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QUADRATIC PROGRAMS ON GRAPHS

BY

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1. Introduction. Throughout this paper, $G=(V, E)$ will denote a simple, finite, undirected graph with $V=\{1, 2, ..., n\}$ the set of vertices and E the set of edges. A stable set in G is a set of pairwise non adjacent vertices. We denote by $\alpha(G)$ the maximum cardinality of a stable set in G. The complement of G is a graph $\overline{G}=(V, \overline{E})$ where vertices i, j are adjacent in $\overline{G}$ ($ij\in\overline{E}$) if and only if $i\neq j$ and i, j are non adjacent in G ($i, j\in E$). Any maximal stable set in $\overline{G}$ is called *clique* in G and $\alpha(\overline{G})$ is denoted by $\omega(G)$.

If p is a positive integer, then a p-colouring of G is a function $c: V\to\{1, 2, ..., p\}$ such that for every $i\in\{1, 2, ..., p\}$, $c^{-1}(i)$ is a stable set in G. The smallest positive integer p such that G has a p-colouring is the chromatic number of G and is denoted by $\chi(G)$.

It is well-known that the problems of computing $\alpha(G)$, $\omega(G)$ or $\chi(G)$ for a given graph are NP-hard. The analytical approaches for these problems usually follow the scheme : transform the problem in a linear program with integer variables (often 0,1), then try to replace the integrity restrictions by linear ones. When this is successful done, polynomial time algorithms and combinatorial min-max relations are obtained for particular classes of graphs.

In this note we shall show that we could renounce to integrity conditions retaining linear restrictions, very simple however, and considering quadratic objective functions. The idea of this approach follows from the work of Motzkin and Strauss [1] who have proved that given a graph $G=(V, E)$

$$\max \{\sum_{ij\in E} x_i x_j \,;\; x\in R^n,\; x_i\geq 0 \text{ and } \sum_{i=1}^{n} x_i=1\}$$

is $(1-1/\omega(G))/2$, and if Q is a clique of maximum cardinality in G then an optimal solution of this quadratic program is given by $x_i=1/\omega(G)$ if i is in Q and $x_i=0$ otherwise. This result allows them to prove Turan's theorem but it couldn't be used to effectively compute $\omega(G)$ because the optimal solutions have non integer components.

On the other hand, if $f(x)=\sum_{i=1}^{n}\sum_{j=i+1}^{n} a_{ij}x_i x_j$ is a real square-free quadratic form and $G=(V, E)$ is an arbitrary graph with vertices set $V=\{1, 2, .., n\}$ then f can be expressed as

$$f(x)=f_{G,a}(x)+f_{\overline{G},a}(x),$$

where $\overline{G}$ is the complement of G and for a graph $H=(V,\ E(H))$ and $a : E(H)\to R$, $f_{H,a}(x)=\sum\limits_{ij\in E(H)} a_{ij}x_ix_j$.

A natural problem is to investigate the possibility of solving a quadratic program having f as objective function and linear restrictions, using a combinatorial approach on a suitable graph G. Examples in this direction are given in sections 2 and 3. By investigating such problems we hope to gain insights relevant for eventual new methods in quadratic optimization.

Finally, let us note that our results suggest that a combinatorial duality for such programs would be possible, with very interesting consequences.

2. Stable sets of maximum cardinality. Let $G=(V,\ E)$ be a graph with $|V|=n$ and $\overline{E}$ the set of edges of $\overline{G}$ the complement of G. Let $f_G : [0,1]^n\to R$ be defined by :

$$f_G(x)=\sum_{ij\in\overline{E}} x_ix_j-n\sum_{ij\in E} x_ix_j.$$

Theorem 1. *The set of optimal solutions for the program*

$$\max\ \{f_G(x)\ ;\ 0\leqslant x_i\leqslant 1,\ i=\overline{1,\ n}\} \tag{1}$$

is the set of incidence vectors of stable sets of maximum cardinality in G. *The optimal value for this program is* $\alpha(G)\ (\alpha(G)-1)/2$.

