subtree of t at occurrence u. Thus, a $\perp$ indicates an occurrence where a rule from R could be applied. If $\mathrm{res}_R(\gamma)=(\bar{t})_R$ has the empty tree as a subtree then we could apply to $\bar{t}$ a rule of R and hence γ would not be a $\leqslant^*$-maximal computation.

Theorem 2. *The condition (1) is valid for any term t, any complete F-algebra A and any term rewriting system R which satisfies* (3).

Proof. Following the remark after Proposition 1, we must only show that for any $\gamma\in\overline{\mathcal{C}}^{\infty}_R(t)$, the relation $\mathrm{res}_R(\gamma)\leqslant\mathrm{res}_R(\bar{\gamma})$ is in fact an equality ($\bar{\gamma}$ denotes the $\leqslant^*$-maximal computation that includes the computation given by Proposition 3). From the definition of $\leqslant$, the above relation can be strict iff $\perp$ is a subtree in $\mathrm{res}_R(\gamma)$. But, by Theorem 1 this is not possible.

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Received 15.XI.1987

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Tomul XXXIII, s. I a. Matematică, 1987, fasc. 4

ON THE INTERRELATIONAL DEPENDENCIES IN DATABASE SYSTEMS

BY

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In the following, we consider the relational model of databases due to Codd [3], who introduced and studied the first dependency, named functional dependency. Let U be a finite and nonempty set. The elements of U are called attributes. For each $A\in U$ a nonempty set — DOM (A) — is given. A tuple (over U) is a mapping $\varphi: U\to\bigcup_{A\in U}\mathrm{DOM}\ (A)$, such that $\varphi(A)\in\mathrm{DOM}\ (A)$.

A relation over U, or U-relation is a set of tuples over U. A relation scheme is an object $R[U]$, where R is the name of the relation scheme or relation symbol. A database scheme $D=\{R_1[U_1], \ldots, R_n[U_n]\}$ is a set of relation schemes. A database over D is a mapping that associates each relation scheme $R_i[U_i]$ with a relation r_i over R_i. We refer to $r=(r_1, r_2, \ldots, r_n)$ as the database. In [1] and [2] R. Fagin studied many classes of dependencies. Using the certain first-order sentences, R. Fagin defined the implicational dependencies (ID) and embedded implicational dependencies (EID). Many dependencies are equivalent to a finite set of implicational (or embedded implicational) dependencies. Also Fagin studied the existence of Armstrong relations for these classes of dependencies and a class of interrelational dependencies.

In this paper we discuss a generalization of the Fagin's interrelational dependencies and show the existence of „Armstrong-like" database in the presence of the new dependencies.

For more information about notations, definitions and results, see [1,4].

A database is relationwise nonempty if every relation in the database is nonempty. In the following, we only consider such databases. Let $r^j=(r_1^j, \ldots, r_n^j)$, $j\in J$ be a nonempty family ($J\neq\varnothing$) of databases over D. The direct product, denoted $\otimes<r^j : j\in J>$, is the database $r=(r_1, \ldots, r_n)$ over D, such that for every $i\in\{1, 2, \ldots, n\}$ we have:

$$(x_1, \ldots, x_{d_i})\in r_i \Leftrightarrow (x_h\in X(r_i^j : j\in J),\ h=\overline{1, d_i}),$$

where d_i is the arity of R_i and X is the Cartesian product.

The domain of the attribute A in the direct product is $X(D_j, j\in J)$, where $D_j=\mathrm{DOM}\ (A)$ in r^j. Let σ be a sentence of first-order logic, which contains the relation symbols $R_1, R_2, \ldots, R_n$. We say that σ is upward faithful (with respect to the direct product) if whenever $(r^j, j\in J)$ is a nonempty family of databases such that σ holds for every r^j, then σ holds for $\otimes<r^j$,

$j\in J>$. We say that σ is downward faithful (with respect to direct product). if whenever $(r^j,\ j\in J)$ is a nonempty family of databases. such that σ holds for $\otimes<r^j,\ j\in J>$. then σ holds for every r^j.

The sentence σ is faithful if and only if it is both upward and downward faithful.

Let P be a relation symbol with the arity d. that represents a relation. We consider a set of individual variables. which represent entries in a relation. The formulas of the form $Pz_1\dots z_d$ or of the form $x=y$. where the z_is and x, y are individual variables. are atomic formulas. We call $Pz_1\dots z_d$ a relational formula and $x=y$ an equality. A formula σ is said to be typed. if for every relation symbol P which appears in it. there is a partition of the set of all individual variables from the atomic formulas of the form $Pz_1\dots z_d$ such that whenever $Rx_1\dots x_d$ and $Ry_1\dots y_d$ are two relational formulas. then for every $i\in\{1.\ 2.\ \dots\ d\}$. x_i and y_i belong the same class C_i. We say that x_i is of type i. Moreover. if the equality appears in the formula. then x and y have the same type corresponding to any relation symbol R. The negation may be in σ only for relational formulas. We say that σ is an interrelational typed formula. if it contains at least two relation symbols and is typed. Let $\mathcal{S}$ be a set of sentences.

