ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA"
Tomul XXXIV, s. I a. Matematică-Informatică, 1988, f. 4

AN ALGORITHM FOR EMBEDDING PLANAR GRAPHS IN SIX PAGES

BY

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1. Introduction. The *pagenumber* of a graph is the minimum number of pages required in a bookembedding of the graph. A *bookembedding* is a linear ordering of the vertices on the "spine" of the book together with an embedding of the edges on the pages so that edges in the same pages do not cross.

The interest in the bookembedding problem comes from different area : fault-tolerant VLSI design [CLR1], [R1], [CLR2], complexity theory [PRST], [GKS], sorting with parallel stacks and single-row routing (see [CLR2]).

The case of planar graphs seems to be basic in the understanding of bookembedding. It was conjectured in [BK] that planar graphs require an unbounded number of pages. The conjecture was disproved by [BS] and [H]. [BS] gives an algorithm for embedding planar graphs in 9 pages. With a different method, [H] derives a 7-page embedding algorithm.

In what follows we will present a 6-page embedding algorithm, building on the 9-page method of B u s s and S h o r [BS].

A couple of months after this manuscript was written M. Y a n n a-k a k i s announced a 4-page algorithm [Y1]. His paper also shows that 4 is a lower bound on the number of pages of planar graphs [Y2].

Our algorithm is based on a recursive decomposition of planar graphs using W h i t n e y Theorem. The ideas behind the algorithm are simple and intuitive and with a potential to develop fast parallel algorithms for bookembedding.

The following is an informal description of the method. Our method builds on the 9-page method of J. B u s s and P. S h o r [BS]. Their basic idea is to use a theorem due to W h i t n e y: *Any triangulated planar graph without separating triangles has a Hamiltonian Circuit (HC).*

The theorem gives us two directions. First of all, a HC for a graph provides with a 2-page embedding of the graph : the order of the vertices on the spine is simply the HC order, the interior edges of HC are embedded in one page, the exterior edges of HC are embedded in a second page, while the edges the HC is passing through can be embedded in either one. This can be considered as the "local" part of the embedding. The second direction derived from the theorem is the "global" one. Namelly, the separating triangles and the way they nest, suggest a decomposition of the graph into levels. A triangle in a triangulated planar graph is *separating* when its removal separates the graph in two components: the "interior" and the "exterior". We decompose the graph G in levels with the intention that at each level, the Whitney theorem applies (to every connected component). Each level has no separating triangles. For example, the first level of G is G^0 and consists in the subgraph of G generated by the vertices not enclosed by any separating triangle. The second level G^1 is the first level of $G—G^0$. And so on. Our Algorithm will use the level-structure of the graph as the global direction. However, the local embedding will be different from the Buss-Shor Algorithm. The points of departure can be summarized as follows. We use a stronger result (than the Whitney Theorem), the Whitney Lemma, which yields a more "flexible" Hamiltonian Circuit. We take advantage of the detailed structure of the interior of a triangle, in order to control the combinatorial structure of the chords and pinch vertices. By a corresponding pre-processing of the levels, the HC facilitates the embedding of inter-level edges.

At each level the connected components will be embedded using their HC in the same 2 pages, while the edges connecting adjacent levels require 4 more pages.

The remainder of the paper is organized as follows. Section 2 contains some definitions. Section 3 gives basic lemmas. In Section 4 a presentation of the Whitney Lemma and the Whitney Theorem is included. Section 5 contains the main result. The last section presents conclusions.

2. Definitions. In this section, we present the basic definitions and notations used in the paper.

Let G be a planar graph embedded in the plane. G is said *triangulated* when no edges can be added without losing planarity. The set of frontier vertices is denoted $\mathrm{Fr}(G)$; $\#\mathrm{Fr}(G)$ denotes the size of $\mathrm{Fr}(G)$. A *frontier edge* is an edge connecting two consecutive frontier vertices. G is *interior-triangulated* if we cannot add edges in the interior of the graph without losing planarity. The frontier of a triangulated graph is a triangle. The frontier of an interior triangulated graph is a polygon (possibly with some of the vertices identified). Fig. 1. presents an interior triangulated graph. The frontier vertices P_i, $1 \leqslant i \leqslant 3$ can be seen as identifying two frontier vertices of a polygonal frontier. These vertices P_1, P_2, P_3 are called *pinch vertices* [H]. A *k-pinch* vertex ($k \geqslant 2$) is a frontier vertex of an interior triangulated graph whose removal separates the graph into k connected components.

