2	3	1

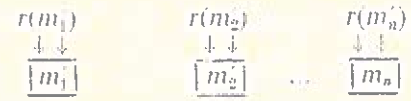
Let us remark that the i-th letter of m is equal to the $f(i)$-th letter of m'. It is obvious that $i(f + 1_1) = i'$. Another equality for t and t' may be written, but we shall do it only in the general case where the understanding is easier.

Let M be a monoid and let $r : M \to S^*$ and $r' : M \to S^*$ be two monoid morphisms.

Let (i, t, m) and (i', t', m') be in $Fl_{M,T}(a, b)$. Suppose $m = m_1 m_2 \ldots m_n$, $m' = m'_1 m'_2 \ldots m'_n$ and $f \in N(n, n)$ is an isomorphism such that $m_i = m'_{f(i)}$ for every $i \in [n]$. Now think of (i, t, m) and (i', t', m') as two flowcharts with n internal vertices which differ only by the order of their internal vertices and try to write the conditions to be imposed on i and i' and on t and t'. Let us think of f aw as an isomorphism between the orders of the internal vertices. If we regard the imputs of the statements in the first flowchart



and in the second flowchart, respectively



we see that $r(f : m'_1, m'_2, \ldots, m'_n)$ is the isomorphism between the inputs of the statements induced by f, therefore i and i' must satisfy

$$i(r(f : m'_1, m'_2, \ldots, m'_n) + 1_c) = i'.$$

If we regard the outputs of the statements, we see that $r'(f; m'_1, m'_2, \ldots, m'_n)$ in the isomorphism between the outputs of the statements induced by f, therefore t and t' must satisfy

$$t(r(f; m'_1, m'_2, \ldots, m'_n) + 1_b) = r'(f; m'_1, m'_2, \ldots, m'_n)\, t'.$$

This is our motivation for definition 2.4 below

2. **Quotient T-modules.** Let T be an S-sorted algebraic theory with iterate.

2.1. **Definition.** *Let Q be a T-module with iterate. We say that $\sim$ is a congruence on Q if for every $a, b \in S^*$ an equivalence relation on $Q(a, b)$ is given such that:*
a) $\alpha \sim \gamma$ *in* $Q(a, b)$ *and* $\beta \sim \delta$ *in* $Q(b, c)$ *imply* $\alpha\beta \sim \gamma\delta$ *in* $Q(a, c)$.
b) $\alpha \sim \gamma$ *in* $Q(a, c)$ *and* $\beta \sim \delta$ *in* $Q(b, c)$ *imply* $\langle\alpha, \beta\rangle \sim \langle\gamma, \delta\rangle$ *in* $Q(ab, c)$,
c) $\alpha \sim \beta$ *in* $Q(a, ab)$ *implies* $\alpha^+ \sim \beta^+$ *in* $Q(a, b)$.

Let us denote by $C(\alpha)$ the equivalence class of $\alpha \in Q(a, b)$.

2.2. **Definition.** *Let $H : T \to Q$ be a T-module with iterate and let $\sim$ be a congruence on Q. The quotient T-module with iterate $H/\sim : T \to Q/\sim$ of Q by $\sim$ is defined by*

$Q/\sim(a, b) = Q(a, b)/\sim$	*for all* $a, b \in S^*$,
composition $C(\alpha)\,C(\beta) = C(\alpha\beta)$	*for all* $\alpha \in Q(a, b)$ *and* $\beta \in Q(b, c)$,
tupling $\langle C(\alpha), C(\beta)\rangle = C(\langle\alpha, \beta\rangle)$	*for all* $\alpha \in Q(a, c)$ *and* $\beta \in Q(b, c)$,
iterate $C(\alpha)^+ = C(\alpha^+)$	*for all* $\alpha \in Q(a, ab)$,
$H/\sim(f) = C(H(f))$	*for all* $f \in T(a, b)$.

The correctness of the composition, tupling and iterate operations in the above definition follows from the conditions a), b) and c) of definition 2.1.

Obviously $C : Q \to Q/\sim$ is a T-module with iterate morphism. It is easy to prove the following property :

2.3. **Proposition.** *If $\sim$ is a congruence on the T-module with iterate Q and if $F : Q \to Q'$ is a morphism of T-modules with iterate such that*

$$\alpha \sim \beta \text{ implies } F(\alpha) = F(\beta).$$

then there exists a unique T-module with iterate morphism $G : Q/\sim \to Q'$ such that $CG = F$.