Let $\Sigma(R_1,\ \dots,\ R_n)$ be a set of first-order sentences $\Sigma(R_1.\ \dots.\ R_n)\subseteq\mathcal{S}$, which contain the relation symbols $R_1.\ R_2.\ \dots.\ R_n$ and $\sigma(R_1.\ \dots.\ R_n)$ be a single first-order sentence. which contains $R_1.\ \dots,\ R_n$. We say that $\Sigma(R_1.\ \dots,\ R_n)$ logically implies $\sigma(R_1.\ \dots.\ R_n)$ or that $\sigma(R_1.\ \dots.\ R_n)$ is a logical consequence of $\Sigma(R_1.\ \dots.\ R_n)$. if whenever r is a database over $D=\{R_1[U_1].\ \dots\ .\ R_n[U_n]\}$. we have :

$$\sigma(R_1.\ \dots.\ R_n)\text{ holds for } r \text{ if } \tau \text{ holds for } r$$

$$(1)\qquad \forall\tau\in\Sigma(R_1,\ \dots.\ R_n).$$

Let $\Sigma^*(R_1,\ \dots,\ R_n)$ be the set of all first-order sentences in $\mathcal{S}$. that are logical consequences of $\Sigma(R_1,\ \dots,\ R_n)$.

The formula $\sigma(R_1,\ \dots,\ R_n)$ is logical consequence with respect to relationwise nonempty databases. if whenever r is a relationwise nonempty database. we have (1).

Let $\Sigma^*_{\text{nonempty}}$ $(R_1.\ \dots.\ R_n)$ be the set of all formulas in $\mathcal{S}$. that are logical consequences of $\Sigma(R_1,\ \dots.\ R_n)$ with respect to relationwise nonempty databases.

We also say r obeys σ instead of „σ holds for r" and r obeys Σ. iff r obeys σ, for every $\sigma\in\Sigma$. The database $r=(r_1.\ \dots.\ r_n)$ over D is called Armstrong database for $\Sigma(R_1.\ \dots.\ R_n)\subseteq\mathcal{S}$. if r obeys $\Sigma^*(R_1.\ \dots,\ R_n)$ and no other sentences in $\mathcal{S}$.

The database $r=(r_1.\ \dots,\ r_n)$ over D is called Armstrong-like database for $\Sigma(R_1,\ \dots,\ R_n)\subseteq\mathcal{S}$. i if r obeys $\Sigma^*_{\text{nonempty}}$ $(R_1.\ \dots.\ R_n)$ and no other sentences in $\mathcal{S}$.

Let $r=(r_1.\ \dots.\ r_n)$ be a database over $D=\{R_1[U_1].\ \dots.\ R_n[U_n]\}$. $U=\bigcup_{i=1}^{n}U_i$, $U=\{A_1.\ \dots,\ A_m\}$ and $\varphi:U\to\bigcup_{i=1}^{m}\mathrm{DOM}\ (A_i)$.

Let $\Phi(x_1.\ \dots.\ x_m)$ be an interrelational typed and quantifier-free formula. where $x_1.\ \dots.\ x_m$ are the individual variables. that appear in atomic formulas from Φ. We assume that $\Phi(x_1.\ \dots.\ x_m)$ contains the relation symbols $R_1.\ \dots\ R_n$.

Let p^j_i be the type of x_i for R_j. $1\leqslant i\leqslant m$. $1\leqslant j\leqslant n$. (x_i appears in $R_jz_1\dots z_d$ on the place p^i_j).

We take $p^j_i=0$ if x_i has no type for R_j. It follows that for every $i\in\{1,\ 2.\ \dots.\ m\}$, there exists $j\in\{1.\ 2.\ \dots.\ n\}$. such that $p^j_i\neq0$.

A variable x_i could have types for many R_j.

Let $p^i_{\beta_{1,i}}.\ \dots.\ p^i_{\beta_{i,q_i}}$. the types of x_i for $R_{\beta_{1,i}}.\ \dots.\ R_{\beta_{i,q_i}}$ respectively ($1\leqslant i\leqslant m$).