Let G have a k-pinch vertex P. If Q, P, Q' are consecutive frontier vertices, the edge (Q, Q') added in the exterior face is a *bridge* for P. Adding bridges "eliminates" the pinch vertices; namely, in the resulting graph, the original pinch vertices lose their pinch attribute. A *chord* in an interior triangulated graph is an edge connecting two nonconsecutive frontier vertices In Fig. 1, P_1 and P_2 are 2-pinch, P_3 is a 3-pinch and e_1, e_2, e_3 are chords.

A *circuit* in G is a sequence of different vertices $v_0, \ldots, v_n$ except that $v_0 = v_n$ and for each i, (v_i, v_{i+1}) is an edge of G, $0 \leqslant i \leqslant n-1$. A *Hamiltonian Circuit* is a circuit containing all the vertices of G.

Graphs and subgraphs with frontier a triangle will play a major role in the sequel. Some notations concerning them are necessary. A *triangle*

T will be given by $T = \{L, M, N\}$. Let T be a triangle of G. Then $\mathrm{int}_G(T)$ denotes the interior of T in G, i.e., the subgraph generated by the interior vertices. Fig. 5.1 presents a triangulated graph G with frontier $T = \{L, M, N\}$ in which the frontier of its interior is drawn. Fig. 5.2 gives $\mathrm{int}_G(T)$. Denote by $\mathrm{Conn}_G(T)$ the set of edges connecting T with (the frontier of) $\mathrm{int}_G(T)$. (As usual, the index G in $\mathrm{int}_G(T)$, $\mathrm{Conn}_G(T)$ is dropped when G can be derived from context.)

A triangle in G is called *separating* when it is not a face boundary, I. e., the removal of the triangle will separate the graph in several ($\geqslant 2$) components. Interior triangulated graphs without separating triangles (w.s.t.) will play an important role.

3. Basic Lemmas. We prove in this section, four lemmas. This first gives a way of eliminating pinch vertices and chords from the initial graph; these two types of structures prevent us from being able to apply the Whitney lemma. The second lemma captures three basic properties of a Hamiltonian Circuit. Lemma 3 gives the 2-page embedding of a graph having a Hamiltonian Circuit. The last lemma presents the structure of the interior of a (triangulated) triangle.

Lemma 1. (Chord and Pinch elimination) *The chords and pinch vertices of a graph can be eliminated by adding some extra edges. The resulting graph contains the initial graph as a subgraph.* □

Proof. 1) *The chords.* Suppose G is interior triangulated. Consider the chord AB. Let ABC, ABD be the two triangles having AB in common and empty interiors. (Note that AB can be common to other triangle, say containing ABC.) Now we draw AB in the exterior face, add one new vertex in ABC, and one in ABD, say E and F. Connect vertex E with A, B, C, and E, and vertex F with A, B, D. This eliminates the chord AB and introduces no other chords. In this way all the chords of G can be eliminated.

2) *The pinch vertices.* Let P be a pinch vertex. Consider Q, P, Q', three consecutive frontier vertices. The edge $e = (Q, Q')$ is not in G because P is pinch. If we add e in the exterior face, it will "cover" P; we shall call it a *bridge of* P. If P is a k-pinch vertex, the addition of $k-1$ bridges for it, transforms it into a non-pinch vertex.

Lemma 2. (Three properties of a HC) *Let G be an interior triangulated graph* w.s.t. *Then the following properties hold:*

2.1. *A* HC *passes through all the bridges.*

2.2. *A* HC *visits the frontier vertices in the frontier order.*

2.3. *The edges of any triangle are either all in the interior or all in the exterior of the* HC. (*The* HC *edges are considered both interior and exterior.*)

Proof. 2.1. If a graph has a pinch vertex P then no HC exists for it. Indeed, the circuit must visit P twice which is not possible. Now suppose that $e = (Q, Q')$ is a bridge for P in G. P is not a pinch in G, but is a pinch in $G—e$. Suppose that G has a HC which is not passing through e. Then $G—e$ has the same HC. This is impossible because it has a pinch vertex P.

