$$<\gamma H(0_b+1_c),\ \alpha><\beta,\ 1_{bc}>=<\gamma,\ \alpha<\beta,\ 1_c>>$$

therefore

$$<\alpha,\ \gamma H(0_b+1_c)><\beta,\ 1_c>=H(S_a^d)<\gamma H(0_b+1_c),\ \alpha><\beta,\ 1_c>=$$

$$=H(S_a^d)<\gamma,\ \alpha<\beta,\ 1_c>>=<<\beta,\ 1_c>,\ \gamma>.$$

4.4. **Proposition.** *Let $H: T\to Q$ be a commutative T-module with iterate. If $\alpha\in Q(a,\ abc)$ and $\beta\in Q(b,\ abc)$ then*

(4.5) $$<\alpha,\ \beta>^{\dagger}=H(S_a^b)(<\beta,\ \alpha>H(S_a^b+1_c))^{\dagger}.$$

Proof. We deduce from (4.1) that

$$H(S_b^a)<\alpha,\ \beta>^{\dagger}=(H(S_b^a)<\alpha,\ \beta>H(S_a^b+1_c))^{\dagger}$$

therefore using proposition 3.2 we obtain (4.5).

The next proof is based on the following property of every T-module with iterate:

$$H(1_a+0_b)<\alpha,\ \beta>^{\dagger}=(\alpha H(S_a^b+1_c)<(\beta H(S_a^b+1_c))^{\dagger},\ 1_{ac}>)^{\dagger}$$

for every $\alpha\in Q(a,\ abc)$ and $\beta\in Q(b,\ abc)$.

4.5. **Proposition.** *Let $H: T\to Q$ be a commutative T-module with iterate. If $\alpha\in Q(a,\ abc)$ and $\beta\in Q(b,\ abc)$ then*

(4.6) $$H(0_a+1_b)<\alpha,\ \beta>^{\dagger}=(\beta<\alpha^{\dagger},\ 1_{bc}>)^{\dagger}.$$

Proof. $H(0_a+1_b)<\alpha,\ \beta>^{\dagger}=H(0_a+1_b)H(S_a^b)(<\beta,\ \alpha>H(S_a^b+1_c))^{\dagger}=$ $=H(1_b+0_a)<\beta H(S_a^b+1_c),\ \alpha H(S_a^b+1_c)>^{\dagger}=(\beta<\alpha^{\dagger},\ 1_{bc}>)^{\dagger}.$

The next proof is based on the following property of every T-module with iterate:

$$(H(0_a)+<\alpha,\ \beta>)^{\dagger}=<\alpha,\ H(1_b+0_c)><\beta^{\dagger},\ 1_c>$$

for every $\alpha\in Q(a,\ bc)$ and $\beta\in Q(b,\ bc)$.

4.6. **Proposition.** *Let $H: T\to Q$ be a commutative T-module with iterate. If $\alpha\in Q(a,\ ac)$ and $\beta\in Q(b,\ ac)$ then*

(4.7) $$(<\alpha,\ \beta>H(1_a+0_b+1_c))^{\dagger}=<H(1_a+0_c),\ \beta><\alpha^{\dagger},\ 1_c>.$$

Proof. $(<\alpha,\ \beta>H(1_a+0_b+1_c))^{\dagger}=H(S_a^b)(<\beta H(1_a+0_b+1_c),\ \alpha H(1_a+$

$$+0_b+1_c)>H(S_a^b+1_c))^{\dagger}=H(S_a^b)(H(0_b)+<\beta,\ \alpha>)^{\dagger}=$$

$$=H(S_a^b)<\beta,\ H(1_a+0_c)><\alpha^{\dagger},\ 1_c>=<H(1_a+0_c),\ \beta><\alpha^{\dagger},\ 1_c>.$$

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EQUIVALENT COMPUTATIONAL SEMANTICS OF CONFLUENT TERM REWRITING SYSTEMS

BY

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§ 1. **Introduction.** In [1], computational semantics of terms and term rewriting systems are studied. In the case of confluent term rewriting systems, two kinds of semantics, for the same term t, are defined. These are denoted $\mathrm{Comp}_{<R,A>}(t)$ and $\mathrm{Val}_{<R,A>}(t)$ where A is the semantical domain (a complete F-algebra) and R a term rewriting system. For the case of noetherian or deterministic systems the problem:

(1) $$(\forall t)(\mathrm{Val}_{<R,A>}(t)=\mathrm{Comp}_{<R,A>}(t))$$

is positively solved. In this paper, we prove that (1) remains valid for the general case of confluent term rewriting systems.

§ 2. **Preliminaries.** The basic formalism is that of G. Boudol [1].