We have : $\bigcup_{i=1}^{m}\{\beta_{1,i},\ \dots\ \beta_{i,q_i}\}=\{1.\ 2.\ \dots.\ n\}$, $q_i\geqslant1$. for every $i\in\{1.\ 2,\ \dots,\ m\}$.

Let $r^l=(r^l_1.\ \dots.\ r^l_n)$. $l\in J$. be a nonempty family of databases over $D=\{R_1[U_1].\ \dots.\ R_n[U_n]\}$ and $\varphi^l:U\to\bigcup_{A\in U}\mathrm{DOM}\ (A)$. $U=\bigcup_{i=1}^{n}U_i=\{A_1.\ \dots,\ A_m\}$.

We write $l\in\{1.\ 2.\ \dots.\ h\}$ instead of $l\in J$ in order to simplify our notations. Now. we define $l=1$ and $q_i\in\varphi^l(A_i)$. for every $i\in\{1.\ 2.\ \dots.\ m\}$; $q=q_1q_2\dots q_m$. We select $t^j_i\in r^j_i$. for every $i\in\{1.\ 2,\ \dots.\ n\}$ and $j\in\{2.\ 3.\ \dots.\ h\}$.

If $t^j_i=(t^j_{i1}.\ t^j_{i2}.\ \dots.\ t^j_{iq_i})$. then we define for every $i\in\{1.\ 2.\ \dots\ n\}$:

$$Q^i_{11}=(q_1.\ t^2_{ik}.\ t^3_{ik}.\ \dots.\ t^h_{ik}).\ \text{where } k=p^1_i.\ \text{if } p^1_i\neq0.$$

$$(2)\qquad Q^i_{21}=(q_2.\ t^2_{ik}.\ t^3_{ik}.\ \dots.\ t^h_{ik}).\ \text{where } k=p^2_i.\ \text{if } p^2_i\neq0.$$

$$\vdots$$

$$Q^i_{m1}=(q_m.\ t^2_{ik}.\ t^3_{ik}.\ \dots.\ t^h_{ik}).\ \text{where } k=p^m_i.\ \text{if } p^m_i\neq0.$$

For the same q and $l\in\{2.\ \dots.\ h\}$, we select $t^j_i\in r^j_i$. $i\in\{1.\ 2.\ \dots,\ n\}$. $j\in\{1,\ 2,\ \dots\ h\}-\{l\}$; Q^i_{sl} are defined similar to Q^i_{s1}. with the exception that q_s will be on the position l in Q^i_{sl} ($1\leqslant s\leqslant m$). instead of the position 1.

Consider (3) $\mathscr{M}_{jl}=\{Q^i_{jl}.$ if $p^j_i\neq0$; $1\leqslant i\leqslant n.\quad 1\leqslant j\leqslant m\}$.

The sets $\mathscr{M}_{jl}$. $1\leqslant j\leqslant m$ correspond to l. $q=q_1q_2\dots q_m$ and $t^j_i\in r^j_i$ $1\leqslant i\leqslant n$ $j\in\{1.\ 2.\ \dots.\ h\}-\{l\}$.

We have $Q^i_{jl}\in\otimes<\varphi^s(A_j).\ s\in\{1.\ 2.\ \dots.\ h\}>$. $1\leqslant j\leqslant m$.

For the interrelational typed and cuantifier-free Φ, let S_Φ be the set of all pairs $(\Psi_1.\Psi_2)$. such that Ψ_1 and Ψ_2 are subformulas in Φ and $\Psi_1\wedge\Psi_2$ is also a subformula in Φ.

We say that $\bar r=\otimes<r^s.\ s\in\{1.\ 2.\ \dots.\ h\}>C$-obeys Φ. if for every $(\Psi_1.\ \Psi_2)\in S_\Phi$. for every $l\in\{1.\ 2,\ \dots.\ h\}$. $q=q_1q_2\dots q_m$. $t^j_i\in r^j_i$, we have :

$(\exists Q_1\in\mathscr{M}_{1l}.\ \dots\ \exists Q_m\in\mathscr{M}_{ml})[\Psi_1(Q_1.\ \dots.\ Q_m)$ holds for $\bar r]$. and $(\exists V_1\in\mathscr{M}_{1l}.\ \dots,\ \exists V_m\in\mathscr{M}_{ml})\ [\Psi_2(V_1.\ \dots.\ V_m)$ holds for $\bar r]$ implies

$$(\exists W_1\in\mathscr{M}_{1l}.\ \dots.\ \exists W_m\in\mathscr{M}_{ml})\ [(\Psi_1\wedge\Psi_2)\ (W_1.\ \dots.\ W_m)\text{ holds for }\bar r]$$

If for any i. $1\leqslant i\leqslant m$. Q_i and V_i have the same values in the component l, namely $q_i=v_i$. then we claim that W_i has as l-component the value q_i.