2.2. Recall that $\mathrm{Fr}(G)$ is the set of frontier vertices of G. If $\#\mathrm{Fr}(G) \leqslant 3$, then in any order the HC visits them, this is the order of the frontier. Suppose $\#\mathrm{Fr}(G) = k > 3$. There exist permutations of $1, 2, \ldots, k$ which are not circular versions of $1, 2, \ldots, k$ (e.g., we can "jump"). We number the frontier vertices

$1, 2, \ldots, k$. By contradiction, suppose that there is a HC not satisfying 2.2. Suppose that HC starts at 1. Let r_1, r_2 be the first two frontier vertices, $r_2 \geqslant r_1+2$ which provides with a jump, i.e.,

$$\mathrm{HC} = 1\alpha_1 \ldots \alpha_n r\beta_1 \ldots \beta_l r_1 \gamma_1 \ldots \gamma_t 1$$

such that in $\beta_1 \ldots \beta_l$ there are no frontier vertices (See Fig. 2). Without loss of generality, we can assume that $1 < r_1$. Because r_1, r_2 is the first pair of this kind starting from 1, the skipped frontier vertex number r_1+1 belongs to $\gamma_1 \ldots \gamma_t$. Consider now the fragments of HC : r_1——r_2 and r_1+1——1 Although they are parts of the HC they intersect. This contradiction completes the proof.

2.3. This follows easily from the fact that every triangle (except the frontier one when the graph is triangulated) has no interior.

Lemma 3. (Two page embedding of a HC passing through a frontier edge.) *If G has a HC passing through a frontier edge then G can be embedded in two pages such that on one page all the frontrer vertices are free (i.e., width = 0).*

Proof. By Lemma 2.2, the HC divides the set of edges of G in several regions. Each pair of consecutive frontier vertices defines a region bounded on one side by the HC fragment going into the interior of the graph, and on the other side by the frontier edge connecting them. This is an *exterior region* (including the boundary edges). In the special case of a frontier edge that is part of the HC, the corresponding exterior region has an empty interior, i.e., it contains only the frontier edge.

The circuit also defines an *interior region*.

The figures 3.1—3.3 present the HC, the interior and exterior regions and the 2-page embedding of it.

The exterior regions are embedded in the upper page. The interior region and the frontier edge XY are embedded in the lower page.

The next Lemma identifies the structure of the interior of a triangle. The structure will be called 3-*petal graph* because of its composition of 3 chains: each chain is made out of pinch-less graphs arranged linearly and "linked" by a common pinch vertex. The 3 chains connect together in a flower-like manner having the center a pinch-less graph or a 3-pinch vertex.

The Fig. 5.2 gives an example of 3-petal graph.

Lemma 4. (The petal structure of a triangle interior.) *The interior of a triangulated triangle is a 3-petal graph.*

Proof. Let $T=\{L, M, N\}$ be the triangle frontier of a triangulated graph G. Let $G'=\mathrm{int}_G(T)$. Clearly G' is connected and the only vertices connected to L, M, N are those from $\mathrm{Fr}\,(G')$.

Let us classify the vertices of $\mathrm{Fr}(G')$ according to their connections to L, M, N. A vertex can have at most 3 connections. There is a partition of $\mathrm{Fr}(G')$ defined in this way; the notation for a set will be with corresponding indices. Some of the sets can be empty.

3. Three connections C_{LMN}. At most one vertex connected to L, M and N

2. Two connections : C_{LM}, C_{MN}, C_{NL}. The vertices connected to both L and M are the elements of C_{LM}. We shall order the vertices of C_{LM} as follows. For any $c \in C_{LM}$, consider the triangle LMc. The family obtained in this way $\{LMc\}_{c \in C_{LM}}$ can be linearly ordered by the relation "subtriangle of", because all of them have a common leg LM. Let LMc_1 be the minimal one and LMc_γ be the maximal one. Similar considerations can be made for the sets C_{MN} and C_{NL}.

1. One connection C_L, C_M, C_N. These three sets contain the remaining vertices of $\mathrm{Fr}\,(G)$.

The pictures 4.1 and 4.2 present the possible situations. The cycles stand for arbitrary planar graphs having a cycle as frontier or (the graph is reduced to) an edge or a vertex. In the above pictures $C_{LM}=\{c_1, c_2, \ldots, c_\gamma\}$, $C_{MN}=\{a_1, a_2, \ldots, a_\alpha\}$, $C_{NL} = \{b_1, b_2, \ldots, b_\beta\}$; C_{LMN} is $\{P\}$ in 3.1 and $\varnothing$ in 3.2 ; C_L, C_M, C_N are easily obtained.

