ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
Tomul XXXIV, s. I a. Matematică-Informatică, 1988, f. 4

SUMAR

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA"
Tomul XXXIV, s. I a. Matematică-Informatică, 1988, f. 4

SUFFICIENT CONDITIONS FOR THE DECIDABILITY OF $s \xrightarrow{*} t$

BY

T. JUCAN, C. MASALAGIU, F. ŢIPLEA

0. Introduction. The aim of this paper is to study the decidability of the problem $s \xrightarrow{*} t$ in term rewriting systems; it has an essential role for finding „optimum" derivations.

1. Definitions and Notations. The general framework follows the line of [4]. For omitted concepts and notations, we refer also to [10].

By $M(F, V)$ we denote the set of *first-order terms* constructed starting from a set V of *variables* and a set $F = \bigcup_{i \geq 0} F_i$ of *functional symbols*. When F and V are known from the context, $M(F, V)$ will be denoted by $\mathcal{T}$. The free monoid over the set of natural numbers, N, will be denoted by N^*, λ being the empty word.

For any term t, we define *its set of occurrences* as a finite subset $O(t)$ of the set N^*, by:

(i) if $t \in V \cup F_0$ then $O(t) = \{\lambda\}$

(ii) if $t = f(t_1, \ldots, t_n)$, $n \geq 1$, then $O(t) = \{\lambda\} \cup \{i \, . \, u / 1 \leq i \leq n, \ u \in O(t_i)\}$.

If $u \in O(t)$, we inductively define *the subterm of t at u*, as the term t/u:

(i) $t/\lambda = t$,

(ii) if $t = f(t_1, \ldots, t_n)$, $n \geq 1$, and $u = i \, . \, u'$, $1 \leq i \leq n$, then $t/u = t_i/u'$.

If $t \in \mathcal{T}$ and $u \in O(t)$ then, for every term s, *the replacement in t of s at u* is the term $t[u \leftarrow s]$, defined by:

(i) $t[\lambda \leftarrow s] = s$,

(ii) if $t = f(t_1, \ldots, t_n)$, $n \geq 1$, and $u = i \, . \, u'$, $1 \leq i \leq n$, then $t[i \, . \, u' \leftarrow s] = f(t_1, \ldots, t_i[u' \leftarrow s], \ldots, t_n)$.

We consider the following functions:

(i) $v : \mathcal{T} \to \mathcal{P}(V)$

$$v(t) = \begin{cases} \emptyset, & \text{if } t \in F_0, \\ t, & \text{if } t \in V, \\ \bigcup_{i=1}^{n} v(t_i), & \text{if } t = f(t_1, \ldots, t_n), \end{cases}$$

$v(t)$ is called *the set of variable symbols occuring in t*;

(ii) $\mathcal{F} : \mathcal{T} \to \mathcal{P}(F)$

$$\mathcal{F}(t) = \begin{cases} \{t\}, & \text{if } t \in F_0, \\ \varnothing, & \text{if } t \in V \\ \{f\} \cup \bigcup_{i=1}^{n} \mathcal{F}(t_i), & \text{if } t = f(t_1, \ldots, t_n). \end{cases}$$

$\mathcal{F}(t)$ is called *the set of functional symbols occuring in t*;

(iii) $|\cdot| : \mathcal{T} \to N$,

$$|t| = \begin{cases} 1, & \text{if } t \in F_0 \cup V, \\ 1 + \sum_{i=1}^{n} |t_i|, & \text{if } t = f(t_1, \ldots, t_n), \end{cases}$$

$|t|$ is called *the length of the term t*;