Now. let r be a database over D. and $\varphi:U\to\bigcup_{i=1}^{m}\mathrm{DOM}\ (A_i)$. $q_i\in\varphi(A_i)$, $1\leqslant i\leqslant m$. $q=q_1q_2\dots q_m$. We say. that r C-obeys Φ. if for every $(\Psi_1.\ \Psi_2)\in S_\Phi$, for every $q = q_1 \ldots q_m$ we have: $\{\exists(q_1, q_2, \ldots q_m)\, [\Psi_1(q_1, \ldots, q_m)$ holds for $r]$ and $\exists(V_1, V_2, \ldots V_m)\, [\Psi_2(V_1, \ldots V_m)$ holds for $r]\}$ implies $\exists(W_1, \ldots, W_m)\, [(\Psi_1 \wedge \Psi_2)(W_1, \ldots W_m)$ holds for $r]$. Moreover, if $(\exists i)$, $i \in \{1, 2, \ldots, m\}$, such that $q_i = V_i$, then we claim that $W_i = q_i$. When σ holds for r, we consider $(\Psi_1, \Psi_2) \in S_\Phi$, iff Ψ_1 holds for r and Ψ_2 holds for r.

We assume that there is a class of models, a class of sentences and a relationship HOLDS between these two classes. If σ is a sentence and r is a model, then HOLDS (σ, r) means that σ „holds for" r or that r „obeys" σ.

Theorem 1. [1]. *Let S be a set of first-order sentences. The following properties of S are equivalent:*

(a) *Existence of a faithful operator. There is an operator $\otimes$ that maps nonempty families of models into models, such that if σ is a sentence in S and $<r_i, i \in I>$ is a nonempty family of models, then σ holds for $\otimes <r_i, i \in I>$, iff σ holds for each r_i.*

(b) *Existence of Armstrong models. Whenever Σ is a consistent subset of S and Σ^* is the set of sentences in S that are logical consequences of Σ, then there is a model (an „Armstrong model") that obeys Σ^* and no other sentences in S.*

In our case, we consider as class of models the class of relationwise nonempty databases. We take as operator $\otimes$, the direct product $\otimes$ of databases.

We say that σ C-holds for r, iff σ holds for r and r C obeys σ. We define HOLDS (σ, r) iff σ C-holds for r, hence in theorem 1, „σ holds for r" means σ C-holds for r.

We say that σ is C-upward faithful (with respect to direct product) if whenever $(r^j, j \in J)$ is a nonempty family of databases, such that σ C-holds for every r^j, then σ C-holds for $\otimes <r^j, j \in J>$.

Also, we say that σ is C-downward faithful (with respect to direct product) if whenever $(r^j, j \in J)$ is a nonempty family of databases, such that σ C-holds for $\otimes <r^j, j \in J>$, then σ C-holds for every r^j.

The formula σ is called C-faithful, iff it is C-upward and C-downward faithful.

A generalized implicational interrelational dependence (GID) is an interrelational typed sentence of the form:

$$(\forall x_1 \ldots x_m)\, ((A_1 \wedge \ldots \wedge A_m) \Rightarrow B).$$

where each A_i is a relational formula, B is atomic, each variable x_i appears in at least one of A_j. Also, x_i is universally quantified.

A generalized embedded implicational interrelational dependence (GEID) is an interrelational sentence of the form:

$$(\forall x_1, \ldots, x_m)\, ((A_1 \wedge \ldots \wedge A_n) \Rightarrow (\exists y_1 \ldots y_r)\, (B_1 \wedge \ldots \wedge B_s)).$$

where each A_i is a relational formula, B is atomic and $A_1 \wedge \ldots \wedge A_n$ is interrelational typed. We assume that $n \geqslant 1$, each x_j appears in at least one of A_i and $r \geqslant 0$, $s \geqslant 1$.

Theorem 2. *Let σ be a sentence of the form $(\forall x_1 \ldots x_m)\, \Phi$, where Φ is an interrelational typed quantifier-free formula.*

Then σ is C-downward faithful.

Proof. Let $(r^j, j = \overline{1, h})$ be a nonempty family of databases. We have, σ C-holds for $\otimes <r^j, j = \overline{1, h}> = \bar{r}$. Let us show that σ C-holds for every r^j, $j = 1, h$. We have σ holds for $\otimes <r^j, j = \overline{1, h}>$ and $\bar{r}$ C-obeys σ. We show that σ holds for every r^j, $j = \overline{1, h}$ and every r^j C-obeys σ.