§ 2.1. **Trees, term rewriting systems and computations.** Let X and $F=\bigcup_{(i\geq 0)}F_i$ be fixed denumerable sets of variables and function symbols (F_i being the set of function symbols of arity i). Let $W_F=\{(f,\ i)\ (\exists k\in N)(f\in F_k\wedge 1\leq i\leq k)\}$ where $(f,\ i)$ will be denoted by f_i. A *tree* (F-tree on X) is then a partial function $t: W_F^*\to X\cup F$ with domain dom (t) which satisfies:

a) $u\in\mathrm{dom}\ (t)\wedge v\leq_{pref} u\Rightarrow v\in\mathrm{dom}\ (t)$;

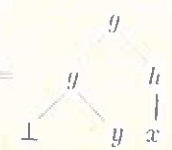
b) $wf_i\in\mathrm{dom}\ (t)\Rightarrow t(w)=f\wedge(\mathrm{dom}\ (t)/w)\cap W_F\subseteq\{f_1,\ f_2,\ \dots,\ f_k\}$.

In the definition above W_F^* is the free monoid generated by W_F (the monoid operation is concatenation and the empty word will be denoted by e). For $L\subseteq W_F^*$, $u,\ v,\ w\in W_F^*$, standard notations of formal language theory were used, and

$$v\leq_{pref} u\Leftrightarrow(\exists v'\in W_F^*)\ (u=vv').$$

$$L/w=\{u\mid u\in W_F^*\wedge wu\in L\}.$$

In this interpretation, the „nodes" of a tree t are the elements of dom (t) and for $u\in\mathrm{dom}\ (t)$, $t(u)$ is the „*label*" of node u. The totally undefined mapping on W_F^* („the empty tree") is denoted by $\perp$ and one-node trees by their label. For example, the tree $t=g(g(\perp,\ y),\ h(x))$, that is

$t=$ 

is a partial mapping having $\mathrm{dom}(t)=\{e,\ g_1,\ g_2,\ g_1g_2,\ g_2h_1\}$ and given by $t(e)=g,\ t(g_1)=g,\ t(g_2)=h,\ t(g_1g_2)=y,\ t(g_2h_1)=x$. The set of trees will be denoted by $M_F^{\omega}(X)$ and the set of finite trees (trees having finite domain) by $M_F(X)$. The standard „less defined" partial order „$\leqslant$" can be defined on $M_F^{\omega}(X)$:

$$t\leqslant t'\Leftrightarrow(\forall u\in W_F^*)\,(u\in\mathrm{dom}\,(t)\Rightarrow u\in\mathrm{dom}\,(t')\wedge t'(u)=t(u)).$$

The set of *terms* ($\leqslant$ — maximal finite trees) will be denoted by $\overline{M}_F(X)$. For $t\in M_F^{\omega}(X)$ the set of *occurrences*, $\mathrm{occ}\,(t)$, is given by:

a) $\mathrm{occ}\,(\perp)=\{e\}$

b) $\mathrm{occ}\,(t)=\mathrm{dom}\,(t)\cup\{wf_i\mid w\in\mathrm{dom}\,(t)\wedge t(w)=f\in F_i\wedge 1\leqslant i\leqslant k\},\ \forall t\neq\perp$

For $u\in\mathrm{occ}\,(t)$ *the subtree of t at occurrence u*, (t/u) is defined by:

a) $\mathrm{dom}\,(t/u)=\mathrm{dom}\,(t)/u$, b) $(t/u)\,(v)=t(uv)$.

If $T\subseteq M_F^{\omega}(X)$ and $t\in M_F^{\omega}(X)$ we consider the sets:

$\mathrm{occ}\,(T,t)=\{u\mid u\in\mathrm{occ}\,(t)\wedge(t/u)\in T\}$

$\mathrm{int}\,(t)=\mathrm{occ}\,(t)\setminus\mathrm{occ}(X,t)$.

We say that t' *is an instance of t* (*t matches* t') iff $t'=s^*(t)$ where $s:X\to M_F^{\omega}(X)$ is a substitution and s^* is its unique extension to $M_F^{\omega}(X)$. t and t' are *unifiable* iff one of them is an instance of the other.

For $t,\ t'\in M_F^{\omega}(X)$ we say that t *overlaps* t' *at occurrence u* iff $u\in\mathrm{int}(t)$ and (t/u) and t' are unifiable.

A *term rewriting system* (TRS) is a finite relation $R\subseteq\overline{M}_F(X)\times\overline{M}_F(X)$ such that every $r=(\theta,\theta')\in R$ ($\theta=\mathrm{lhs}(r)$, $\theta'=\mathrm{rhs}\,(r)$) satisfies:

a) $\theta\notin X$,

b) $\mathrm{var}(\theta')\subseteq\mathrm{var}(\theta)$,

c) θ is linear,

where for $t\in M_F^{\omega}(X)$, $\mathrm{var}(t)=\{x\in X\mid(\exists u\in\mathrm{occ}\,(t))\,((t/u)=x)\}$ and t is *linear* iff every variable has at most one occurrence in t. The elements of R are called *rules*.