2.4. **Definition.** *Let (i, t, m) and (i', t', m') be two elements of $Fl_{M, T}(a, b)$. We say that ((i, t, m) is equivalent to (i', t', m')) and we write $(i, t, m) \sim \sim (i', t', m')$ if there exists a positive integer n, an isomorphism $f \in X(n, n)$ and $m_1, m_2, \ldots, m_n, m'_1, m'_2, \ldots, m'_n \in M$ such that*

$$m = m_1 m_2 \ldots m_n, \quad m' = m'_1 m'_2 \ldots m'_n, \quad m_j = m'_{f(j)} \text{ for every } j \in [n].$$

$$i(r(f ; m'_1, m'_2, \ldots, m'_n) + 1_a) = i' \text{ and}$$

$$t(r(f ; m'_1, m'_2, \ldots, m'_n) + 1_b) = r'(f ; m_1, m_2, \ldots, m_n)\, t'.$$

In this definition we may replace every element m_k or m'_k by a new product. For example if we replace m_k by $m^k_1 m^k_2 \ldots m^k_j = m_k$ then we replace $m'_{f(k)}$ by $m^k_1 m^k_2 \ldots m^k_j$ and f by the isomorphism $g \in X(n-1+j, n-1+j)$ defined by

$$g(u) = \begin{cases} f(u) & \text{if} \quad u < k \text{ and } f(u) < f(k) \\ f(u) + j - 1 & \text{if} \quad u < k \text{ and } f(u) > f(k) \\ f(k) + u - k & \text{if } k \le u < k + j \\ f(u - j + 1) & \text{if } k + j \le u \quad \text{and } f(u - j + 1) < f(k) \\ f(u - j + 1) + j - 1 & \text{if } k + j \le u \quad \text{and } f(u - j + 1) > f(k). \end{cases}$$

It follows that

$$r(f ; m'_1, m'_2, \ldots, m'_n) = r(g ; m'_1, \ldots, m'_{f(k)-1}, m^k_1, m^k_2, \ldots, m^k_j, m'_{f(k)+1}, \ldots, m'_n)$$

therefore all the conditions of definition 2.4 are again fulfilled.

We suppose in the sequel that the monoid M is equidivisible [2]. The following property is equivalent to the definition of equidivisibility : the reader may take it as a definition even if the original definition is more elegant. If

$$m'_1 m'_2 \ldots m'_n = m''_1 m''_2 \ldots m''_p,$$

then there exists $m_1, m_2, \ldots, m_q$, $0 = i_0 < i_1 < \ldots < i_{n-1} < i_n = q$ and $0 = j_0 < j_1 < \ldots < j_{p-1} < j_p = q$ such that

$$m'_k = m_{i_{k-1}+1} m_{i_{k-1}+2} \ldots m_{i_k} \quad \text{for every } k \in [n] \text{ and}$$

$$m''_k = m_{j_{k-1}+1} m_{j_{k-1}+2} \ldots m_{j_k} \quad \text{for every } k \in [p].$$

2.5. **Proposition.** *If the monoid M is equidivisible then the relation $\sim$ from definition 2.4 is a congruence on $Fl_{M, T}$.*

Proof. To show that $\sim$ is an equivalence relation on $Fl_{M, T}(a, b)$ we prove only transitivity.

Let us suppose $(i, t, m) \sim (i', t', m')$ with the same notation as in definition 2.4 and $(i', t', m') \sim (i'', t'', m'')$. From the above remark and equidivisibility we may accept without loss of generality that we have only one decomposition of m' as a product, i.e. there exists an isomorphism $g \in X(n, n)$ and the equality $m'' = m''_1 m''_2 \ldots m''_n$ such that

$$m'_i = m''_{g(i)} \text{ for every } i \in [n],$$

$$i'(r(g ; m''_1, m''_2, \ldots, m''_n) + 1_a) = i'' \text{ and}$$

$$t'(r(g ; m''_1, m''_2, \ldots, m''_n) + 1_b) = r'(g ; m'_1, m'_2, \ldots, m'_n)\, t''$$

Then, using (1.1) we deduce

$$i(r(fg ; m''_1, m''_2, \ldots, m''_n) + 1_a) = i(r(f ; m'_1, m'_2, \ldots, m'_n) +$$

$$+ 1_a)(r(g ; m''_1, m''_2, \ldots, m''_n + 1_a) = i'(r(g ; m''_1, m''_2, \ldots, m''_n) + 1_a) = i'' \text{ and}$$

$$t(r(fg ; m''_1, m''_2, \ldots, m''_n) + 1_b) = t(r(f ; m'_1, m'_2, \ldots, m'_n) +$$

$$+ 1_b)(r(g ; m''_1, m''_2, \ldots, m''_n) + 1_b) = r'(f ; m_1, m_2, \ldots, m_n)\, t'(r(g ; m''_1, m''_2, \ldots, m''_n) +$$

$$+ 1_b) = r'(f ; m_1, m_2, \ldots, m_n) r'(g ; m'_1, m'_2, \ldots, m'_n)\, t'' = r'(fg ; m_1, m_2, \ldots, m_n)\, t''.$$

We still have to prove that conditions a), b) and c) of definition 2.1 are fulfilled.