For the first part, we assume the contrary; let r^i be the database for which σ fails. By relabeling (if $i \neq 1$), assume that for r^1 σ fails. Then there exist $q_1, q_2, \ldots, q_m$, such that $\neg\Phi(q_1, \ldots, q_m)$ holds for r^1. Let r^j be $(r_1^j, \ldots, r_n^j)$. We have $r_i^j \neq 0$ by definition. For $l = 1$, $q = q_1 q_2 \ldots q_m$ and $t_i^j \in r_i^j$, $i \in \{1, 2, \ldots, n\}$, $j \in \{2, 3, \ldots, h\}$, we consider $Q_{11}^i, Q_{21}^i, \ldots, Q_{m1}^i$ from (2) and $\mathcal{M}_j = \mathcal{M}_{j1}$ from (3), $1 \leqslant j \leqslant m$.

Let us denote $Q_j^i = Q_{j1}^i$, $1 \leqslant j \leqslant m$, $1 \leqslant i \leqslant n$.

First, we show by induction on the structure of formulas:

$$(4) \quad \begin{gathered} (\exists Q_1 \in \mathcal{M}_1, \ldots, \exists Q_m \in \mathcal{M}_m)\, [\Psi(Q_1, \ldots, Q_m) \text{ holds for } \bar{r}] \\ \text{implies } \Psi(q_1, \ldots, q_m) \text{ holds for } r^1. \end{gathered}$$

where Ψ is a cuantifier-free, interrelational typed formula.

Case a. Ψ is a relational formula $P_i x_{j_1} x_{j_2} \ldots x_{j_{d_i}}$. It follows that x_{jk} has the type k for P_i, $1 \leqslant k \leqslant d_i$. Let us denote $d = d_i$.

Since the left part of (4) is true, there exist

$$Q_{j_1}^i \in \mathcal{M}_{j_1}, \ldots, Q_{j_d}^i \in \mathcal{M}_{j_d}, \text{ such that}$$

$$S = (Q_{j_1}^i, \ldots, Q_{j_d}^i) \text{ is a tuple of } \bar{r}.$$

But $\quad Q_{i_1}^i = (q_{j_1}, t_{i_1}^2, \ldots, t_{i_1}^h)$

$$\vdots$$

$$Q_{i_d}^i = (q_{j_d}, t_{i_d}^2, \ldots, t_{i_d}^h)$$

We have $\quad (t_{i1}^2, \ldots, t_{id}^2) = t_i^2 \in r_i^2$

$$\vdots$$

$$(t_{i1}^h, \ldots, t_{id}^h) = t_i^h \in r_i^h.$$

It follows that:

$$S \text{ is a tuple of } \bar{r} \text{ iff } (q_{j_1}, q_{j_2}, \ldots, q_{j_d}) \in r_i^1.$$

that means Ψ holds for r^1.

Also, it results that (4) is true for $\Psi = \neg P z_1 \ldots z_d$.

Case b. Ψ is $x = y$. There exist i, j, such that $x = x_i$, $y = x_j$. We have:

$$\exists Q_i \in \mathcal{M}_i, \; \exists Q_j \in \mathcal{M}_j \text{ such that } Q_i = Q_j.$$

It follows $q_i = q_j$.

Assume that (4) is true for Ψ_1 and Ψ_2. Let us show that (4) is true for $\Psi = \Psi_1 \wedge \Psi_2$. We have:

$$(\exists Q_1 \in \mathcal{M}_1, \ldots, \exists Q_m \in \mathcal{M}_m)\, [\Psi(Q_1, \ldots, Q_m) \text{ holds for } \bar{r}]$$

implies $\quad \{(\exists Q_1 \in \mathcal{M}_1, \ldots, \exists Q_m \in \mathcal{M}_m)\, [\Psi_1(Q_1, \ldots, Q_m)$ holds for $\bar{r}]$

and $\quad (\exists Q_1 \in \mathcal{M}_1, \ldots, \exists Q_m \in \mathcal{M}_m)\ [\Psi_2(Q_1, \ldots, Q_m)$ holds for $\bar{r}]\}$

implies $\quad \{[\Psi_1(q_1, \ldots, q_m)$ holds for $r^1]$ and

$[\Psi_2(q_1, \ldots, q_m)$ holds for $r^1]\}$ implies

$[\Psi(q_1, q_2, \ldots, q_m)$ holds for $r^1]$.

For $\Psi = \Psi_1 \vee \Psi_2$, (4) is true, if it is true for Ψ_1 or Ψ_2.