Given a TRS R and a term t, we put:

$$\mathrm{lhs}(R)=\bigcup_{r\in R}\mathrm{lhs}\,(r),$$

$$\mathrm{lhr}(R)=\bigcup_{r\in R}\mathrm{lhr}\,(r)$$

$$\mathrm{occ}_R(t)=\{u\mid u\in\mathrm{occ}(t)\wedge(\exists\theta\in\mathrm{lhs}\,(R))\,(\exists s:X\to M_F^{\omega}(X))\,((t/u)=s^*(\theta))\},$$

$$\mathrm{red}_R(t)=\{(u,\ r)\mid u\in\mathrm{occ}_R(t)\wedge(\exists s:X\to M^{\omega}(X))\,(t/u)=s^*(\mathrm{lhs}\,(r))\}.$$

The elements of $\mathrm{red}_R(t)$ are called *redexes* for t. A redex (u,r) says that rule can be applied for the subtree (t/u) of t. In order to avoid a too fine definition of computation, we introduce the notion of consistent set of redexes.

A redex (u,r_1) *overlaps* a redex (v,r_2) iff there exists $w\in W_F^*$ such that $v=uw$ and $\mathrm{lhs}\,(r_1)$ overlaps $\mathrm{lhs}\,(r_2)$ at occurence w. For $t\in\overline{M}_F(X)$ a set $\sigma\subseteq\mathrm{red}_R(t)$ is *consistent* iff any two elements of σ do not overlap.

For $t,t'\in M_F^{\omega}(X)$ the *replacement of the subtree at occurrence u in t by t'* (denoted $[t'/u]\,t$ or $t\,[u\leftarrow t']$) is defined by:

a) $\mathrm{dom}\,([t'/u]\,t)=(\mathrm{dom}\,(t)\setminus\{u\}\,W_F^*)\cup\{u\}\,\mathrm{dom}\,(t')$,

b) $[t'/u]\,t(v)=t(v)$ if $v\in\mathrm{dom}\,(t)\wedge u\nleq_{\mathrm{pref}} v$

c) $[t'/u]\,t(uw)=t'(w)$ if $w\in\mathrm{dom}\,(t')$.

Given any $R\subseteq M_F^{\omega}(X)\times M_F^{\omega}(X)$, the *one step rewriting relation* determined by R, denoted $\underset{R}{\to}$, is defined by:

$$t\underset{R}{\to}t'\Leftrightarrow(\exists r=(\theta,\theta')\in R)\,(\exists s:X\to M_F^{\omega}(X))$$

$$(\exists u\in\mathrm{occ}\,(t))\,(t/u)=s^*(\theta)\wedge t'=[s^*(\theta')/u](t)).$$

A consistent set of redexes may be viewed as a partial mapping from $\mathrm{occ}\,(t)$ to R (indicating what rule is to be applied at some occurrence) whose domain is

$$\mathrm{dom}\,(\sigma)=\{u\mid u\in\mathrm{occ}(t)\wedge(\exists r\in R)\,((u,r)\in\sigma)\}.$$

The *result* of the application of a consistent set of redexes σ to a term t (denoted by $t\cdot\sigma$) is recursively defined by:

a) $t\cdot\varnothing=t$,

b) If $\sigma\neq\varnothing$ then $t\cdot\sigma=t'\cdot\sigma'$ where $\sigma'=\sigma\setminus\{(w,\ \sigma(w))\}$,

and w is the least element of $\mathrm{dom}\,(\sigma)$

(e.g. via lexicographic order) and $t\xrightarrow[R]{(w,\sigma(w))}t'$.

We shall write $t\overset{\sigma}{\underset{R}{\to}}t'$ if σ is a consistent set of redexes for t and $t'=t\cdot\sigma$.

Let R be a TRS and $t\in\overline{M}_F(X)$. A *computation* of t in R is a sequence $\gamma=(\sigma_i)_{i\in I}$ such that $\forall\,i\in I$, σ_i is a consistent set of redexes for $t\cdot\sigma_0\sigma_1\ldots\sigma_{i-1}$ ($t\cdot\sigma_0\ldots\sigma_{i-1}$ is a shorthand for $(((t\cdot\sigma_0)\cdot\sigma_1),\ldots),\ \sigma_{i-1})$. The index set I may be N or an initial segment of N. If I is finite, $\gamma=(\sigma_i)_{i\in I}$ is a finite computation of t in $t'=t\cdot\sigma_0\sigma_1\ldots\sigma_{|\gamma|-1}$, of *length* $|\gamma|$, where $|\gamma|$ is the integer such that $I=\{i\mid i\in N\wedge i<|\gamma|\}$. The set of finite computations of t in R is denoted $\mathcal{C}_R(t)$, the empty computation by ε and for $\gamma\in\mathcal{C}_R(t)$ we shall write:

$$\gamma:t\overset{*}{\underset{R}{\to}}t'\quad\text{or}\quad t=t_0\overset{\sigma_0}{\to}t_1\overset{\sigma_1}{\to}\ \ldots\ \overset{\sigma_{|\gamma|-1}}{\longrightarrow}t_{|\gamma|}=t'.$$

If $I=N$, the computation $\gamma=(\sigma_i)_{i\in I}$ is infinite, the set of infinite computation being denoted by $\mathcal{C}_R^{\omega}(t)$ and for $\gamma\in\mathcal{C}_R^{\omega}(t)$ we write:

$$(2)\quad\gamma:t=t_0\overset{\sigma_0}{\to}t_1\overset{\sigma_1}{\to}t_2\overset{\sigma_2}{\to}\ldots\overset{\sigma_i}{\to}t_{i+1}\overset{\sigma_{i+1}}{\to}\ldots$$

We put $\mathcal{C}_R^{\infty}(t)=\mathcal{C}_R^{\omega}(t)\cup\mathcal{C}_R(t)$. A step $(t_k\overset{\sigma_k}{\to}t_{k+1})$ in a computation may thus be viewed as being realized by a number of „parallel" rewritings. The *size* of a computation γ is the number of nonempty steps and is denoted by $\|\gamma\|$. Because we are interested in the results of computations, a notion of „equivalent computations" is necessary. To start with, some other notions are in order.