4. The Whitney lemma and theorem. The following lemma was proved by W h i t n e y as a technical result in order to obtain his famous theorem. Our interest in the slightly stronger result of the Lemma (instead of the Theorem) is a basic point of departure from the B u s s-S h o r Algorithm. The W h i t n e y Lemma will provide us with more flexibility in the embedding.

Lemma (W h i t n e y 1931, [W, p. 380]). *Let G be a triangulated planar graph without separating triangles, and R a circuit in it. Let A, B be two distinct vertices of R, dividing R into two parts R_1 and R_2, in each of which we include both A and B. Suppose*

(1) *No pair of vertices of R_1 are joined by an edge interior to R, and*

(2) *Either no pair of vertices of R_2 are joined by an interior edge in R, or else there is a vertex C in R_2 distinct from A and B, dividing R_2 into two parts R_3 and R_4 in which we include C, such that no pair of vertices of R_3 and no pair of vertices of R_4 are joined by an interior edge.*

Then we can draw a path from A to B, passing only along edges of and inside R, and passing through each vertex of and inside R once and only once.

The following informal description of the Lemma is taken from the same paper.

"In brief, if we can divide the circuit T into either two or three parts, such that in any part, including end vertices, no pair of vertices touch each other inside R [in our terminology ≪joined by an interior edge≫], we can then draw the required curve from anyone end vertex to any other end vertex on these parts".

From the lemma, his theorem follows easily :

Whitney Theorem. *Every triangulated planar graph without separating triangles has a Hamiltonian Circuit.*

5. Main result. In order to present our main result, i.e., an algorithm for embedding planar graphs in 6 pages, we first derive a lemma which gives the invariant, on the base of which our algorithm is proved to be correct.

The invariant property refers to triangles and is elaborated in what follows. Our embedding of a planar graph in 6 pages will use φ_0, φ_1, φ_2 the *upper* pages and ψ_0, ψ_1, ψ_2 the *lower* pages. The construction is symmetrical, due to a symmetry imposed by the Hamiltonian Circuit construction (the interior of HC is embedded in one page say φ_0, and the exterior in ψ_0, see Fig. 3.3).

An equivalent way of saying that 3 upper pages are available is that we have one upper page only UP on which we can use three colors to draw edges: no edges that cross can be drawn with the same color. Similarly for the lower pages, LOW. Now suppose that a triangle T is embedded entirely in one page, say φ_0. The *area of* T, denoted T^a in this embedding is the area defined in UP by the largest edge of T and the spine. Similarly for a triangle embedded in ψ_0 and its area T^a in LOW. Up to three colors can be used in this area.

Now, we call T^a *mono-color* if T^a has only edges colored with one color.

Property (M).

Let the triangle T *be embedded such that:*

1. T *is entirely embedded in* φ_0 *or entirely in* ψ_0.
2. T^a *is mono-color.*

Lemma 5. *Let* G *be a graph* w.s.t. *and with frontier triangle* T.

If T *has property* (M) *then* int (T) *can be embedded such that any triangle of* int (T) *has property* (M).

The total number of pages required is 4.

Proof. We construct the desired embedding of int (T) in 8 steps. A corresponding generic sequence of figures is given in Fig. 5.1—5.8.

1. Let $T=\{L, M, N\}$. By Lemma 4, int (T) has a 3-petal structure.
2. The frontier of int(T) is connected to L, M, N using edges from Conn (T).
3. By Lemma 1, we eliminate the pinch vertices; that is, we add some bridge edges.
4. By Lemma 1, we eliminate the chords in some regions. That is, we prepare the graph to meet the hypotheses of W h i t n e y Lemma; A, B, C are the decomposition points. Dashed regions are the exterior-face regions where the chords may lie. The resulting graph satisfies the W h i t n e y Lemma hypotheses. Therefore it has a HC. By Lemma 2.1, HC passes through all the bridges.
5. Consider the bridge $e=(Q, R)$. By Lemma 3, HC and (Q, R) provide us with a 2-page embedding, having the frontier vertices free (including those covered by bridges).
6. The fact that the frontier vertices are free implies that the chords can be accommodated without conflicts.
7. The 3-page embedding of Conn (T) is given together with the chord embedding and the embedding of T. The following table gives the pages in which the connection s are embedded. The notations are those of Lemma 4: e.g., C_{LM} means the frontier vertices of int (T) connected to both L and M. Also in the table "i" stands for φ_i, $0 \leqslant i \leqslant 2$.
8. The transformations realized by our algorithm can be seen in the nonembedded version.