Now, we show :

$$[\Psi(q_1, q_2, \ldots, q_m) \text{ holds for } r^1] \text{ implies}$$

(5)

$$(\exists Q_1 \in \mathcal{M}_1, \ldots, \exists Q_m \in \mathcal{M}_m)\ [\Psi(Q_1, \ldots, Q_m) \text{ holds for } \bar{r}]$$

When Ψ is $Pz_1 \ldots z_d$ or $\neg Pz_1 \ldots z_d$ the proof of (5) is similar to that of (4).

Let Ψ be $x = y$. There exist P_i, such that x and y have the same type for P_i, namely p. There exist i_1, i_2 such that $x = x_{i_1}$ and $y = x_{i_2}$. We have $q_{i_1} = q_{i_2}$. Then there are $Q_{i_1} \in \mathcal{M}_{i_1}$, $Q_{i_2} \in \mathcal{M}_{i_2}$ such that $Q_{i_1} = (q_{i_1}, t^2_{ip}, \ldots, t^h_{ip})$

$$Q_{i_2} = (q_{i_2}, t^2_{ip}, \ldots, t^h_{ip}), \text{ hence } Q_{i_1} = Q_{i_2}.$$

Let suppose (5) for Ψ_1 and Ψ_2. To show it for $\Psi = \Psi_1 \wedge \Psi_2$, we can write :

$[\Psi(q_1, \ldots, q_m)$ holds for $r^1]$ implies $\{[\Psi_1(q_1, \ldots, q_m)$

holds for $r^1]$ and $[\Psi_2(q_1, \ldots, q_m)$ holds for $r^1]\}$ implies

$\{(\exists Q_1 \in \mathcal{M}_1, \ldots, \exists Q_m \in \mathcal{M}_m)\ [\Psi_1(Q_1, \ldots, Q_m)$ holds for $\bar{r}]$

and $(\exists V_1 \in \mathcal{M}_1, \ldots, \exists V_m \in \mathcal{M}_m)\ [\Psi_2(Q_1, \ldots, Q_m)$ holds for $\bar{r}]\}$

((5) is true for Ψ_1 and Ψ_2)

implies

$$(\exists T_1 \in \mathcal{M}_1, \ldots, \exists T_m \in \mathcal{M}_m)\ [\Psi(T_1, \ldots, T_m) \text{ holds for } \bar{r}]$$

$$(\bar{r}\ C\text{-obeys } \Psi).$$

(The first component of T_i is q_i, $i = \overline{1, m}$) implies $\Psi(q_1, q_2, \ldots, q_m)$ holds for r^1 (by (4)).

For $\Psi = \Psi_1 \vee \Psi_2$ (5) is true if it is true for Ψ_1 or Ψ_2. From (4) and (5) it results :

$$(\exists Q_1 \in \mathcal{M}_1, \ldots, \exists Q_m \in \mathcal{M}_m)\ [\Psi(Q_1, \ldots, Q_m) \text{ holds for } \bar{r}] \text{ iff}$$

(6)

$$[\Psi(q_1, \ldots, q_m) \text{ holds for } r^1]$$

The relation (6) is true for Φ instead of Ψ. Since $[\neg\Phi(q_1, \ldots, q_m)$ holds for $r^1]$, it follows $(\forall Q_1 \in \mathcal{M}_1, \ldots \forall Q_m \in \mathcal{M}_m)\ [\neg\Phi_1(Q_1, \ldots, Q_m)$ holds for $\bar{r}]$.

But this contradicts that $(\forall x_1 \ldots x_m)(\Phi(x_1, \ldots, x_m))$ holds for $\bar{r}$. We also must show :

(7) $\quad$ every r^j C-obeys σ if $\bar{r}$ C-obeys σ.

We consider $j = 1$. For $j = \overline{2, h}$ the proof is analogous.

$$(\exists q_1, \ldots, q_m)\ [\Psi_1(q_1, \ldots, q_m) \text{ holds for } r^1] \text{ and}$$

$$(\exists v_1, \ldots, v_m)\ [\Psi_2(v_1, \ldots, v_m) \text{ holds for } r^1].$$

imply (by (5))

$$\exists Q_1 \in \mathcal{M}_1, \ldots \exists Q_m \in \mathcal{M}_m)\ [\Psi_1(Q_1, \ldots, Q_m) \text{ holds for } \bar{r}] \text{ and}$$

$$(\exists V_1 \in \mathcal{M}_1, \ldots, \exists V_m \in \mathcal{M}_m)\ [\Psi_2(V_1, \ldots, V_m) \text{ holds for } \bar{r}].$$

which implyes ($\bar{r}$ C-obeys σ)

$$(\exists W_1 \in \mathcal{M}_1, \ldots, \exists W_m \in \mathcal{M}_m)\ [(\Psi_1 \wedge \Psi_2)(W_1, \ldots, W_m) \text{ holds for } \bar{r}]$$

and which implies (by (4)) :

$$(\Psi_1 \wedge \Psi_2)(z_1, \ldots, z_m) \text{ holds for } r^1,$$

where z_i is the first component if W_i, $1 \leqslant i \leqslant m$.