(iv) $\# : (F \cup V) \times \mathcal{T} \to N$ by

$$\#(\alpha, t) = \begin{cases} 0, & \text{if } \alpha \in F_0 \cup V \text{ and } t \in F_0 \cup V \text{ and } \alpha \neq t \\ & \text{or } \alpha \in F - F_0 \text{ and } t \in F_0 \cup V, \\ 1, & \text{if } \alpha \in F_0 \cup V \text{ and } t \in F_0 \cup V \text{ and } \alpha = t, \\ \sum_{i=1}^{n} \#(\alpha, t_i), & \text{if } \alpha \in F - F_0,\ t = f(t_1, \ldots, t_n), \\ & \text{and } \alpha \neq t, \\ 1 + \sum_{i=1}^{n} \#(\alpha, t_i), & \text{if } \alpha \in F - F_0,\ t = f(t_1, \ldots, t_n), \\ & \text{and } \alpha = f \end{cases}$$

$\#(\alpha, t)$ is called *the number of occurrences of α in t*.

The *domain* of a substitution $\sigma : \mathcal{T} \to \mathcal{T}$ will be denoted by Dom (σ). Let $\Sigma = \{\sigma \mid \sigma : \mathcal{T} \to \mathcal{T}, \sigma \text{ a substitution}\}$ and let $\varepsilon \in \Sigma$ be the identity on $\mathcal{T}$.

A term rewriting system (TRS) is a finite set of pairs of terms, called rules, $R = \{(\alpha_i, \beta_i) / 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ such that for $(\alpha, \beta) \in R$, $v(\beta) \subseteq v(\alpha)$. For $1 \leqslant i \leqslant n$, the rule (α_i, β_i) is usually denoted by $\alpha_i \to \beta_i$ or r_i: $\alpha_i \to \beta_i$ or r_i; α_i is its *left-hand side* (lhs (r_i)) and β_i is its *right-hand side* (rhs (r_i)). The sum $|r_i| = |\alpha_i| + |\beta_i|$ is called *the length* of the rule $r_i : \alpha_i \to \beta_i$. Let $\mathcal{R}$ be the class of TRS's. For two terms, s and t, we say that *s is rewritten into t in R* (in one step), $s \underset{R}{\to} t$, if there exist $u \in O(s)$, $\alpha_i \to \beta_i \in R$ and $\sigma \in \Sigma$ such that $s/u = \sigma(\alpha_i)$ and $t = s[u \leftarrow \sigma(\beta_i)]$. Occasionally, we keep track of the information by writing $s \xrightarrow[{[u,\, r_i,\, \sigma]R}]{} t$. As usual, $\overset{+}{\underset{R}{\to}}$ is the transitive, and $\overset{*}{\underset{R}{\to}}$ the reflexive and transitive closure of $\underset{R}{\to}$. A TRS R is *regular* iff for every $r : \alpha \to \beta \in R$, $v(\alpha) = v(\beta)$.

A *R-derivation from s* is a sequence (possibly infinite) of steps:

$$\gamma : s \xrightarrow[{[u_1,\, r_{i_1},\, \sigma_1]R}]{} s_1 \to \ldots \xrightarrow[{[u_m,\, r_{i_m},\, \sigma_m]R}]{} s_m \to \ldots$$

and a *R-derivation from s to t* is a finite sequence

$$\gamma : s \xrightarrow[{[u_1,\, r_{i_1},\, \sigma_1]R}]{} s_1 \to \ldots \xrightarrow[{[u_m,\, r_{i_m},\, \sigma_m]R}]{} s_m = t.$$

Let $\mathcal{D}_R^{\infty}(s)$ be the set of R-derivations from s, $\mathcal{D}_R(s)$ be the set of R-derivations from s of finite length (*the length of the finite R-derivation* γ, denoted $|\gamma|$, equals its number of steps) and $\mathcal{D}_R(s,t)$ be the set of R-derivations from s to t.

For $\gamma \in \mathcal{D}_R(s)$, $(\gamma \in \mathcal{D}_R(s, t))$, we shall denote by *rule* (γ) the set of rules used in γ. For $R \in \mathcal{R}$, γ a finite R-derivation and $\mathcal{D}$ a set of such γ, occ (r,γ) will denote the number of occurrences of $r \in R$ in γ and occ $(r,\mathcal{D})$ the number $\sum_{\gamma \in \mathcal{D}}$ occ (r, γ).