Comments. i) In the special case when part of a petal or even the whole interior is reduced to pinch vertices only it is not difficult to carry step 3 to cover all the pinch vertices.

ii) As far as page requirement is concerned, Conn (T) needs 3 pages: φ_0, φ_1, φ_2; the outside chords need 2 pages: φ_0 and φ_1; int (T) needs 2 pages: φ_0 and ψ_0; the triangle T needs one page: φ_0. Total 4 pages.

Conn to vertices	L	M	N
A	0	1	—
B	—	1	1
D	0	—	1
C_L	0	—	—
C_M	—	1	—
C_N	—	—	1
C_{LM}	0	1	—
C_{NL}	0	—	1
C_{MN}	—	1	2
C_{LMN}	0	1	2

iii) Any triangle of int (T) is entirely embedded in φ_0 or ψ_0 by Lemma 2.3 and its region is mono-color, so property (M) holds.

We are now ready for the main result of this paper.

Theorem. *Every planar graph is embeddable in 6 pages.*

Proof. Let $G=(V, E)$ be a planar graph. Without loss of generality we can assume that G is triangulated having as frontier the triangle T_0 (If G is not triangiulated, we just add edges until G is; to get the embedding of G, we will remove the extra edges from the final embedding.) The construction is exemplified in Fig. 7.1—7.5.

We decompose now G in levels according to the following

Definition [BS]. *Let* V_0 *be the set of vertices of* G *which are not enclosed by any triangle of* G. *Suppose that* $V_0, V_1, \ldots, V_k$ *have been defined and let* $V^{k+1}=V-\bigcup_{i=0}^{k} V_i$, *and* G^{k+1} *be the subgraph of* G *induced by* V^{k+1}. *Define* $V_{k+1} \subseteq V^{k+1}$ *to be the vertices of* G^{k+1} *not enclosed by a triangle of* G^{k+1}. *Let* G_k *be the subgraph induced by* V_k. *Each* G_k *is a level of* G. *A connected component of any* G_k *is a section of* G. *Intuitively, for a graph* K *let* K^{Δ} *be the result of removing all the interiors of the separating triangles of* K. *Note that* K^{Δ} *is* w.s.t. *If* T *is a separating triangle of* K, int $(T)^{\Delta}$ *is a section of the next level of* K: *i.e., if* T *was part of level* l, int $(T)^{\Delta}$ *is a section of level* $l+1$.

A level is the collection of sections defined in this way for all the separating triangles of the previous level.

Now the embedding of a planar graph will be defined recursively on levels. With a level, we embed each section int $(T)^{\Delta}$ of it and its connections to the corresponding triangle T, embedded in the previous step (as part of the previous level). This last task is provided by the algorithm of Lemma 5 described formally by:

Procedure ET (T).

input: T a triangulated planar graph having T as (triangle) frontier; an embedding of the triangle T satisfying property (M).

output: The embedding of int (T) in 4 pages; any triangle of int (T) has property (M).

We will embed G in 6 pages φ_i, ψ_i, $0 \leqslant i \leqslant 2$. using the following recursive algorithm. It uses ET as a subroutine.

Procedure EMBEDDING (T)

{input: a triangulated planar graph T}

```
{output : a 6 page embedding of T}
begin

if T is w.s.t.  then ET (T)
                else
                  begin
                  ET (T^Δ) ;
                  for any separating triangle T' of T do
                      EMBEDDING (T') ;
                  end ;
end.
```

Comment. The procedure ET (T') embeds the interior of T'. It assumes that T' is embedded in one page with the vertices in order L', M', N'. Then the interior is embedded between L' and M'.

Calling ET here, we have to be precise about the place of embedding the interior of T' because at the call, between L' and M', on the spine can be other vertices. Therefore, let L_0' be the next vertex on the spine after L'. We embed the interior between L' and L_0'. In the case of two (or more) triangles having the same L' as the leftmost vertex on the spine, we still embed their interiors between L' and L_0' but the order will be "the largest first". See Figure 6.