Let us suppose $(i, t, m) \sim (i', t', m')$ with the same notation as in definition 2.4 and $(j, u, q) \sim (j', u', q')$ in $Fl_{M, T}(b, c)$, i.e. there exist $p \geq 1$, an isomorphism $g \in X(p, p)$ and $q_1, q_2, q_3, \ldots, q_p, q'_1, q'_2, \ldots, q'_p \in M$ such that

$$q = q_1 q_2 \ldots q_p, \quad q' = q'_1 q'_2 \ldots q'_p, \quad q_i = q'_{g(i)} \text{ for every } i \in [p].$$

$$j(r(g ; q'_1, q'_2, \ldots, q'_p) + 1_c) = j' \text{ and}$$

$$u(r(g ; q'_1, q'_2, \ldots, q'_p) + 1_c) = r'(g ; q_1, q_2, \ldots, q_p)\, u'.$$

Therefore :

$$mq = m_1 m_2 \ldots m_n q_1 q_2 \ldots q_p, \quad m'q' = m'_1 m'_2 \ldots m'_n q'_1 q'_2 \ldots q'_p$$

and the isomorphism $f + g \in X(n + p, n + p)$ fulfils the necessary conditions to show that $(i, t, m)(j, u, q) \sim (i', t', m')(j', u', q')$:

$$i(1_{r(m)} + j)(r(f + g ; m'_1, m'_2, \ldots, m'_n, q'_1, q'_2, \ldots, q'_p) + 1_c) = i(r(f ; m'_1, m'_2, \ldots, m'_n) +$$

$$+ 1_a)(1_{r(m')} + j(r(g ; q'_1, q'_2, \ldots, q'_p) + 1_c)) = i'(1_{r(m')} + j') \text{ and}$$

$$< t(1_{r(m)} + j), 0_{r(m)} + u > (r(f + g ; m'_1, m'_2, \ldots, m'_n, q'_1, q'_2, \ldots, q'_p) + 1_c) =$$

$$= < t(r(f ; m'_1, m'_2, \ldots, m'_n) + 1_b)(1_{r(m')} + j(r(g ; q'_1, q'_2, \ldots, q'_p) + 1_c)),$$

$$0_{r(m')} + u(r(g ; q'_1, q'_2, \ldots, q'_p) + 1_c) > = < r'(f ; m_1, m_2, \ldots, m_n)\, t'(1_{r(m')} + j'),$$

$$0_{r(m)} + r'(g ; q_1, q_2, \ldots, q_p)\, u' > = r'(f + g ; m_1, m_2, \ldots, m_n, q_1, q_2, \ldots, q_p) <$$

$$< t'(1_{r(m')} + j'), 0_{r(m')} + u' >.$$

Let us suppose $(i, l, m)\sim(i', l', m')$ with the same notation as in definition 2.1 and $(j, u, q)\sim(j', u', q')$ in $Fl_{M, T}(c, b)$ with notation similar to the above one. The isomorphism $f+g\in N(n+p, n+p)$ fulfills the necessary conditions to show that $<(i, l, m), (j, u, q)>\sim<(i', l', m'), (j', u', q')>$:

$$<i(1_{r(m)}+0_{r(q)}+1_b), 0_{r(m)}+j>(r(f+g : m_1', \dots, m_n', q_1', \dots, q_p')+1_b)=$$

$$=<i(r(f : m_1', \dots, m_n')+1_b)(1_{r(m')}+0_{r(q')}+1_b), 0_{r(m')}+j(r(g: q_1', \dots, q_p')+1_b)>=$$

$$=<i'(1_{r(m')}+0_{r(q')}+1_b), 0_{r(m')}+j'> \text{ and}$$

$$<l(1_{r(m)}+0_{r(q)}+1_b), 0_{r(m)}+u>(r(f+g : m_1', \dots, m_n', q_1', \dots, q_p')+1_b)=$$

$$=<r'(f : m_1', \dots, m_n')\,l'(1_{r(m')}+0_{r(q')}+1_b), 0_{r(m')}+r'(g : q_1, \dots, q_p')\,u'>=$$

$$=r'(f+g : m_1, \dots, m_n, q_1, \dots, q_p)<l'(1_{r(m')}+0_{r(q')}+1_b), 0_{r(m')}+u'>.$$