2. Substitution closed balance TRS's. In a forthcoming paper we intend to define and study some optimization problems in term rewriting systems, of the following type:

given $R \in \mathcal{R}$, $s, t \in \mathcal{T}$, find a R-derivation from s to t, of „minimum cost", if there exist such R-derivations.

In connection with these kinds of problems, the question if $s \overset{*}{\to} t$ is decidable or, at least, partially decidable becomes of principal interest. Generally, the answer to this question is negative [4], [5].

In the following, the omitted proofs can be found in [10].

Definition 2.1. *Let $R = \{\alpha_i \to \beta_i / 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ be a TRS.*

(i) *We call the balance of the rule $r_i : \alpha_i \to \beta_i \in R$ the number $B(r_i) = |\alpha_i| - |\beta_i|$.*

(ii) *We say that R is with substitution closed balance iff for every $r_i \in R$ and every $\sigma \in \Sigma$, $B(r_i) = B(\sigma(r_i))$ (where $B(\sigma(r_i)) = |\sigma(\alpha_i)| - |\sigma(\beta_i)|$).*

In the following, we shall denote by $\mathcal{R}_{scb}$ the class of the term rewriting systems with substitution closed balance and we shall prove some properties of these types of TRS's.

Proposition 2.2. *Let $R \in \mathcal{R}$. Then, $R \in \mathcal{R}_{scb}$ iff for every $r \in R$ and every $x \in V$, $\#(x, \text{lhs}(r)) = \#(x, \text{rhs}(r))$.*

Corollary 2.3. *If $R \in \mathcal{R}_{scb}$, then R is regular.*

Proof. From Proposition 2.2.

Proposition 2.4. *Let $R \in \mathcal{R}_{scb}$ and $s, t \in \mathcal{T}$ be such that $s \overset{*}{\underset{R}{\to}} t$. Then, for every $x \in V$, $\#(x, s) = \#(x, t)$.*

Theorem 2.5. *Let $R = \{\alpha_i \to \beta_i / 1 \leqslant i \leqslant n,\ n \geqslant 1\} \in \mathcal{R}_{scb}$ and $s \in \mathcal{T}$. Then, for every $t \in \mathcal{T}$ such that $\mathcal{D}_R(s, t) \neq \varnothing$ and every $\gamma \in \mathcal{D}_R(s, t)$ we have:*

$$\sum_{i=1}^{n} \text{occ}\,(r_i, \gamma)\, B(r_i) = |s| - |t|. \tag{1}$$

Proof. Let $R = \{\alpha_i \to \beta_i / 1 \leqslant i \leqslant n,\ n \geqslant 1\} \in \mathcal{R}_{scb}$ and $s, t \in \mathcal{T}$ be such that $\mathcal{D}_R(s, t) \neq \varnothing$ and $\gamma \in \mathcal{D}_R(s, t)$. We prove the statement of the theorem by induction on $k=|\gamma|$. We may assume, without loss of generality, that $k\geq 1$.

Basis. $\gamma: s\underset{R}{\rightarrow}t$ $(k=1)$. Then, there exist $u\in O(s)$, $r_i:\alpha_i\rightarrow\beta_i\in R$ and $\sigma\in\Sigma$ such that :

(i) $s/u=\sigma(\alpha_i)$,

(ii) $t=s[u\rightarrow\sigma(\beta_i)]$.

Moreover, $\mathrm{occ}(r_i,\gamma)=1$ and $\mathrm{occ}(r_j,\gamma)=0$ for every $j\neq i$. Then, $\sum_{i'=1}^{n}\mathrm{occ}(r_{i'},\gamma)B(r_{i'})=B(r_i)$. From (i) and (ii) we deduce that $|s|-|t|=(|s|-|s/u|+|\sigma(\alpha_i)|)-(|s|-|s/u|+|\sigma(\beta_i)|)=|\sigma(\alpha_i)|-|\sigma(\beta_i)|=B(\sigma(r_i))=B(r_i)$ since $R\in\mathcal{R}_{scb}$. Hence, (1) is true.