Let us remark that conflicts still may occur in the embedding of the interiors of separating triangles at a fixed level. ET (T^{Δ}) embeds T^{Δ} in two pages: φ_0 and ψ_0. Any triangle of T^{Δ} can be a separating one in T, so we have to analyze in which way embedding all their interiors (back) will not conflict. First of all, if T' is a triangle of T^{Δ} embedded in φ_0, then $\mathrm{Conn}_T(T')$ will be embedded in the upper pages φ_0, φ_1, φ_2. If T'' is embedded in ψ_0, then the lower pages will accomodate $\mathrm{Conn}_T(T'')$, so no conflict can be between the connections of T' and T''. The embedding of their interiors are performed in disjoint regions of the spine, so they are also conflict-free. The only place where a potential conflict can be is when we embed the connections of two triangles T_1, T_2 of T^{Δ} both embedded in the same part, say both in φ_0. Suppose that the vertices of the triangles are on the spine in order L_1, M_1, N_1, respectively L_2, M_2, N_2. If $L_1 \neq L_2$ no conflict can occur because of different regions of embedding for Conn. However, if $L_1 = L_2$ then the regions for Conn intersect but by the nature of the embedding no conflict occur within a page. Figure 6 gives the generic picture.

6. Conclusions. We have presented an algorithm for embedding a planar graph in 6 pages. Two pages in the construction are used less than the other four (see Fig. 5.7). A problem that remains for the present algorithm is to eliminate the two sparse pages, if at all possible.

The complexity of the algorithm is dominated by the Whitney Theorem construction and is a small polynomial in the number of vertices. However, the approach seems to be suitable for a parallel approach as most of the steps (except the Whitney Theorem Construction) are "reducible" to graph connectivity. On the other hand, the Whitney theorem was proved by an induction argument of a sequential nature. It remains to be seen if the proof holds for a divide-and-conquer approach.

The algorithm developed by M. Yannakakis embeds a planar graph in 4 pages; its time is linear in the number of vertices. His algorithm uses a different methed building on Heath's 7-page algorithm [H]. Certainly the two algorithms present two completely different approaches to the bookembedding of planar graphs, and therefore they add specific information to our understanding of the bookembedding problem.

Acknowledgements

First of all, I want to thank Tom Leighton for enlightening conversations about the problem, useful suggestions and encouragement.

It is a great pleasure to thank Jonathan Buss and Peter Shor for their criticism, valuable ideas and continuous enthusiasm.

The paper benefited in different stages by comments and suggestions from Alan Cobham, Irina Bercovici, Val Breazu-Tannen, Dan Dougherty, Lenny Heath and Susan Landau. Their interest in the paper is gratefully acknowledged.

Alan Cobham and Albert Meyer encouraged me to consider the publication of this paper after M. Yannakakis announced his result.

Last but not least, I am very pleased to express my gratitude to Albert Meyer, whose continuous support, advice, strong influence and encouragement extended over the past three years, helped me overcome difficult moments.

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Received 5. IV.88

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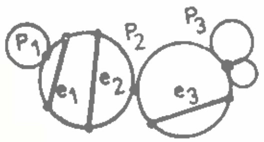