If $\gamma:t\overset{*}{\underset{R}{\to}}t'$, $\gamma':t'\overset{\sigma_0'}{\to}t_1\overset{\sigma_1'}{\to}\ldots$, the *composition of* γ *with* γ', denoted by $\gamma;\gamma'$, is the computation $t\overset{*}{\to}t'\overset{\sigma_0'}{\to}t_1\overset{\sigma_1'}{\to}\ldots$ If t is a term, $w\in\mathrm{occ}\,(t)$ and $(u,r)\in\mathrm{red}_R(t)$, we define the *residual* and *trace* of w by (u,r) (denoted respectively by $w[u,r]$ and $w<u,r>$):

a) if $w \neq u$ or $w <_{pref} u$ then $w[u.r] = w < u.r > = \{w\}$

b) if $u \leqslant_{pref} w$ then :

b1) if $\exists v \in \mathrm{int}\ (\mathrm{lhs}\ (r))$ such that $w = uv$, then $w[u.r] = \emptyset$ and $w < u.r > = \{u\}$

b2) if $\exists v \in \mathrm{occ}\ (X, \mathrm{lhs}\ (r))$ and $\exists w'$ such that $w = uvw'$ then $w[u.r] = w < u.r > = u \cdot \{v' \mid v' \in \mathrm{occ}\ (\mathrm{rhs}\ (r)) \wedge (\mathrm{rhs}(r)/v') = (\mathrm{lhs}\ (r)/v)\} \cdot w'$

These notions can naturally be extended to finite computations and sets of occurrences. Namely, given $t \in \overline{M}_F(X)$, $\gamma \in \mathcal{C}_R(t)$, $W \subseteq \mathrm{occ}\ (t)$ the residuals and traces of W by γ (denoted $W[\gamma]$ and $W < \gamma >$) are the sets given by :

(a) $$W[\gamma] = \bigcup_{w \in W} w[\gamma],\quad W < \gamma > = \bigcup_{w \in W} w < \gamma >$$

(b) $w[\varepsilon] = \{w\} = w < \varepsilon >$.

(c) $w[\sigma : \gamma] = (w[\sigma])[\gamma]$, $w < \sigma : \gamma > = (w < \sigma >) < \gamma >$.

(d) $w[\emptyset] = \{w\} = w < \emptyset >$.

(e) if $|\sigma| > 0$ then $w[\sigma] = (w[u.r])[\sigma']$.

$w < \sigma > = (w < u, r >) < \sigma' >$ where $(u.r) \in \sigma$, u is the last element of dom (σ) (with respect to lexicographic order) and $\sigma' = \sigma \setminus \{(u, r)\}$.

If σ, σ' are consistent sets of redexes for t we say that they are *compatible* (written $\sigma \uparrow \sigma'$) iff $\sigma \cup \sigma'$ is also a consistent set of redexes for t. For such sets σ, σ' of redexes we can formally define „what remains to be done of the first after applying the other" :

$$\sigma[\sigma'] = \{(w', r) \mid (\exists w)\ ((w.r) \in \sigma \wedge w' \in w[\sigma'])\}$$

$\sigma[\sigma']$ is called the *residual of σ by σ'*.

Lemma 1 ([*1*], p. *191*). *For any σ, σ' compatible consistent sets of redexes for t, $\sigma[\sigma']$ is a consistent set of redexes for $t \cdot \sigma'$. $\gamma \in \mathcal{C}_R(t)$ is relative to a consistent set of redexes σ for t* iff $\gamma = \varepsilon$ or $\gamma = \sigma' : \gamma'$, $\sigma' \subseteq \sigma$, $\sigma' \neq \emptyset$ *and γ' is relative to $\sigma[\sigma']$*. We say that γ *realizes* σ (or is a *development* of σ) if γ is *relative to* σ and (dom (σ)) $[\gamma] = \emptyset$. If γ realizes σ, nothing of σ remains to be done after it, γ performing all the rewritings indicated by σ.