Induction step. Assume that for every $s'\in\mathcal{T}$ such that $\mathcal{D}_R(s,s')\neq\emptyset$ and every $\gamma'\in\mathcal{D}_R(s,s')$ such that $1<|\gamma'|=l\leq k$, we have :

$$\sum_{i=1}^{n}\mathrm{occ}\,(r_i,\gamma')\,B(r_i)=|s|-|s'|. \tag{2}$$

Let $t\in\mathcal{T}$ be such that $\mathcal{D}_R(s,t)\neq\emptyset$ and $\gamma\in\mathcal{D}_R(s,t)$ be such that $|\gamma|=k+1$. Then, there exists $s'\in\mathcal{T}$ such that

$$\gamma: s\overset{*}{\underset{R}{\rightarrow}}s'\underset{R}{\rightarrow}t.$$

Let γ' be the computation $\gamma': s\overset{*}{\underset{R}{\rightarrow}}s'$, and since $|\gamma'|=k$ relation (2) is true by the induction hypothesis. For the derivation $\gamma'': s'\underset{R}{\rightarrow}t$ we have :

$$B(r_j)=|s'|-|t| \tag{3}$$

where r_j is the rule used in this derivation. We have

$$\mathrm{occ}(r_i,\gamma)=\begin{cases}\mathrm{occ}\,(r_i,\gamma')+1, & \text{if } i=j,\\ \mathrm{occ}\,(r_i,\gamma'), & \text{if } i\neq j,\end{cases}$$

We combine (2) and (3) to deduce :

$$\sum_{i=1}^{n}\mathrm{occ}\,(r,\gamma)\,B(r_i)=|s|-|t|.$$

This completes the proof of the theorem.

Corollary 2.6. *Let $R=\{\alpha_i\rightarrow\beta_i/1\leq i\leq n,\ n\geq 1\}\in\mathcal{R}_{scb}$ and $s\in\mathcal{T}$. Then, for every $t\in\mathcal{T}$,* if $s\overset{*}{\underset{R}{\rightarrow}}t$ *there exist $c_1,\ldots,c_n\in N$ such that :*

$$\sum_{i=1}^{n}c_iB(r_i)=|s|-|t|. \tag{4}$$

Proof. Let $R=\{\alpha_i\rightarrow\beta_i/1\leq i\leq n,\ n\geq 1\}\in\mathcal{R}_{scb}$ and $s,t\in\mathcal{T}$ be such that $s\overset{*}{\underset{R}{\rightarrow}}t$. Then $\mathcal{D}_R(s,t)\neq\emptyset$ and for every $\gamma\in\mathcal{D}_R(s,t)$ we have :

$$\sum_{i=1}^{n}c_iB(r_i)=|s|-|t|.$$

where $c_i=\mathrm{occ}\,(r_i,\gamma)$, $1\leq i\leq n$. Hence there exist $c_1,\ldots,c_n\in N$ such that (4) holds.

Let $R\in\mathcal{R}$ and $s\in\mathcal{T}$. Let $F(R\,;\,s)$ denote *the set of functional symbols occuring in R and s* :

$$F(R\,;\,s)=\mathcal{F}(s)\cup\bigcup_{r\in R}\mathcal{F}(\mathrm{lhs}\,(r))\cup\bigcup_{r\in R}\mathcal{F}(rhs\,(r)).$$

Definition 2.7. *Let $R=\{\alpha_i\rightarrow\beta_i/1\leq i\leq n,\ n\geq 1\}$ and $f\in F$. We call the balance of f in the rule r_i, $1\leq i\leq n$, the number $B_i(f)=\#(f,\alpha_i)-\#(f,\beta_i)$.*