Let us suppose $(i, l, m)\sim(i', l', m')$ in $Fl_{M, T}(a, ab)$, i.e. there exist $n\geqslant 1$, an isomorphism $f\in N(n, n)$ and $m_1, m_2, \dots, m_n, m_1', m_2', \dots, m_n'$ such that

$$m=m_1m_2\dots m_n,\ m'=m_1'm_2'\dots m_n',\ m_i=m_{f(i)}' \text{ for each } i\in[n],$$

$$i(r(f ; m_1', m_2', \dots, m_n')+1_{ab})=i' \text{ and}$$

$$l(r(f ; m_1', m_2', \dots, m_n')+1_{ab})=r'(f : m_1, m_2, \dots, m_n)\,l'.$$

The following calculation proves that $(i, l, m)^{\dagger}\sim(i', l', m')^{\dagger}$:

$$(i(S^a_{r(m)}+1_b))^{\dagger}(r(f : m_1', m_2', \dots, m_n')+1_b)=(i(S^a_{r(m)}+1_b)(1_a+$$

$$+r(f ; m_1', m_2', \dots, m_n')+1_b))^{\dagger}=(i(r(f ; m_1', m_2', \dots, m_n')+1_{ab})(S^a_{r(m')}+1_b))^{\dagger}=$$

$$=(i'(S^a_{r(m')}+1_b))^{\dagger} \text{ and } l(S^a_{r(m)}+1_b)<(i(S^a_{r(m)}+1_b))^{\dagger}, 1_{r(m)b}>$$

$$(r(f ; m_1m_2, \dots, m_n)+1_b)=l(S^a_{r(m)}+1_b)(1_a+r(f : m_1, m_2, \dots, m_n)+$$

$$1_b)<(i'(S^a_{r(m')}+1_b))^{\dagger}, 1_{r(m')b}>=l(r(f : m_1, m_2, \dots, m_n)+1_{ab})(S^a_{r(m')}+$$

$$+1_b)<(i'(S^a_{r(m')}+1_b))^{\dagger}, 1_{r(m')b}>=r'(f ; m_1, m_2, \dots, m_n)\,l'(S^a_{r(m')}+$$

$$+1_b)<(i'(S^a_{r(m')}+1_b))^{\dagger}, 1_{r(m')b}>.$$

Let $CFl_{M, T}$ be the *quotient* of $Fl_{M, T}$ by $\sim$. $CSt : T\to CFl_{M, T}$ is structural functor and $C_M : M\to CFl_{M, T}$ the standard interpretation of M in $CFl_{M, T}$, i.e. the composition of I_M by canonical T-module with iterate morphism $C : Fl_{M, T}\to Fl_{M, T}/\sim$.

3. **Commutative T-modules.** 3.1. **Definition.** *A T-module $H : T\to Q$ is said to be commutative if*

$$H(S^b_a)(\alpha+\beta)=(\beta+\alpha)\,H(S^c_d),$$

for every $\alpha\in Q(a, c)$ and $\beta\in Q(b, d)$.

The T-module T with structural functor 1_T is commutative.

3.2. **Proposition.** *A T-module with iterate $H : T\to Q$ is commutative if and only if*

$$H(S^b_a)<\alpha, \beta>=<\beta, \alpha>,$$

for every $\alpha\in Q(b, c)$ and $\beta\in Q(a, c)$.

Proof. Let us suppose that Q is commutative, $\alpha\in Q(b, c)$ and $\beta\in Q(a, c)$. As $<\alpha, \beta>=(\alpha+\beta)\,H(<1_c, 1_c>)$ we deduce

$$H(S^b_a)<\alpha, \beta>=H(S^b_a)(\alpha+\beta)\,H(<1_c, 1_c>)=(\beta+\alpha)\,H(S^c_c<1_c, 1_c>)=$$

$$=(\beta+\alpha)\,H(<1_c, 1_c>)=<\beta, \alpha>.$$

Let us prove the other implication. If $\alpha\in Q(a, c)$ and $\beta\in Q(b, d)$ then

$$H(S^a_b)(\alpha+\beta)=H(S^a_b)<\alpha H(1_c+0_d), \beta H(0_c+1_d)>=<\beta H(0_c+$$

$$+1_d), \alpha H(1_c+0_d)>=<\beta H((1_d+0_c)S^c_d), \alpha H((0_d+1_c)S^c_d)>=$$