We introduce the following relations on $\mathcal{C}_R(t)$:

(i) $\gamma \sim \gamma' \Leftrightarrow \gamma, \gamma'$ are developments for the same consistent set of redexes σ for t, or $\|\gamma\| = \|\gamma'\| = 0$;

(ii) $$\gamma \approx \gamma' \Leftrightarrow \gamma = \gamma_1 : \delta : \gamma_2 \wedge \gamma' = \gamma_1 : \delta' : \gamma_2 \wedge \delta \sim \delta'.$$

The transitive closure of „$\approx$" is denoted by „$\equiv$" and it is an equivalence relation (*equivalence by permutation*). Other relations on $\mathcal{C}_R(t)$ can now be defined :

(i) $$\gamma \sqsubseteq_c \gamma' \Leftrightarrow (\exists \delta)\ (\gamma' = \gamma : \delta).$$

(ii) $$\gamma \leqslant \gamma' \Leftrightarrow (\exists \delta)\ (\gamma \sqsubseteq_c \delta \wedge \delta \equiv \gamma').$$

The relations „$\sqsubseteq_c$" and „$\leqslant$" can naturally be extended to $\mathcal{C}^{\omega}_R(t)$ ([1], p. 202—203), the extensions being denoted by „$\sqsubseteq_c^{\omega}$" and „$\leqslant^{\omega}$" and they are a order resectively a preorder on $\mathcal{C}^{\omega}_R(t)$. The equivalence relation induced by the preorder „$\leqslant^{\omega}$" is denoted „$\equiv^{\omega}$" and it is the desired equivalence of computations.

$\mathcal{C}^{\omega}_R(t)/\equiv^{\omega}$ has many interesting properties but we need especially the following one :

Lemma 2 ([1], p. 206). *For all $\gamma \in \mathcal{C}^{\omega}_R(t)$ there exists $\gamma' \in \mathcal{C}^{\omega}_R(t)$ such that γ' is $\leqslant^{\omega}$-maximal (i.e $\gamma' \leqslant^{\omega} \delta \Rightarrow \delta \equiv^{\omega} \gamma'$) and $\gamma \leqslant^{\omega} \gamma'$.*

The set of $\leqslant^{\omega}$-maximal computations will be denoted by $\overline{\mathcal{C}}^{\omega}_R(t)$. Intuitively, if $\gamma \in \overline{\mathcal{C}}^{\omega}_R(t)$, then „nothing remains to do after it".

§ 2.2. **Complete F-algebras and semantics of computations.** A *domain* is a complete partial ordered set, that is a triple $< D, \sqsubseteq, \perp >$ where :

(i) D is the carrier set ;

(ii) $\sqsubseteq \subseteq D \times D$ is a partial order ;

(iii) $\perp \in D$ is the least element ;

(iv) any directed set $X \subseteq D$ has a least upper bound in D (lub), denoted $\bigsqcup X$.

A *complete F-algebra* is a tuple $A = < D, \sqsubseteq, \perp, \{f_A \mid f \in F\} >$ where $< D, \sqsubseteq, \perp >$ is a domain, F is a set of function symbols and for any $f \in F_k$, f_A is a continuous function $f_A : D^k \to D$. If $A = < D, \sqsubseteq, \perp, \{f_A \mid f \in F\} >$ is a complete F-algebra, any mapping $e : X \to D$ may be uniquely extended to e^{ω}: $M^{\omega}_F(X) \to D$. The *interpretation of a term $t \in M^{\omega}_F(X)$ in A* is the mapping $t_A : D^X \to D$ given by $t_A(e) = e^{\omega}(t)$. Now, *the A-semantics* of the language over $F \cup X$ whose *abstract syntax* is $M^{\omega}_F(X)$ is the mapping $\lambda_A : M^{\omega}_F(X) \to D^{D^X}$ given by $\lambda_A(t) = t_A$.

For any F and X we can consider as a particular semantical domain the complete F-algebra $H = < M^{\omega}_F(X), \leqslant, \perp, \{f_H \mid f \in F\} >$, namely the free complete F-algebra generated by $F \cup X$. The semantics λ_H is then the *symbolic semantics.*

Now, having a semantics for $M^{\omega}_F(X)$ we can define semantics for TRS's R and computational semantics for terms. Namely, if A is a complete F-algebra and $R \subseteq \overline{M}_F(X) \times \overline{M}_F(X)$ is a TRS, $\lambda_A : M^{\omega}_F(X) \to D^{D^X}$ is also a semantics for R iff :

(3) $$\lambda_A(\theta) = \theta_A = \perp,\ \textit{for any}\ \theta \in \mathrm{lhs}\ (R).$$

From (3) the following can be proved :

Lemma 3 ([1], p. 211). *If $t \xrightarrow[R]{*} t'$ then $t_A \sqsubseteq t'_A$ and hence $\lambda_A(t) \sqsubseteq \lambda_A(t')$.*

Given $\gamma \in \mathcal{C}^{\omega}_R(t)$ as in (2), the *result of γ in the algebra A* is the element :

$$\mathrm{res}_A(\gamma) = \bigsqcup \{\lambda_A(t_n) \mid n \in N\} .$$

The *computational semantics* of a term t in the TRS R under the interpretation A is given by :

(4) $$\mathrm{Comp}_{< R, A >}(t) = \cup \{\mathrm{res}_A(\gamma) \mid \gamma \in \mathcal{C}^{\omega}_R(t)\} .$$

A TRS R is *confluent* iff

$$(\forall t)\ (\forall t_1)\ (\forall t_2)\ (t \xrightarrow[R]{*} t_1 \wedge t \xrightarrow[R]{*} t_2 \Rightarrow (\exists \hat{t})\ (t_1 \xrightarrow[R]{*} \hat{t} \wedge t_2 \xrightarrow[R]{*} \hat{t})).$$

For confluent TRS's we can define another computational semantics for a term t :

(5) $$\mathrm{Val}_{< R, A >}(t) = \bigsqcup \{\lambda_A(t') \mid t \xrightarrow[R]{*} t'\} .$$

For noetherian (or deterministic) TRS's clearly the two semantics coincide for any term t because $\mathcal{C}_R^{\infty}(t)$ has exactly one element (namely the „normal form" of t). In the next section we prove that (1) remains valid for general confluent term rewriting systems.