Fig. 1

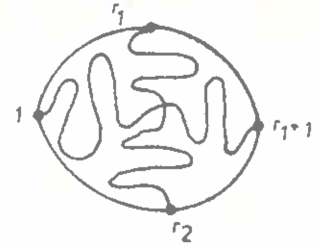
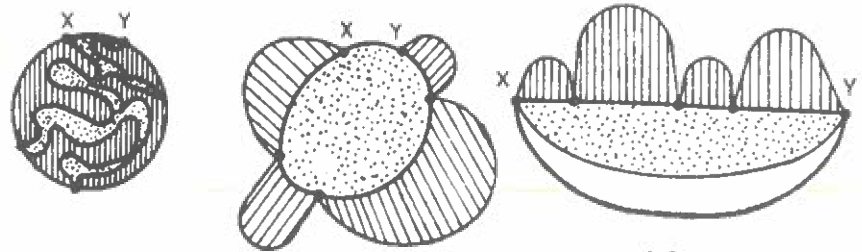
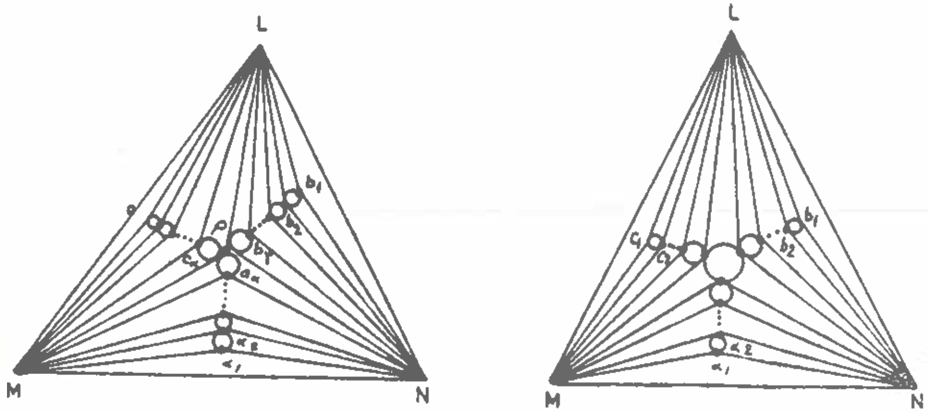


Fig. 2



3.1 3.2 3.3

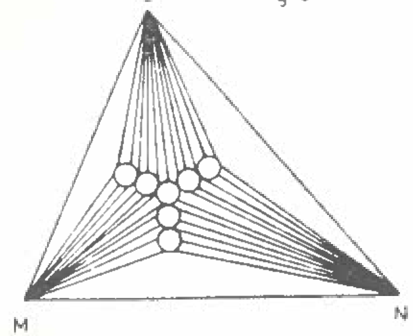
Fig. 3



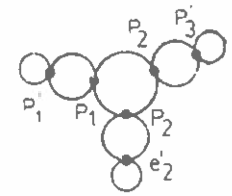
4.1 4.2

Fig. 4

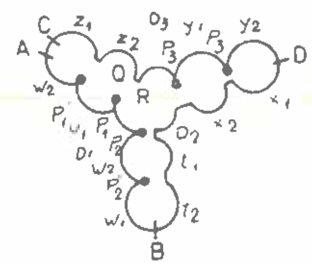
Fig 5



5.1



5.2



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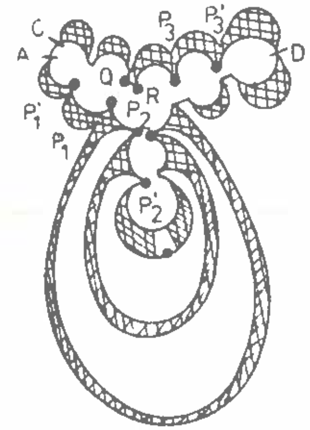

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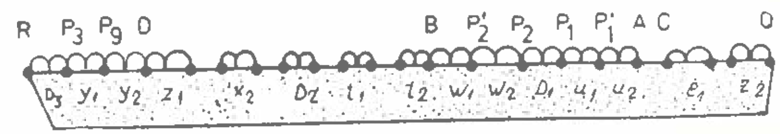

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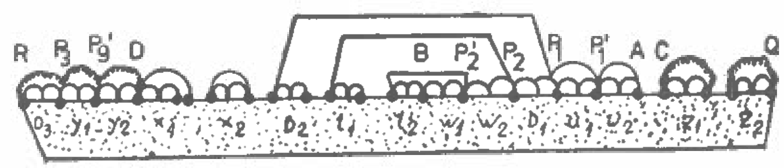

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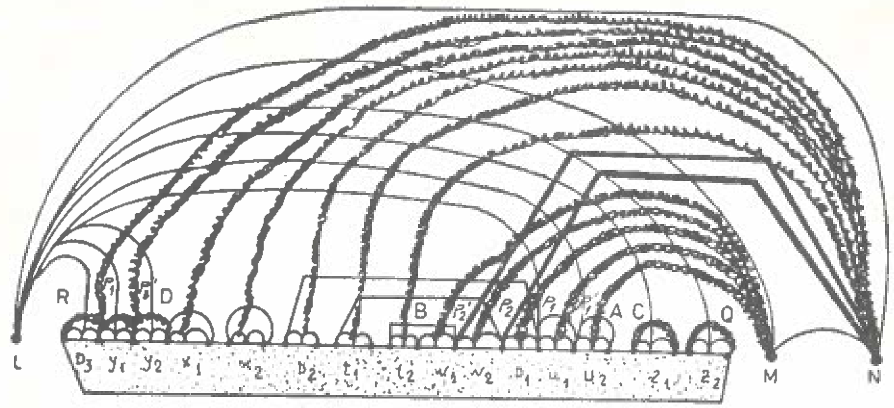