Proposition 2.8. *Let $R=\{\alpha_i\rightarrow\beta_i/1\leq i\leq n,\ n\geq 1\}\in\mathcal{R}_{scb}$ and $f\in F$. Then, for every $\sigma\in\Sigma$ and every $i\in\{1,\ldots,n\}$, $B_i(f)=\#(f,\sigma(\alpha_i))-\#(f,\sigma(\beta_i))$.*

Theorem 2.9. *Let $R=\{\alpha_i\rightarrow\beta_i/1\leq i\leq n,\ n\geq 1\}\in\mathcal{R}_{scb}$, $s\in\mathcal{T}$ and $f\in F(R\,;\,s)$. Then, for every $t\in\mathcal{T}$ such that $\mathcal{D}_R(s,t)\neq\emptyset$ and every $\gamma\in\mathcal{D}_R(s,t)$ we have :*

$$\sum_{i=1}^{n}\mathrm{occ}\,(r_i,\gamma)\,B_i(f)=\#(f,s)-\#(f,t). \tag{5}$$

Proof. Let $R=\{\alpha_i\rightarrow\beta_i/1\leq i\leq n,\ n\geq 1\}\in\mathcal{R}_{scb}$, $s,t\in\mathcal{T}$ be such that $\mathcal{D}_R(s,t)\neq\emptyset$, $f\in F(R\,;\,s)$ and $\gamma\in\mathcal{D}_R(s,t)$. We prove the theorem by induction on $k=|\gamma|$. We may assume, without loss of generality, that $k\geq 1$.

Basis. $\gamma: s\underset{R}{\rightarrow}t$ $(k=1)$. Then, there exist $u\in O(s)$, $r_i:\alpha_i\rightarrow\beta_i\in R$ and $\sigma\in\Sigma$ such that :

(i) $s/u=\sigma(\alpha_i)$,

(ii) $t=s[u\rightarrow\sigma(\beta_i)]$.

There are three cases to consider :

a) if $\#(f,s)=0$, then $\#(f,\sigma(\alpha_i))=0$. We have $\#(f,t)>0$, iff $\#(f,\sigma(\beta_i))>0$, and in this case $\#(f,t)=\#(f,\sigma(\beta_i))$. Hence, $\#(f,s)-\#(f,t)=\#(f,\sigma(\alpha_i))-\#(f,\sigma(\beta_i))=\#(f,\alpha_i)-\#(f,\beta_i)$ (by Proposition 2.8) ;

b) if $\#(f,s)>0$ and $\#(f,s/u)=0$, then $\#(f,\sigma(\alpha_i))=0$ and $\#(f,t)=\#(f,s)-\#(f,s/u)+\#(f,\sigma(\beta_i))$. Hence, $\#(f,s)-\#(f,t)=\#(f,\sigma(\alpha_i))-\#(f,\sigma(\beta_i))=\#(f,\alpha_i)-\#(f,\beta_i)$ (by Proposition 2.8);

c) if $\#(f,s)>0$ and $\#(f,s/u)>0$, then $\#(f,\sigma(\alpha_i))=\#(f,s/u)$ and $\#(f,t)=\#(f,s)-\#(f,s/u)+\#(f,\sigma(\beta_i))=\#(f,s)-\#(f,\sigma(\alpha_i))+\#(f,\sigma(\beta_i))$. By Proposition 2.8 we deduce that $\#(f,s)-\#(f,t)=\#(f,\alpha_i)-\#(f,\beta_i)$.

Since $\mathrm{occ}\,(r_i,\gamma)=1$ and $\mathrm{occ}\,(r_j,\gamma)=0$ for every $j\neq i$, we have $\sum_{i'=1}^{n}\mathrm{occ}(r_{i'},\gamma)B_{i'}(f)=B_i(f)=|s|-|t|$ (from the cases a, b, c).