$$=<\beta H(1_d+0_c), \alpha H(0_d+1_c)>H(S^c_d)=(\beta+\alpha)\,H(S^c_d).$$

3.3. **Proposition.** *The T-module $CFl_{M, T}$ is commutative.*

Proof. To show that $CFl_{M, T}$ is commutative, it is enough to prove that $(i, l, m)\in Fl_{M, T}(a, c)$ and $(i', l', m')\in Fl_{M, T}(b, c)$ imply

$$St(S^b_a)<(i', l', m'), (i, l, m)>\sim<(i, l, m), (i', l', m')>.$$

Let us recall that

$$<(i, l, m), (i', l', m')>=(<i(1_{r(m)}+0_{r(m')}+1_c), i'(0_{r(m)}+1_{r(m')c})>,$$

$$<l(1_{r(m)}+0_{r(m')}+1_c), l'(0_{r(m)}+1_{r(m')c})>, mm').$$

It is easy to see that

$$St(S^b_a)<(i', l', m'), (i, l, m)>=(<i(0_{r(m')}+1_{r(m)c}), i'(1_{r(m')}+0_{r(m)}+1_c)>,$$

$$<l'(1_{r(m')}+0_{r(m)}+1_c), l(0_{r(m')}+1_{r(m)c})>, m'm)$$

Let $f\in N(2, 2)$ be the isomorphism defined by $f(1)=2$ and $f(2)=1$. As $r(f : m, m')=S^{r(m)}_{r(m')}$ and $r'(f : m, m')=S^{r'(m)}_{r'(m')}$ it follows that

$$<i(0_{r(m')}+1_{r(m)c}), i'(1_{r(m')}+0_{r(m)}+1_c)>(r(f : m, m')+1_c)=$$

$$=<i(1_{r(m)}+0_{r(m')}+1_c), i'(0_{r(m)}+1_{r(m')c})> \text{ and}$$

$$<l'(1_{r(m')}+0_{r(m)}+1_c), l(0_{r(m')}+1_{r(m)c})>(r(f, m, m')+1_c)=$$

$$=r'(f : m, m')<l(1_{r(m)}+0_{r(m')}+1_c), l'(0_{r(m)}+1_{r(m')c})>.$$

Then

$$CSt(S^b_a)<C(i', l', m'), C(i, l, m)>=<C(i, l, m), C(i', l', m')>$$

therefore $CFl_{M, T}$ is commutative.

3.4. **Proposition.** *Let $H : T\to Q$ be a commutative T-module. If $g\in N(n, n)$ is an isomorphism and $\alpha_i\in Q(a^i, b^i)$ for every $i\in[n]$ then*

$$H(1_S(g : a^1, a_2, \dots, a^n))(\alpha_1+\alpha_2+\dots+\alpha_n)=(\alpha_{g(1)}+\alpha_{g(2)}+\dots+$$

$$+\alpha_{g(n)})\,H(1_S(g : b^1, b_2, \dots, b^n)).$$

Proof. We regard the above identity as a property of the isomorphism g and we prove that all the isomorphisms from N having this property form a subcategory of N which is closed under sum.

If $g, f \in N(n, n)$ are isomorphisms with the above property then

$$H(1_{s_a}(gf : a^1, a^2, \ldots, a^n))(\alpha_1 + \alpha_2 + \ldots + \alpha_n) =$$

$$H(1_{s_a}(g : a^{f(1)}, a^{f(2)}, \ldots, a^{f(n)}))(\alpha_{f(1)} + \alpha_{f(2)} + \ldots + \alpha_{f(n)})$$

$$H(1_{s_a}(f : b^1, b, \ldots, b_2^{\,n})) = (\alpha_{f(g(1))} + \ldots + \alpha_{f(g(n))})\, H(1_{s_a}(g : b^{f(1)}, b^{f(2)}, \ldots, b^{f(n)}))$$

$$H(1_{s_a}(f : b^1, b^2, \ldots, b^n)) = (\alpha_{(fg)(1)} + \alpha_{(gf)(2)} + \ldots +$$

$$+ \alpha_{(gf)(n)})\, H(1_{s_a}(gf : b^1, b^2, \ldots, b^n)).$$

If $g \in N(n, n)$ and $f \in N(p, p)$ are isomorphisms with the above property and $\alpha_i \in Q(a^i, b^i)$ for every $i \in [n+p]$ then

$$H(1_{s_a}(g+f : a^1, a^2, \ldots, a^{n+p}))(\alpha_1 + \alpha_2 + \ldots + \alpha_{n+p}) =$$

$$= H(1_{s_a}(g : a^1, a^2, \ldots, a^n))(\alpha_1 + \alpha_2 + \ldots + \alpha_n) +$$