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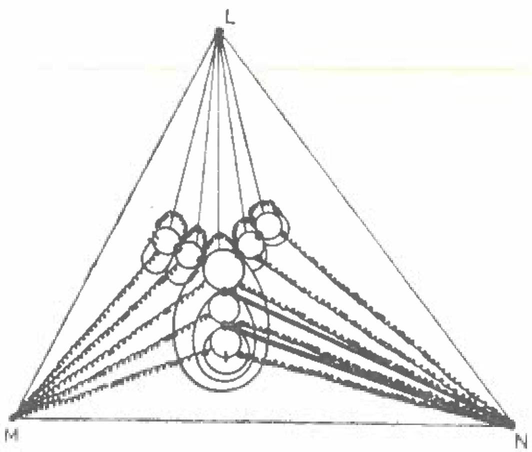

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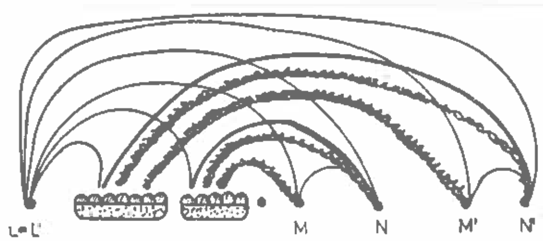
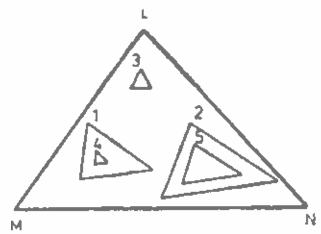
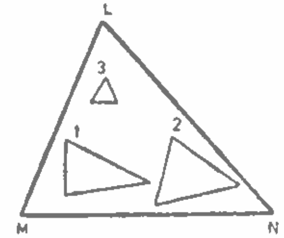


Fig 6

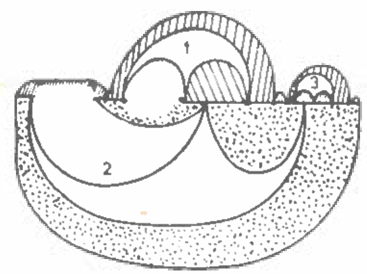
Fig 7



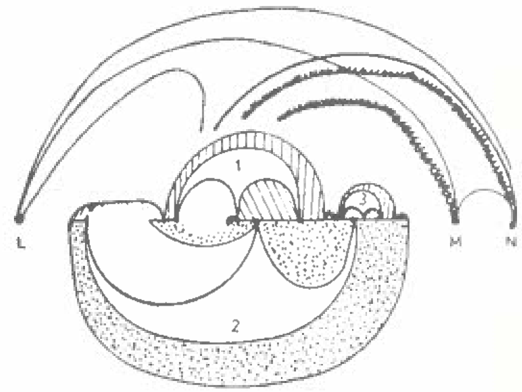
7.1 A two-level graph G



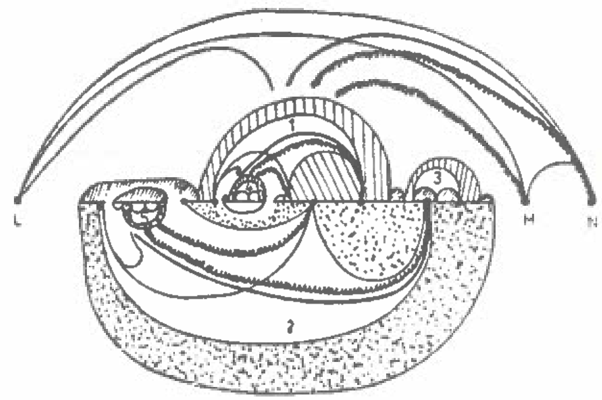
7.2 G_0 the first level



7.3 The embeding of G_0



7.4 Embeding Conn and T={L,M,N}



7.5 Embeding the two sections of the next level G_1