Induction step. Suppose that for every $s'\in\mathcal{T}$ such that $\mathcal{D}_R(s,s')\neq\emptyset$ and every $\gamma'\in\mathcal{D}_R(s,s')$ such that $1<l=|\gamma'|\leq k$, we have :

$$\sum_{i=1}^{n}\mathrm{occ}\,(r_i,\gamma')\,B_i(f)=\#(f,s)-\#(f,s'). \tag{6}$$

Let $t \in \mathcal{T}$ be such that $\mathcal{D}_R(s, t) \neq \emptyset$ and $\gamma \in \mathcal{D}_R(s, t)$ be such that $|\gamma| = k+1$. Then, there exists $s' \in \mathcal{T}$ such that:

$$\gamma : s \overset{*}{\underset{R}{\to}} s' \underset{R}{\to} t.$$

Let γ' be the derivation $\gamma' : s \overset{*}{\underset{R}{\to}} s'$, and since $|\gamma'| = k$, relation (6) is true by the induction hypothesis. For the derivation $\gamma'' : s' \underset{R}{\to} t$ we have:

$$B_j(f) = \#(f, s') - \#(f, t), \tag{7}$$

where r_j is the rule used in this derivation.
Since

$$\mathrm{occ}\,(r_i, \gamma) = \begin{cases} \mathrm{occ}\,(r_i, \gamma') + 1, & \text{if } i = j, \\ \mathrm{occ}\,(r_i, \gamma'), & \text{if } i \neq j, \end{cases}$$

by combining (5) and (7) we deduce that

$$\sum_{i=1}^{n} \mathrm{occ}\,(r_i, \gamma)\, B_i(f) = |s| - |t|,$$

which completes the proof of the theorem.

Corollary 2.10. *Let* $R = \{\alpha_i \to \beta_i / 1 \leq i \leq n,\ n \geq 1\} \in \mathcal{R}_{scb}$ *and* $s \in \mathcal{T}$. *Then, for every* $t \in \mathcal{T}$, *if* $s \overset{*}{\underset{R}{\to}} t$ *there exist* $c_1, \ldots, c_n \in N$ *such that for every* $f \in F(R\,;\, s)$ *we have:*

$$\sum_{i=1}^{n} c_i B_i(f) = \#(f, s) - \#(f, t). \tag{8}$$

Proof. Let $R = \{\alpha_i \to \beta_i / 1 \leq i \leq n,\ n \geq 1\} \in \mathcal{R}_{scb}$, $s, t \in \mathcal{T}$ be such that $s \overset{*}{\underset{R}{\to}} t$ and $f \in F(R\,;\, s)$. Then, $\mathcal{D}_R(s, t) \neq \emptyset$ and from Theorem 2.9, for every $\gamma \in \mathcal{D}_R(s, t)$ we have:

$$\sum_{i=1}^{n} \mathrm{occ}\,(r_i, \gamma)\, B_i(f) = \#(f, s) - \#(f, t). \tag{9}$$

Let $c_i = \mathrm{occ}\,(r_i, \gamma)$, $1 \leq i \leq n$, and since the c_i's do not depend on f, we deduce that (9) holds, for every $f \in F(R\,;\, s)$.

Hence, if $s \overset{*}{\underset{R}{\to}} t$ then there exist $c_1, \ldots, c_n \in N$, such that for every $f \in F(R\,;\, s)$, (8) holds.

Remark 2.11. Let $R = \{\alpha_i \to \beta_i / 1 \leq i \leq n,\ n \geq 1\} \in \mathcal{R}_{scb}$ and $s, t \in \mathcal{T}$ be such that $s \underset{R}{\to} t$. From Corollary 2.6 and Corollary 2.10 we deduce that there exist $c_1, \ldots, c_n \in N$ such that:

$$\begin{cases} \sum_{i=1}^{n} c_i B(r_i) = |s| - |t| \\ \sum_{i=1}^{n} c_i B_i(f) = \#(f, s) - \#(f, t), \text{ for every } f \in F(R\,;\, s). \end{cases} \tag{10}$$