When $q_i = v_i$, then because the first component of Q_i is q_i, and the first component of V_i is v_i, we have the first component of W_i is q_i, hence $z_i = q_i$. This completes the proof.

Theorem 3. *Let σ be a sentence of the form $(\forall x_1 \ldots x_m)\ \Phi(x_1, \ldots, x_m)$, where Φ is an interrelational typed quantifier-free formula. Then σ is C-upward faithful.*

Proof. It is known that σ is upward faithful (theorem 2.4 [1]). It remains to show :

$$(\sigma\ C\text{-holds for every } r^i,\ i \in \{1, 2, \ldots, h\})$$

$$\text{implies}$$

(8)

$$\bar{r} = \otimes < r^i,\ i = \overline{1, h} > C\text{-obeys } \sigma$$

We take $l = 1$. For $l = \overline{2, h}$ the proof is similar.

It results :

$\{(\exists Q_1 \in \mathcal{M}_{1l}, \ldots, \exists Q_m \in \mathcal{M}_{ml})\ [\Psi_1(Q_1, \ldots, Q_m)$ holds for $\bar{r}]$,

and

$(\exists V_1 \in \mathcal{M}_{1l}, \ldots, \exists V_m \in \mathcal{M}_{ml})\ [\Psi_2(V_1, \ldots, V_m)$ holds for $\bar{r}]\}$,

implies

$\{(\Psi_1(q_1, \ldots, q_m)$ holds for $r^1)$ and

$(\Psi_2(v_1, \ldots, v_m)$ holds for $r^1)\}$ implies

$$(\exists w_1, \ldots, w_m)\ [(\Psi_1 \wedge \Psi_2)(w_1, \ldots, w_m) \text{ holds for } r^1].$$

Because Φ holds for every r^i, $i = \overline{2, h}$, we have Ψ_1 holds for every r^i and Ψ_2 holds for every i, for $(\Psi_1, \Psi_2) \in S_\Phi$ (by the assumption on S_Φ).

Then we have for each $i \in \{2, \ldots, h\}$

$\exists(z_{1_i}, \ldots, z_{m_i})\ [(\Psi_1 \wedge \Psi_2)(z_{1i}, \ldots, z_{mi})$ holds for $r^i]$

Let us define :

$$\varphi^1(A_i) = w_i,\ \varphi^j(A_i) = z_{ij},\ j = \overline{2, h},\ 1 \leqslant i \leqslant m.$$

$$r^j_0 = (r^j_{01}, \ldots, r^j_{0n}),\ j = \overline{1, h}$$

where r^j_{0k} contains one tuple with values w_k, for $j = 1$ and z_{ij} for $j > 1$.

We denote $W_i = (w_i, z_{i2}, \ldots, z_{ih})$ $1 \leqslant i \leqslant m$.

Because $\Psi_1 \wedge \Psi_2$ is upward faithful, we have:

$$(\Psi_1 \wedge \Psi_2)(W_1, \ldots, W_m) \text{ holds for } \otimes < r_0^j,\ j = \overline{1, h} > .$$

This implies

$$(\Psi_1 \wedge \Psi_2)(W_1, \ldots, W_m) \text{ holds for } \otimes < r^j,\ j = \overline{1, h} > .$$

If Q_i and V_i have the same values in the first component, it results that $q_i = v_i$, hence $w_i = q_i$.

Corollary 4. *Let σ be a sentence of the form $(\forall x_1, \ldots x_m)\Phi(x_1, \ldots, x_m)$, where Φ is an interrelational typed quantifier-free formula. Then σ is C-faithful.*

The proof results from theorems 2 and 3.

Theorem 5. *Let σ be a sentence of the form:*

$$(\forall x_1, \ldots x_m)\ (\Phi \Rightarrow (\exists y_1 \ldots y_r)\gamma) .$$

where Φ is an interrelational typed quantifier-free formula, γ is a positive quantifier-free formula. Moreover, Φ does not contain atomic formula of the form $x = y$. Then σ is C-downward faithful.

Proof. Let $(r^j,\ j = \overline{1, h})$ be a nonempty family of databases. We have σ C-holds for $\otimes < r^j,\ j = \overline{1, h} > = \bar{r}$. To show that σ C-holds for every r^j, $j = \overline{1, h}$. We have:

(9) σ holds for $\bar{r}$ and $\bar{r}$ C-obeys σ.