$$H(1_{s_a}(f : a^{n+1}, a^{n+2}, \ldots, a^{n+p}))(\alpha_{n+1} + \alpha_{n+2} + \ldots + \alpha_{n+p}) = (\alpha_{g(1)} +$$

$$+ \alpha_{g(2)} + \ldots + \alpha_{g(n)})\, H(1_{s_a}(g : b^1, b^2, \ldots, b^n)) + (\alpha_{n+f(1)} + \alpha_{n+f(2)} + \ldots +$$

$$+ \alpha_{n+f(p)})\, H(1_{s_a}(f, b^{n+1}, b^{n+2}, \ldots, b^{n+p})) = (\alpha_{(g+f)(1)} + \alpha_{(g+f)(2)} + \ldots +$$

$$\alpha_{(g+f)(n+p)})\, H(1_{s_a}(g+f) : b^1, b^2, \ldots, b^{n+p})).$$

Let $f \in N(2, 2)$ be the isomorphism defined by $f(1) = 2$ and $f(2) = 1$. As Q is commutative, the isomorphism f has the above property. But every subcategory of N containing f and being closed under sum contains all the isomorphisms, therefore every isomorphism has the above property.

4. The free commutative T-module with iterate. Let M be an equidivisible monoid, let $r : M \to S^*$ and $r' : M \to S^*$ be two monoid morphisms and let T be an S-sorted algebraic theory with iterate.

We recall that if $\alpha \in Q(a, ab)$ is a morphism in a T-module with iterate $H : T \to Q$ and if $f \in T(c, a)$ is a isomorphism then

$$H(f)\,\alpha^\dagger = (H(f)\,\alpha H(f^{-1} + 1_b))^\dagger. \tag{4.1}$$

4.1. Theorem. *$CFl_{M,\,T}$ is the free commutative T-module with iterate generated by M.*

Proof. We have to prove that for every interpretation I of M in a commutative T-module with iterate $H : T \to Q$ there exists a unique T-module with iterate morphism $G : CFl_{M,\,T} \to Q$ such that $C_M G = I$.

As $Fl_{M,\,T}$ is freely generated by M there exists a unique T-module with iterate morphism $F : Fl_{M,\,T} \to Q$ such that $I_M F = I$.

To apply proposition 2.3 we shall prove that $(i, t, m) \sim (i', t', m')$ in $Fl_{M,\,T}(a, b)$ implies $F(i, t, m) = F(i', t', m')$.

Let us recall that $F(i, t, m) = H(i) < (I(m)\, H(t))^\dagger, 1_b >$. With the same notations as in definition 2.4, using in turn the above remark and proposition 3.1 we deduce

$$H(r(f : m_1', m_2', \ldots, m_n'))\,(I(m')H(t'))^\dagger =$$

$$= (H(r(f : m_1', m_2', \ldots, m_n'))\, I(m')\, H(t'(r(f : m_1', m_2', \ldots, m_n')^{-1} + 1_b)))^\dagger =$$

$$= (H(1_{s_a}(f : r(m_1'), r(m_2'), \ldots, r(m_n')))\,(I(m_1') + I(m_2') + \ldots + I(m_n'))$$

$$H(t'(r(f : m_1', m_2', \ldots, m_n')^{-1} + 1_b)))^\dagger = ((I(m_1) + I(m_2) + \ldots +$$

$$+ I(m_n))\, H(1_{s_a}(f : r'(m_1'), r'(m_2'), \ldots, r'(m_n')))$$

$$H(t'(r(f : m_1', m_2', \ldots, m_n')^{-1} + 1_b)))^\dagger =$$

$$= (I(m) H(r'(f : m_1', m_2', \ldots, m_n')\, t'(r(f : m_1', m_2', \ldots, m_n')^{-1} + 1_b)))^\dagger = (I(m)\, H(t))^\dagger.$$

Therefore

$$F(i', t', m') = H(i') < (I(m')\, H(t'))^\dagger, 1_b > =$$

$$= H(i) < H(r(f : m_1', m_2', \ldots, m_n'))\,(I(m')\, H(t'))^\dagger, 1_b > =$$

$$= H(i) < (I(m)\, H(t))^\dagger, 1_b > = F(i, t, m).$$

We deduce from proposition 2.3 the existence of a unique T-module with iterate morphism $G : CFl_{M,\,T} \to Q$ such that $CG = F$. It follows that $C_M G = I_M C G = I_M F = I$.

We finish this paper with some identities in a commutative T-module with iterate which are not true in some T-module with iterate.