§ 3. **Equivalent computational semantics.** Let $\gamma_1, \gamma_2 \in \mathcal{C}_R^{\infty}(t)$ be given by:

$$
(6) \qquad \begin{cases} \gamma_1 : t = t_0^1 \xrightarrow[R]{\sigma_0^1} t_1^1 \xrightarrow[R]{\sigma_1^1} \dots \xrightarrow[R]{\sigma_n^1} t_{n+1}^1 \xrightarrow[R]{t_{n+1}^1} \dots \\ \gamma_2 : t = t_0^2 \xrightarrow[R]{\sigma_0^2} t_1^2 \xrightarrow[R]{\sigma_1^2} \dots \xrightarrow[R]{\sigma_n^2} t_{n+1}^1 \xrightarrow[R]{t_{n+1}^2} \dots \end{cases}
$$

where R is a given confluent TRS.

A central concept for what follows is that of *semantical preorder*. Let $A = \langle D, \sqsubseteq, \perp, \{f_A \mid f \in F\}\rangle$ be a complete F-algebra and γ_1, γ_2 like in (6).

Definition 1. *The computation γ_1 semantically precedes the computation γ_2 (written $\gamma_1 \sqsubseteq_s \gamma_2$) iff $(\forall n \in N)(\exists k \in N)\ (\lambda_A(t_n^1) \sqsubseteq \lambda_A(t_k^2))$.*

In the definition above N is replaced by initial segments of N if any of the computations is finite.

Proposition 1. *The relation „$\sqsubseteq_s$" is a preorder on $\mathcal{C}_R^{\infty}(t)$.*

Proof. Reflexivity and transitivity of „$\sqsubseteq_s$" follows directly from the definition.

Proposition 2. *For any $\gamma_1, \gamma_2 \in \mathcal{C}_R^{\infty}(t)$ there exists $\gamma \in \mathcal{C}_R^{\infty}(t)$ such that $\gamma_1 \sqsubseteq_s \gamma$ and $\gamma_2 \sqsubseteq_s \gamma$.*

Proof. Let $\gamma_1, \gamma_2 \in \mathcal{C}_R^{\infty}(t)$ be given by (6). We can construct $\gamma : t \xrightarrow[R]{*} \bar{t}_1 \xrightarrow[R]{*} \bar{t}_2 \xrightarrow{*} \dots$ using the confluence of R and following the scheme:

$$
\begin{array}{l}
\gamma_1 : t \xrightarrow{\sigma_0^1} t_1^1 \xrightarrow{\sigma_1^1} t_2^1 \xrightarrow{\sigma_2^1} \dots \xrightarrow{\sigma_{n-1}^1} t_n^1 \xrightarrow{\sigma_n^1} t_{n+1}^1 \xrightarrow{\sigma_{n+1}^1} \dots \\
\qquad\qquad \searrow^{*} \qquad \searrow^{*} \\
\qquad\qquad\quad \bar{t}_1 \xrightarrow{*} \bar{t}_2 \xrightarrow{*} \bar{t}_3 \xrightarrow{*} \dots \\
\qquad\qquad \nearrow^{*} \qquad \nearrow^{*} \\
\gamma_2 : t \xrightarrow{\sigma_0^2} t_1^2 \xrightarrow{\sigma_1^2} t_2^2 \xrightarrow{\sigma_2^2} \dots \xrightarrow{\sigma_{n-1}^2} t_n^2 \xrightarrow{\sigma_n^2} t_{n-1}^2 \xrightarrow{\sigma_{n+1}^2} \dots
\end{array}
$$

The result then follows using Lemma 3.

Proposition 3. *There exists a computation $\bar{\gamma} \in \mathcal{C}_R^{\infty}(t)$ such that for any $\gamma \in \mathcal{C}_R^{\infty}(t)$ we have $\gamma \sqsubseteq_s \bar{\gamma}$.*

Proof. The set of occurrences of a term is finite and R is finite. Hence $\mathcal{C}_R^{\infty}(t)$ is a denumerable set, $\mathcal{C}_R^{\infty}(t) = \{\gamma_1, \gamma_2, \dots, \gamma_n, \dots\}$. We shall construct $\bar{\gamma}$ using the confluence of R and a well-known diagonalization scheme:

$$
\begin{array}{l}
\gamma_1 : t \to t_1^1 \to t_2^1 \to t_3^1 \to \dots \to t_{n-1}^1 \to t_n^1 \to \dots \\
\qquad\quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \\
\gamma_2 : t \to t_1^2 \to t_2^2 \to t_3^2 \to \dots \to t_{n-1}^2 \to t_n^2 \to \dots \\
\qquad\quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \\
\gamma_3 : t \to t_1^3 \to t_2^3 \to t_3^3 \to \dots \to t_{n-1}^3 \to t_n^3 \to \dots \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\gamma_n : t \to t_1^n \to t_2^n \to t_3^n \to \dots \to t_{n-1}^n \to t_n^n \to \dots \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{array}
$$