Proof. (a) Let $x^0(=x_1^0,\ x_2^0,\ \ldots,\ x_n^0)$ be a feasible solution of (1). We denote by $x^0[x_k^0\mid y]$ the vector obtained by replacing the coordinate x_k with $y\in[0,1]$. Clearly, this is a feasible solution and a simple calculus gives :

$$f_G(x^0)-f_G(x^0[x_k^0\mid y])=(x_k^0-y)\frac{\partial f_G}{\partial x_k}(x^0).$$

(b) Consider x^0 an optimal solution for the quadratic program (1), (of course, there exists such a solution), and suppose that $0<x_k<1$ for some k. Then

$$f_G(x^0)-f_G(x^0[x_k^0\mid 1])=(x_k^0-1)\frac{\partial f_G}{\partial x_k}(x^0)\geqslant 0$$

$$f_G(x^0)-f_G(x^0[x_k^0\mid 0])=x_k^0\frac{\partial f_G}{\partial x_k}(x^0)\geqslant 0.$$

These two relations imply $\dfrac{\partial f_G}{\partial x_k}(x^0)=0$ and therefore $x^0[x_k^0\mid 1]$, $x^0[x_k^0\mid 0]$ are optimal solutions.

(c) We deduce from (b) that for each optimal solution x^0 with non integer components there is an integer solution $\overline{x}^0$ such that $f_G(\overline{x}^0)=f_G(x^0)$.

(d) Let x^0 be an integer optimal solution. Consider $S=\{i\in V\ ;\ x_i^0=1\}$. Then, S is a stable set in G. Indeed, if $p,\ q\in S$ are such that $pq\in E$, then :

$$f_G(x^0)-f_G(x^0[x_p^0\mid 0])=\sum_{jp\in\overline{E}} x_j^0-n\sum_{jp\in E} x_j^0<0,$$

since, $x_q^0=1$ and $pq\in E$ imply $\sum\limits_{jp\in E} x_j^0>1$ while $\sum\limits_{jp\in\overline{E}} x_j^0\leqslant n-1$. This is in contradiction with the hypothesis that x^0 is an optimal solution. Hence, S is a stable set and a simple calculus shows that $f_G(x^0)=|S|\ (|S|-1)/2$. On the other hand, the incidence vector of any stable set is a feasible solution, so we conclude that S is a maximum stable set.

(e) Let x^0 be an optimal solution for (1) and $P(x^0)=\{i\in V\ ;\ 0<x_i^0<1\}$, $S(x^0)=\{i\in V\ ;\ x_i^0\neq 0\}$. Clearly $P(x^0)\subseteq S(x^0)$. We assert that $S(x^0)$ is a stable set in G. Indeed, if this is not true, then there exist $p,\ q\in S(x^0)$ such that $pq\in E$ and one of the following cases holds :

(i) $q\in S(x^0)-P(x^0)$. We deduce as in (d) that $x^0[x_p^0\mid 0]$ is a solution that satisfies $f_G(x^0[x_p^0\mid 0])>f_G(x^0)$, hence a contradiction.

(ii) $q\in P(x^0)$. From (b) there is a solution $\overline{x}^0$ such that $\overline{x}_q^0=1$, $f_G(\overline{x}^0)=f_G(x^0)$ and $S(\overline{x}^0)=S(x^0)$. Now we are in case (i) and thus we can obtain a contradiction.

Therefore, our assertion is true, hence $S(x^0)$ is a stable set and we have :

$$f_G(x^0)\leqslant|S(x^0)|\ \ (|S(x^0)|-1)/2.$$

The equality holds iff $P(x^0)=\varnothing$, so, x^0 is an optimal solution if and only if x^0 is the incidence vector of some stable set of maximum cardinality in G.