If there exist $c_1, \ldots, c_n \in N$ such that (10) hold, it does not mean that $s \overset{*}{\underset{R}{\to}} t$. But, every n-tuple $(c_1, \ldots, c_n) \in N^n$ which verifies (10) induces a finite set of R-derivations from s, $\mathcal{D}_R(c_1, \ldots, c_n\,;\, s) = \{\gamma \in \mathcal{D}_R(s) \mid |\gamma| \leq \leq \sum_{i=1}^{n} c_i$ and $\mathrm{occ}\,(r_i, \gamma) \leq c_i\}$. (10) is a linear system of $m = |F(R\,;\, s)| + 1$ equations with $n = |R|$ unknowns. Because this system depends on R, s and t we shall denote it by $(S(R\,;\, s, t))$ or simply by (S).

We shall denote the set of solutions in N^n of this system by Sol $(S(R\,;\, s, t))$ or Sol (S).

Let us note that $\mathcal{D}_R(s, t) \subseteq \bigcup_{(c_1, \ldots, c_n) \in \mathrm{Sol}(S)} \mathcal{D}_R(c_1, \ldots, c_n;\, s)$ and hence, to determine all the possible R-derivations from s to t, we must determine Sol (S).

To begin with, we can make the following remarks:

1) if there exists $x \in V$ such that $\#(x, s) \neq \#(x, t)$ then, $s \overset{*}{\underset{R}{\not\to}} t$ (by Proposition 2.4);

2) if the system (S) is not compatible or does not have any solution in N^n, then, $s \overset{*}{\underset{R}{\not\to}} t$ (by Corollary 2.6 and Corollary 2.10).

The set Sol (S) can be finite or infinite.

If Sol (S) is finite we shall prove that the problem $s \overset{*}{\underset{R}{\to}} t$ is decidable. Otherwise, the problem $s \overset{*}{\underset{R}{\to}} t$ is partially decidable: we shall prove this statement in another paper.

In the following, let $R = \{\alpha_i \to \beta_i / 1 \leq i \leq n,\ n \geq 1\} \in \mathcal{R}_{scb}$ and $s \in \mathcal{T}$. For every $t \in \mathcal{T}$ we consider the system $(S(R\,;\, s, t))$ in the form

$$AX = B, \tag{11}$$

where

$$A = \begin{pmatrix} B(r_1) & B(r_2) & \cdots & B(r_n) \\ B_1(f_1) & B_2(f_1) & \cdots & B_n(f_1) \\ \ldots\ldots & & & \\ B_1(f_{m-1}) & B_2(f_{m-1}) & \cdots & B_n(f_{m-1}) \end{pmatrix}$$

$$X = (c_1, \ldots, c_n)^T$$

$$B = (b, b_1, \ldots, b_{m-1})$$

$$b = |s| - |t|$$

$$b_i = \#(f_i, s) - \#(f_i, t), \quad 1 \leq i \leq m-1$$

$$F(R\,;\, s) = \{f_1, \ldots, f_{m-1}\}.$$

In connection with the set Sol (S), the following question appears:
— can we decide if Sol (S) is finite or infinite?

The answer to this question is affirmative, because the following holds:

Theorem 2.12. *Let* $R=\{\alpha_i\to\beta_i/1\leqslant i\leqslant n,\ n\geqslant 1\}\in\mathcal{R}_{\mathrm{scb}}$ *and* $s, t\in\mathcal{T}$. Sol (S) *is finite iff the optimization problem*

$$(\bar{P})\qquad\begin{cases}\max\ (c_1+\ldots+c_n),\\ AX=B,\\ X=N^n,\end{cases}$$

has a (finite) optimum.

Proof. Let $R=\{\alpha_i\to\beta_i/1\leqslant i\leqslant n,\ n\geqslant 1\}\in\mathcal{R}_{\mathrm{scb}}$ and $s,\ t\in\mathcal{T}$. If Sol (S) is finite then the solution $(c_1^0, \ldots, c_n^0)\in$ Sol (S), which verify

$$c_1^0+\ldots+c_n^0=\max\ \{(c_1+\ldots+c_n)\mid (c_1, \ldots, c_n)\in \mathrm{Sol}\ (S)\}$$

is an optimum for $(\bar{P})$.