More precisely, from $t \xrightarrow{*} t_1^1$ and $t \xrightarrow{*} t_1^2$, there exists $\bar{t}_1$, such that $t_1^1 \xrightarrow{*} \bar{t}_1$ and $t_1^2 \xrightarrow{*} \bar{t}_1$ and hence $t \xrightarrow{*} \bar{t}_1$. From $t \xrightarrow{*} \bar{t}_1$ and $t \xrightarrow{*} t_1^3$, there exists $\bar{t}_2$ such that $\bar{t}_1 \xrightarrow{*} \bar{t}_2$ and $t_1^3 \xrightarrow{*} \bar{t}_2$ and hence $t \xrightarrow{*} \bar{t}_1 \xrightarrow{*} \bar{t}_2$, a.s.o. The result again follows from Lemma 3.

Corollary. *The computation $\bar{\gamma}$ given by Proposition 3 may be included in a $\leqslant^c$-maximal computation which has the same result in A as $\bar{\gamma}$.*

Proof. By Lemma 2, given $\bar{\gamma} \in \mathcal{C}_R^{\infty}(t)$ there exists $\gamma \in \overline{\mathcal{C}}_R(t)$ with $\bar{\gamma} \leqslant^c \gamma$. Hence, for any $\bar{\delta} \in \mathcal{C}_R(t)$, $\bar{\gamma} = \bar{\delta} : \bar{\gamma}'$, there exists $\delta \in \mathcal{C}_R(t)$ such that $\gamma = \delta : \gamma'$ and $\bar{\delta} \equiv^c \delta$. We have then $\delta : t \xrightarrow{\sigma} \bar{t}_n$, $\delta : t \xrightarrow{\sigma_1} t_k$, $\bar{t}_n = t_k$ and so the following condition is satisfied: $(\forall n \in N)(\exists k \in N)(\lambda_A(\bar{t}_n) = \lambda_A(t_k))$. It follows that $\bar{\gamma} \sqsubseteq_s \gamma$. But from $\gamma \sqsubseteq_s \bar{\gamma}$ we conclude $\mathrm{res}_A(\bar{\gamma}) = \mathrm{res}_A(\gamma)$ and $\gamma \in \overline{\mathcal{C}}_R^{\infty}(t)$.

From this corollary we can consider that the computation $\bar{\gamma}$ given by Proposition 3 is $\leqslant^c$-maximal.

Proposition 4. *If R is a confluent TRS, then for any $t \in \overline{M}_F(X)$ there exist $\bar{\gamma} \in \overline{\mathcal{C}}_R^{\infty}(t)$ such that $\mathrm{res}_A(\bar{\gamma}) = \mathrm{Val}_{\langle R, A\rangle}(t)$.*

Proof. Let $M_1 = \{\lambda_A(\bar{t}_n) \mid n \in N\}$ and $M_2 = \{\lambda_A(t') \mid t \xrightarrow[R]{*} t'\}$, where $\bar{\gamma} : t \to \bar{t}_1 \to \bar{t}_2 \to \dots$ is given by Proposition 3 and is $\leqslant^c$-maximal. From $t \xrightarrow[R]{*} \bar{t}_n$, $m \in N$, we have
$\lambda_A(\bar{t}_n) \in M_2$, $\forall n \in N$ and so $\mathrm{res}_A(\bar{\gamma}) \sqsubseteq \mathrm{Val}_{\langle R, A\rangle}(t)$. For the converse, let us consider any t' such that $\gamma' : t \xrightarrow[R]{*} t'$. By Lemma 2, there exists $\gamma \in \mathcal{C}_R^{\infty}(t)$, $\gamma : t \xrightarrow{*} t_1 \xrightarrow{*} \dots$, such that $\gamma' \leqslant^c \gamma$. By the definition of „$\leqslant^c$", there exists $n \in N$ such that $\lambda_A(t') = \lambda_A(t_n)$. Because $\gamma \sqsubseteq_s \bar{\gamma}$, there exists $k \in N$ such that $\lambda_A(t') = \lambda_A(t_n) \sqsubseteq \lambda_A(\bar{t}_k)$. Taking the lub in the preceding relation we obtain

$$\mathrm{Val}_{\langle R, A\rangle}(t) \sqsubseteq \mathrm{res}_A(\bar{\gamma}).$$