Note that the theorem remains true if

$$f_G(x)=\sum_{ij\in\overline{E}} x_ix_j-(\Delta(G)+1)\sum_{ij\in E} x_ix_j.$$

3. Colourings. Let $G=(V,\ E)$ be a graph and p a positive integer. We denote by $\lambda(G,\ p)$ the maximum cardinality of a vertex set $U\subseteq V$ with property that the subgraph $[U]_G$ induced by U in G has a p-colouring. Obviously, G is p-colourable if and only if $\lambda(G,\ p)=|V|$. Let $G_p=(V_p,\ E_p)$ defined as follows :

$$V_p=\{(v,\ i)\ ;\ v\in V,\ 1\leqslant i\leqslant p\}\ \text{ and}$$

$$E_p=\{(v,\ i)\ (w,\ j)\ \text{with}\ v=w\ \text{and}\ i\neq j\ \text{or}\ vw\in E\ \text{and}\ i=j\}\ .$$

Note that, if $S=\{(v_1,\ i_1),\ (v_2,\ i_2),\ \ldots,\ (v_k,\ i_k)\}$ is a stable set in G_p then $v_i\neq v_j$ if $i\neq j$ and the function $c : \{v_1,\ v_2,\ \ldots,\ v_k\}\to\{1,\ 2,\ \ldots,\ p\}$ given by $c(v_j)=i_j$ is a p-colouring of the graph induced by $\{v_1,\ v_2,\ \ldots,\ v_k\}$ in G. Conversely, if c is a p-colouring of the vertex set $\{v_1,\ v_2,\ \ldots,\ v_k\}$, then $\{(v_j,\ c(v_j))\ ;\ j=\overline{1,\ k}\}$ is a stable set in G_p. Therefore, we have $\lambda(G,\ p)=\alpha(G_p)$ and if we take $f_{G_p} : [0,1]^{np}\to R$ defined as in previous paragraph by

$$f_{G_p}(x)=\sum_{i<j}\Big(\sum_{v\neq w} x_{vi}x_{wj}-np\sum_{v\in V} x_{vi}x_{vj}\Big)+\sum_i\Big(\sum_{vw\in\overline{E}} x_{vi}x_{wi}-np\sum_{vw\in E} x_{vi}x_{wi}\Big)$$

(where $i,\ j$ run from 1 to p and $v,\ w$ on V), we obtain :

Theorem 2. $\max \{f_{G_p}(x) \,;\, x_{vi} \in [0, 1],\ v \in V,\ 1 \leqslant i \leqslant p\}$ *is* $\lambda(G, p)\,(\lambda(G, p)-1)/2$ *and all optimal solutions have integer components. If* x *is an optimal solution then the set of vertices* $v \in V$*, such that there exists* i_v *with* $x_{vi_v}=1$*, induces a* p*-colourable subgraph of maximum order in* G.

Another approach of the vertex colouring problem is to introduce the parameter $\mu(G, p)$ defined as the minimum cardinality of a set A of edges with property that if we remove A from E we obtain a p-colourable graph. Clearly, G has a p-colouring iff $\mu(G, p)=0$.

Let $g_G : [0, 1]^{np} \to R$ be the Motzkin-Strauss function of the graph G_p:

$$g_G(x) = \sum_{i=1,p} \sum_{vw \in E} x_{vi}x_{wi} + \sum_{v \in V} \sum_{i \neq j} x_{vi}x_{vj}.$$

Theorem 3. *The optimal value for the quadratic program*

$$\min g_G(x) \tag{2}$$

$$0 \leqslant x_{vi} \leqslant 1,\quad \sum_{i=1,p} x_{vi}=1, \text{ for all } v \in V \text{ and } i=\overline{1, p}$$

is $\mu(G, p)$ *and each optimal solution has integer components.*

Proof. Observe that, if x is a feasible solution then g_G can be written:

$$g_G(x) = \sum_{i=1,p} \sum_{vw \in E} x_{vi}x_{wi} + \sum_{v \in V} \Big(1 - \sum_{i=1,p} x_{vi}^2\Big).$$