4.2. Proposition. *Let $H : T \to Q$ be a commutative T-module with iterate.*

a) If $\alpha \in Q(a, b)$, $\beta \in Q(c, d)$, $\gamma \in Q(b, e)$ and $\delta \in Q(d, e)$ then

$$(\alpha + \beta) < \gamma, \delta > = < \alpha\gamma, \beta\delta > \tag{4.2}$$

b) If $\alpha \in Q(a, b)$, $\beta \in Q(c, d)$, $\gamma \in Q(b, e)$ and $\delta \in Q(d, f)$ then

$$(\alpha + \beta)(\gamma + \delta) = \alpha\gamma + \beta\delta. \tag{4.3}$$

Proof. We know that if either β or γ is in the image of H then the above identities are true in every T-module with iterate. The proof of the first identity is

$$(\alpha + \beta) < \gamma, \delta > = (\alpha + 1_c)(1_b + \beta) < \gamma, \delta > =$$

$$= (\alpha + 1_c)\, H(S_b^c)(\beta + 1_b)\, H(S_d^b) < \gamma, \delta > = (\alpha + 1_c)\, H(S_b^c)(\beta + 1_b) < \delta, \gamma > =$$

$$= (\alpha + 1_c)\, H(S_b^c) < \beta\delta, \gamma > = (\alpha + 1_c) < \gamma, \beta\delta > = < \alpha\gamma, \beta\delta >.$$

The proof of the second identity is

$$(\alpha + \beta)(\gamma + \delta) = (\alpha + \beta) < \gamma H(1_e + 0_f), \delta H(0_e + 1_f) > =$$

$$= < \alpha\gamma H(1_e + 0_f), \beta\delta H(0_e + 1_f) > = \alpha\gamma + \beta\delta.$$

4.3. Proposition. *Let $H : T \to Q$ be a commutative T-module with iterate. If $\alpha \in Q(a, bc)$, $\gamma \in Q(d, c)$ and $\beta \in Q(b, c)$ then*

$$< \alpha, \gamma H(0_b + 1_c) > < \beta, 1_c > = < \alpha < \beta, 1_c >, \gamma >. \tag{4.4}$$

Proof. We know that in every T-module with iterate

$$<\gamma H(0_b+1_c),\ \alpha><\beta,\ 1_c> = <\gamma,\ \alpha<\beta,\ 1_c>>$$

therefore

$$<\alpha,\ \gamma H(0_b+1_c)><\beta,\ 1_c> = H(S_a^d)<\gamma H(0_b+1_c),\ \alpha><\beta,\ 1_c> =$$
$$= H(S_a^d)<\gamma,\ \alpha<\beta,\ 1_c>> = <<\beta,\ 1_c>,\ \gamma>.$$

4.4. **Proposition.** *Let $H : T \to Q$ be a commutative T-module with iterate. If $\alpha \in Q(a,\ abc)$ and $\beta \in Q(b,\ abc)$ then*

$$<\alpha,\ \beta>^{\dagger} = H(S_a^b)(<\beta,\ \alpha> H(S_a^b+1_c))^{\dagger}. \tag{4.5}$$

Proof. We deduce from (4.1) that

$$H(S_b^a)<\alpha,\ \beta>^{\dagger} = (H(S_b^a)<\alpha,\ \beta> H(S_a^b+1_c))^{\dagger}$$

therefore using proposition 3.2 we obtain (4.5).

The next proof is based on the following property of every T-module with iterate :

$$H(1_a+0_b)<\alpha,\ \beta>^{\dagger} = (\alpha H(S_a^b+1_c)<(\beta H(S_a^b+1_c))^{\dagger},\ 1_{ac}>)^{\dagger}$$

for every $\alpha \in Q(a,\ abc)$ and $\beta \in Q(b,\ abc)$.

4.5. **Proposition.** *Let $H : T \to Q$ be a commutative T-module with iterate. If $\alpha \in Q(a,\ abc)$ and $\beta \in Q(b,\ abc)$ then*

$$H(0_a+1_b)<\alpha,\ \beta>^{\dagger} = (\beta<\alpha^{\dagger},\ 1_{bc}>)^{\dagger}. \tag{4.6}$$

Proof. $H(0_a+1_b)<\alpha,\ \beta>^{\dagger} = H(0_a+1_b)H(S_a^b)(<\beta,\ \alpha>H(S_a^b+1_c))^{\dagger} =$
$= H(1_b+0_a)<\beta H(S_a^b+1_c),\ \alpha H(S_a^b+1_c)>^{\dagger} = (\beta<\alpha^{\dagger},\ 1_{bc}>)^{\dagger}.$

The next proof is based on the following property of every T-module with iterate :

$$(H(0_a)+<\alpha,\ \beta>)^{\dagger} = <\alpha,\ H(1_b+0_c)><\beta^{\dagger},\ 1_c>$$

for every $\alpha \in Q(a,\ bc)$ and $\beta \in Q(b,\ bc)$.