Remark. To prove that condition (1) is true in any complete F-algebra which satisfies (3) it suffices to prove (1) in the free complete F-algebra H. In this algebra the symbolic semantics $\mathrm{res}_H(\gamma)$ is an element of $M_F^{\infty}(X)$ and „$\sqsubseteq$" is the less defined order „$\leqslant$". Because $\mathrm{Comp}_{\langle R, A\rangle}(t) = \cup\,\{\mathrm{res}_A(\gamma) \mid \gamma \in \overline{\mathcal{C}}_R(t)\}$ and we have proved that there exists $\bar{\gamma} \in \overline{\mathcal{C}}_R^{\infty}(t)$ such that $\mathrm{Val}_{\langle R, A\rangle}(t) = \mathrm{res}_A(\bar{\gamma})$, it remains to prove the validity of the condition:

$$(7) \qquad (\forall \gamma \in \overline{\mathcal{C}}_R^{\infty}(t))\ (\mathrm{res}_H(\gamma) = \mathrm{res}_H(\bar{\gamma})).$$

But, from the definition of „$\sqsubseteq_s$" and the properties of $\bar{\gamma}$ (given by Proposition 3) we have $\mathrm{res}_H(\gamma) \leqslant \mathrm{res}_H(\bar{\gamma})$, for any $\gamma \in \mathcal{C}_R^{\infty}(t)$. It remains to show that this relation is not strict.

Theorem 1. *For any $\gamma \in \overline{\mathcal{C}}_R^{\infty}(t)$, the empty tree is not a subtree of $\mathrm{res}_H(\bar{\gamma})$.*

Proof. For any elementary step $t' \xrightarrow{\sigma} t''$ of a computation γ ($\sigma = \{(u, r)\}$, $r \in R$), if $\perp$ is a subtree in $\lambda_H(t')$ then (t/u) is unifiable with lhs (r), $\perp$ being the subtree of l at occurrence u. Thus, a $\perp$ indicates an occurrence where a rule from R could be applied. If $\mathrm{res}_H(\gamma)=(\bar{l})_H$ has the empty tree as a subtree then we could apply to $\bar{l}$ a rule of R and hence γ would not be a $\leqslant^{\omega}$-maximal computation.

Theorem 2. *The condition (1) is valid for any term t, any complete F-algebra A and any term rewriting system R which satisfies* (5).

Proof. Following the remark after Proposition 1, we must only show that for any $\gamma\in\overline{\mathcal{O}}_R^{\infty}(t)$, the relation $\mathrm{res}_H(\gamma)\leqslant\mathrm{res}_H(\bar{\gamma})$ is in fact an equality ($\bar{\gamma}$ denotes the $\leqslant^{\omega}$-maximal computation that includes the computation given by Proposition 3). From the definition of $\leqslant$, the above relation can be strict iff $\perp$ is a subtree in $\mathrm{res}_H(\gamma)$. But, by Theorem 1, this is not possible.

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ON THE INTERRELATIONAL DEPENDENCIES IN DATABASE SYSTEMS

BY

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In the following, we consider the relational model of databases due to Codd [3], who introduced and studied the first dependency, named functional dependency. Let U be a finite and nonempty set. The elements of U are called attributes. For each $A\in U$ a nonempty set — DOM (A) — is given. A tuple (over U) is a mapping $\varphi: U\to\bigcup_{A\in U}\mathrm{DOM}(A)$, such that $\varphi(A)\in\mathrm{DOM}(A)$.

A relation over U, or U-relation is a set of tuples over U. A relation scheme is an object $R[U]$, where R is the name of the relation scheme or relation symbol. A database scheme $D=\{R_1[U_1], \ldots, R_n[U_n]\}$ is a set of relation schemes. A database over D is a mapping that associates each relation scheme $R_i[U_i]$ with a relation r_i over R_i. We refer to $r=(r_1, r_2, \ldots, r_n)$ as the database. In [1] and [2] R. Fagin studied many classes of dependencies. Using the certain first-order sentences, R. Fagin defined the implicational dependencies (ID) and embedded implicational dependencies (EID). Many dependencies are equivalent to a finite set of implicational (or embedded implicational) dependencies. Also Fagin studied the existence of Armstrong relations for these classes of dependencies and a class of interrelational dependencies.

In this paper we discuss a generalization of the Fagin's interrelational dependencies and show the existence of „Armstrong-like" database in the presence of the new dependencies.

For more information, about notations, definitions and results, see [1,4].

A database is relationwise nonempty if every relation in the database is nonempty. In the following, we only consider such databases. Let $r^j=(r_1^j, \ldots, r_n^j)$, $j\in J$ be a nonempty family ($J\neq\varnothing$) of databases over D. The direct product, denoted $\otimes<r^j : j\in J>$, is the database $r=(r_1, \ldots, r_n)$ over D, such that for every $i\in\{1, 2, \ldots, n\}$ we have:

$$(x_1, \ldots, x_{d_i})\in r_i\Leftrightarrow(x_h\in X(r_i^j : j\in J),\ h=\overline{1, d_i}).$$

where d_i is the arity of R_i and X is the Cartesian product.

The domain of the attribute A in the direct product is $X(D_j, j\in J)$, where $D_j=\mathrm{DOM}(A)$ in r^j. Let σ be a sentence of first-order logic, which contains the relation symbols $R_1, R_2, \ldots, R_n$. We say that σ is upward faithful (with respect to the direct product) if whenever $(r^j, j\in J)$ is a nonempty family of databases such that σ holds for every r^j, then σ holds for $\otimes<r^j$.