Let x^0 be an optimal solution of (2) and suppose that it has a non integer component, say x^0_{vi}. Since $\sum_{k=1,p} x^0_{vk}=1$, there exists j, $j \neq i$ and $0 < x^0_{vj} < 1$. Let us consider $\bar{x}^0$ the vector with components:

$$x^0_{wk} = \begin{cases} x^0_{vi} - y & \text{if } w=v \text{ and } k=i \\ x^0_{vj}+y & \text{if } w=v \text{ and } k=j \\ x^0_{wk} & \text{otherwise} \end{cases}$$

where $-x^0_{vi} \leqslant y \leqslant x^0_{vi}$. Obviously, $\bar{x}^0$ is a feasible solution and we have

$$g_G(x^0) - g_G(\bar{x}^0) = 2y^2 + y\left(\frac{\partial g_G}{\partial x_{vi}}(x^0) - \frac{\partial g_G}{\partial x_{vj}}(x^0)\right).$$

It is clear that y can be chosen such that $g_G(x^0) > g_G(\bar{x}^0)$ which contradicts that x^0 is an optimal solution. Therefore, all optimal solutions have integer components. Moreover, if x^0 is such a solution, and A is the set of edges vw so that there exists i with $x^0_{vi}=x^0_{wi}=1$, then $G-A$ has a p-colouring given by $c(v)=i$, and $g_G(x^0)=|A|$. On the other hand, if A is a set of edges such that $G-A$ has a p-colouring $c : V \to \{1, 2, \ldots, p\}$ then, taking $x^A_{vc(v)}=1$ for all $v \in V$ and $x^A_{vi}=0$ otherwise, we obtain a feasible solution x^A for (2) with $g_G(x^A) \leqslant |A|$. This proves that the minimum value of (2) is $\mu(G, p)$.

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SYSTOLIC NETWORKS FOR NUMERICAL DIFFERENTIATION BY USING RICHARDSON-ROMBERG EXTRAPOLATION METHOD

BY

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1. Introduction. Systolic arrays processors offer an effective way to use a large number of processors in parallel in order to obtain a high computation rate, low parts count, modularity, simplicity and small size [5]—[8]. In recent years many systolic array algorithms have been proposed in order to solve problems in the field of numerical analysis [8].

In [3] they are presented some systolic arrays for estimating derivatives based on the Lagrange polynomial interpolation with unequally-spaced arguments.

This paper presents linear and two-dimensional systolic arrays implementing the Richardson-Romberg extrapolation method applied to central and one-sided divided differences in order to estimate the derivatives of many functions choosen as in [4].

In Section 2 we tre at the case of central divided differences and propose a linear systolic array and a two-dimensional systolic array. We prove the correctness of the working of these arrays and analyse their performances.

In Section 3, we discuss similar results for the case of one-sided divided differences.

2. The case of central divided differences. As described in [4], suppose that $f : D \to R$, $D \subset R$. Given a positive integer n and a step size h_1, they are computed

$$h_k = (n-k+1)\, h_1/n,\quad k=2, \ldots, n, \tag{1}$$

and the central divided differences

$$T_{0,\,k} = [f(x+h_k) - f(x-h_k)]\, c_k,\quad k=1, \ldots, n. \tag{2}$$

where $c_k = 1/(2h_k)$, $k=1, \ldots, n$.

The clock tick (CT) is defined as the time to perform a division or both a multiplication and an addition. The time is denoted by t which is a count of the number of CTs. We shall suppose that each processor is active during each pulse number. The processors used to compute (2) are given in Figure 1 (a)—(b).

They work so that: (a) $y(t+1) = -x(t)$ and (b) $z(t+1) = (x(t) + y(t))\,u(t)$. The cell CDD to compute (2) is depicted in Figure 1 (c). If L is