Conversely, let $(c_1^0, \ldots, c_n^0)$ be an optimum for $(\bar{P})$.
Then, Sol $(S)\leqslant\{(c_1, \ldots, c_n)\in N^n \mid c_i\leqslant c_1^0+\ldots+c_n^0,\ 1\leqslant i\leqslant n\}$ and hence Sol (S) is finite.

Because the methods of solving the problem $(\bar{P})$ from Theorem 2.12 allow to decide whether it has a finite optimum or not [6], we deduce that it is decidable whether Sol (S) is finite or infinite.

Then, we can prove:

Theorem 2.13. *Let* $R\in\mathcal{R}_{\mathrm{scb}}$ *and* $s, t\in\mathcal{T}$. *If* Sol (S) *is finite, then the problem* $s\overset{*}{\underset{R}{\to}}t$ *is decidable.*

Proof. Let $R\in\mathcal{R}_{\mathrm{scb}}$ and $s, t\in\mathcal{T}$ be such that Sol (S) is finite. Then, the set $\mathcal{D}_R(s, t)$ is finite since $\mathcal{D}_R(s,t)\subseteq\bigcup\limits_{(c_1, \ldots, c_n)\in \mathrm{Sol}\ (S)}\mathcal{D}_R(c_1, \ldots, c_n; s)$. Hence, the problem $s\overset{*}{\to}t$ is decidable.

Remark 2.14. Following J. Borosh, B. Treybig [1], if the system $AX=0$ has no nontrivial solutions in N^n, then every solution of (11) satisfies $c_i\leqslant M$, $i=\overline{1, n}$ where M is the maximum of the absolute values of all minors of order r of (A/B) ((A/B) is the augmented matrix of (11) and $r=\mathrm{rank}\ (A)$). So, in this case (which includes the case when (11) has a unique solution), the problem $s\overset{*}{\to}t$ is decidable.

REFERENCES

1. Borosh I., Treybig L.B. — *Bounds on Positive Integral Solutions of Linear Diophantine Equations*, Proc. of the AMS, vol. 55, No. 2, 1976, pp. 299—304.
2. Boudol G. — *Computational Semantics of Terms Rewriting Systems, Algebraic Methods in Semantics*, M. Nivat, J. Reynolds (eds.), Cambridge Univ. Press, 1986, pp. 169—237.
3. Garey M. R., Johnson D.S. — *Computers and Intractability / A Guide to the Theory of NP Completeness*, W. H. Freeman and Comp. Publ., San Francisco, 1979.
4. Huet, G., Oppen D.C. — *Equations and Rewrite Rules: A Survey, Formal Language Theory: Perspectives and Open Problems*, R. V. Book (ed.), Academic Press, 1979, pp. 349—405.

5. Huet G., Levy J. J. — *Computations in Nonambiguous Linear Rewriting Systems*, Rapport Laboria 359, INRIA, 1979.
6. Knuth D. E. — *Art of Computer Programming*. Vols. I and 2, Addison-Wesley, Reading, Mass. 1969.
7. Mehlhorn K. — *Data Structures and Algorithms*, Vol. 2, EATCS Monographs on Theoretical Computer Science, Springer-Verlag, 1984.
8. Shani S. — *Computationally Related Problems*, SIAM Journal of Computing, vol. 3 no. 4, 1974, pp. 262—282.
9. Niven I., Zuckerman H. S. — *An Introduction to the Theory of Numbers*, J. Wiley & Sons Inc., 1960.
10. Ţiplea F.L., Jucan T., Masalagiu C. — *Term Rewriting Systems and P/T — nets* (this volume).

Received 15.V. 88

Seminarul Matematic „Al. Myller"
Universitatea „Al. I. Cuza" Iaşi