We must show:

(10) σ holds for every r^j, $j = \overline{1, h}$ and every r^j C-obeys σ.

We proceed as in the proof of Theorem 2. We assume not, (for the first part of (10)). Assume that σ fails for r^1. There exist $q_1, q_2, \ldots, q_m$, such that $\Phi(q_1, \ldots, q_m)$ holds for r^1 and there are no values $s_1, \ldots, s_r$ for which $\gamma(q_1, \ldots, q_m, s_1, \ldots s_r)$ holds for r^1. With the same notations as in Theorem 2, taking Φ instead of Ψ in (5) we have:

$$(\exists Q_1 \in \mathcal{M}_1, \ldots, \exists Q_m \in \mathcal{M}_m)\ [\Phi(Q_1, \ldots, Q_m) \text{ holds for } \bar{r}]$$

iff

(11) $\Phi(q_1, \ldots, q_m)$ holds for r^1

where the first component of Q_i is q_i, $1 \leqslant i \leqslant m$.

It follows

(12) $(\exists Q_1 \in \mathcal{M}_1, \ldots, \exists Q_m \in \mathcal{M}_m)\ [\Phi(Q_1, \ldots, Q_m) \text{ holds for } \bar{r}]$

Since σ holds for $\bar{r}$, from (12) we have:

(13) $\exists(S_1, \ldots, S_r)\ [\gamma(Q_1, \ldots, Q_m, S_1, S_2, \ldots, S_r) \text{ holds for } \bar{r}]$.

Let s_i be the first component of S_i, $1 \leqslant i \leqslant r$. As in the proof of Theorem 2, we have:

(14)
$$[\gamma(q_1, \ldots, q_m, s_1, \ldots, s_r) \text{ holds for } r^1] \text{ if}$$
$$[\gamma(Q_1, \ldots, Q_m, S_1, \ldots S_r) \text{ holds for } \bar{r}] .$$

From (13) and (14) we obtain:

$\gamma(q_1, \ldots, q_m, s_1, \ldots, s_r)$ holds for r^1, which is a contradiction.

The second part of (10) follows as in Theorem 2.

Theorem 6. *Let σ be a sentence of the form*

$$(\forall x_1 \ldots x_m)\ (\Phi \Rightarrow (\exists y_1 \ldots y_r)\gamma),$$

where Φ is an interrelational typed quantifier-free formula, γ is a positive quantifier-free formula.

Moreover, Φ contains no formula of the form $x = y$. Then σ is C-upward faithful.

Proof. The fact that σ is upward faithful results as in the proof of Theorem 2.1 [1] and the fact that $\bar{r} = \otimes < r^j,\ j = \overline{1, h} >$ C-obeys σ follows as in the proof of Theorem 3.

Let $\Sigma(R_1, \ldots, R_n)$ be a set of first-order sentences which contain the relation symbols $R_1, \ldots, R_n$ and $\sigma(R_1, \ldots, R_n)$ be a single sentence. We say that σ is C-logical consequence of Σ if every database that C-holds Σ also C-holds σ. We say that σ is C-logical consequence of Σ with respect to relationwise nonempty databases, if every such database which C-holds Σ, also C-holds σ.

Corollary 7. *Let σ be a sentence of the form in Theorem 5. Then σ is C-faithful.*

The proof follows from Theorem 5 and 6.

Theorem 8. *Let $\Sigma(R_1, \ldots, R_n)$ be a set of GIDs, and let $\Sigma_{nonempty}(R_1, \ldots, R_n)$ be the set of GIDs that are C-logical consequences of Σ with respect to relationwise nonempty databases. Then there exists a database that C-holds Σ and no other GIDs over $R_1, \ldots, R_n$.*

Proof. In Theorem 1, let S be the set of all GIDs, over $R_1, \ldots, R_n$, let a „model" be a relationwise nonempty database, and $\otimes$ be the direct product and HOLDS was precised. By Corollary 4, we have (a) in Theorem 1. This implies (b).

Theorem 9. *Let $\Sigma(R_1, \ldots, R_n)$ be a set of GEIDs and let $\Sigma_{nonempty}(R_1, \ldots, R_n)$ be the set of GEIDs that are C-logical consequences of Σ with respect to relationwise nonempty databases. Then there is a database that C-holds Σ and no other GEIDs over $R_1, \ldots, R_n$.*

The proof is similar to that of Theorem 8 using Corollary 7 instead of Corollary 4.

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Received 15.XI.1987

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