4.6. **Proposition.** *Let $H : T \to Q$ be a commutative T-module with iterate. If $\alpha \in Q(a,\ ac)$ and $\beta \in Q(b,\ ac)$ then*

$$(<\alpha,\ \beta>H(1_a+0_b+1_c))^{\dagger} = <H(1_a+0_c),\ \beta><\alpha^{\dagger},\ 1_c>. \tag{4.7}$$

Proof. $(<\alpha,\ \beta>H(1_a+0_b+1_c))^{\dagger} = H(S_a^b)(<\beta H(1_a+0_b+1_c),\ \alpha H(1_a+$

$$+0_b+1_c)>H(S_a^b+1_c))^{\dagger} = H(S_a^b)(H(0_b)+<\beta,\ \alpha>)^{\dagger} =$$

$$= H(S_a^b)<\beta,\ H(1_a+0_c)><\alpha^{\dagger},\ 1_c> = <H(1_a+0_c),\ \beta><\alpha^{\dagger},\ 1_c>.$$

REFERENCES

1. C a z ă n e s c u V. E., U n g u r e a n u C. — *Again on advice on structuring compilers and proving them correct.* INCREST Preprint series in mathematics No. 75/1982.
2. M c K n i g h t J. D., J r. and S t o r e y A. J. — *Equidivisible semigroups*, J. Algebra 12(1969), 24—48.

Received 10.V.1987

Faculty of Mathematics
University of Bucharest
București, România

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
Tomul XXXIII, s. I a, Matematică, 1987, fasc. 4

EQUIVALENT COMPUTATIONAL SEMANTICS OF CONFLUENT TERM REWRITING SYSTEMS

BY

T. JUCAN and C. MASALAGIU

§ 1. **Introduction.** In [1], computational semantics of terms and term rewriting systems are studied. In the case of confluent term rewriting systems, two kinds of semantics, for the same term t, are defined. These are denoted $\mathrm{Comp}_{<R,A>}(t)$ and $\mathrm{Val}_{<R,A>}(t)$ where A is the semantical domain (a complete F-algebra) and R a term rewriting system. For the case of noetherian or deterministic systems the problem :

$$(\forall t)(\mathrm{Val}_{<R,A>}(t) = \mathrm{Comp}_{<R,A>}(t)) \tag{1}$$

is positively solved. In this paper, we prove that (1) remains valid for the general case of confluent term rewriting systems.

§ 2. **Preliminaries.** The basic formalism is that of G. B o u d o l [1].

§ 2.1. **Trees, term rewriting systems and computations.** Let X and $F = \bigcup_{i \geq 0} F_i$ be fixed denumerable sets of variables and function symbols (F_i being the set of function symbols of arity i). Let $W_F = \{(f,\ i) \mid (\exists k \in N)(f \in F_k \wedge 1 \leq i \leq k)\}$ where $(f,\ i)$ will be denoted by f_i. A *tree* (F-tree on X) is then a partial function $t : W_F^* \to X \cup F$ with domain dom (t) which satisfies :

a) $u \in \mathrm{dom}\,(t) \wedge v \leq_{\mathrm{pref}} u \Rightarrow v \in \mathrm{dom}\,(t)$;

b) $wf_i \in \mathrm{dom}\,(t) \Rightarrow t(w) = f \wedge (\mathrm{dom}\,(t)/w) \cap W_F \subseteq \{f_1,\ f_2,\ \ldots,\ f_k\}$.

In the definition above W_F^* is the free monoid generated by W_F (the monoid operation is concatenation and the empty word will be denoted by e). For $L \subseteq W_F^*$, $u,\ v,\ w \in W_F^*$, standard notations of formal language theory were used, and

$$v \leq_{\mathrm{pref}} u \Leftrightarrow (\exists v' \in W_F^*)(u = vv').$$
$$L/w = \{u \mid u \in W_F^* \wedge wu \in L\}.$$

In this interpretation, the „nodes" of a tree t are the elements of dom (t) and for $u \in \mathrm{dom}\,(t)$, $t(u)$ is the „*label*" of node u. The totally undefined mapping on W_F^* („the empty tree") is denoted by $\perp$ and one-node trees by their label. For example, the tree $t = g(g(\perp,\ y),\ h(x))$, that is

$$t = \begin{tikzcd}[column sep=small] & & g \arrow[dl, dash] \arrow[dr, dash] & \\ & g \arrow[dl, dash] \arrow[dr, dash] & & h \arrow[d, dash] \\ \perp & & y & x \end{tikzcd}$$