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BEHAVIOURISTIC AUTOMATA
II. THE SYNTHESIS PROBLEM

BY

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0. Introduction. The theory of Mealy and/or Moore automata and the theory of Rabin-Scott automata have developed quite independently of each other, although there are certain obvious links between them. In this series of papers we try to explain the relationships between the two types of automata by providing a common generalization of them and by recapturing a few basic results of the two theories within the new framework.

The first idea behind our generalization is that in the study of both types of automata we are interested in their *behaviour*, which in the case of Mealy automata means the sequences of outputs produced by various sequences of inputs, while in the case of Rabin-Scott automata the point of interest consists in the languages accepted by the automaton when various states are chosen as sources. Another natural idea is to treat automata as heterogeneous algebras. As a matter of fact both ideas have already appeared in the literature; see [5], pp. 296, 302 [3], [4] for the former idea and [1], [2] for the latter. However our approach is different, firstly because our automata are more general and secondly because whereas we start with the behaviour idea as in [3—5], we use the universal-algebra tools as in [1], [2]. We hope that this keeps our approach close to the spirit of the conventional theories and may result in teaching suggestions as well.

See the introduction to [7] for more comments and references to the literature.

1. Behaviouristic automata. We summarize below the prerequisites, definitions and results of [7].

A *semiautomaton* is a heterogeneous algebra (S, I, δ), where $\delta : S \times I \to S$ and the set S is finite. The *transition function* δ is extended to $\delta^* : S \times I^* \to S$, which satisfies

$$\delta^*(s, \lambda) = s, \tag{1.1}$$

$$\delta^*(s, w_1 w_2) = \delta^*(\delta^*(s, w_1), \ w_2) \tag{1.2}$$

for every $s \in S$ and $w_1, w_2 \in I^*$, where X^* stands for the set of all words over the alphabet X and λ is the empty word.

A *Mealy automaton* is a heterogeneous algebra $\mathfrak{A} = (S, I, O, \delta, \mu)$, where (S, I, δ) is a semiautomaton and $\mu : S \times I \to O$. The extension $\mu^* : S \times I^* \to O^*$ of the *output function* μ satisfies the identity

(2) $$\mu^*(s, w_1w_2)=\mu^*(s, w_1)\mu^*(\delta^*(s, w_1), w_2).$$

A *Rabin-Scott automaton* is a quadruple $\mathfrak{A}=(S, I, \delta, F)$, where (S, I, δ) is a semiautomaton and F is a non-empty subset of S (the subset of *final states*). For each $s\in S$, the set $L(\mathfrak{A}, s)=\{w\in I^* \mid \delta^*(s, w)\in F\}$ is the *language* generated by the automaton in state s.

Now let I and Ω be two fixed non-empty sets and $\Phi:\Omega^2\dashrightarrow\Omega$ a fixed partially defined function. By an *I-Ω-Φ - automaton* or a *behaviouristic automaton* or simply an *automaton* we mean a 5-tuple $\mathfrak{A}=(S, I, \Omega, \delta, \beta)$, where (S, I, δ) is a semiautomaton and

$$\beta: S\to\Omega^{I^*}$$

(the *behaviour function*) is a function such that for every $s\in S$ and $w\in I^*$,

(3) $$\beta(\delta^*(s, w))(w')=\Phi(\beta(s)(w), \beta(s)(ww'))$$

holds for all words w of length $|w|\leqslant 1$.

Then Mealy automata (S, I, O, δ, μ) coincide with those behaviouristic automata for which $\Omega=O^*$, $\Phi(\omega_1, \omega_1\omega_2)=\omega_2$ and Φ is not defined otherwise, S is finite and $\beta(s)(i)\in O$ for every $s\in S$, $i\in I$[1]; the transformation is given by $\beta(s)(w)=\mu^*(s, w)$ for every $s\in S$, $w\in I^*$. Also, Rabin-Scott automata (S, I, δ, F) coincide with those behaviouristic automata for which $\Omega=\{0, 1\}$, $\Phi(x, y)=y$ for every $x, y\in\{0, 1\}$, and S is finite; the transformation is given by $\beta(s)(w)=1\Leftrightarrow\delta^*(s, w)\in F$.

We have sketched the algebraic background of behaviouristic automata, starting with the concept of *homomorphism* between two automata, $\mathfrak{A}=(S, I, \Omega, \delta, \beta)$ and $\mathfrak{A}'=(S', I, \Omega, \delta', \beta')$: by this term is meant a map $h: S\to S'$ such that

(4) $$h(\delta^*(s, w))=\delta'^*(h(s), w),$$

(5) $$\beta'(h(s))=\beta(s)$$

hold for every $s\in S$ and $w\in I^*$. The corresponding concept of *congruence* of an automaton is defined as follows: an equivalence $\equiv$ on S such that

(6) $$s\equiv s_1\Rightarrow\delta(s, i)\equiv\delta(s_1, i) \;\;\&\;\; \beta(s)=\beta(s_1)$$

for every $s, s_1\in S$ and $i\in I$. In particular if $h: S\to S'$ (if $h: S\to S'$ is an automaton homomorphism) then the relation $\ker h$ defined by

(7) $$s_1 \ker h\, s_2\Leftrightarrow h(s_1)=h(s_2)$$

is an equivalence (a congruence). The *quotient* $\hat{\mathfrak{A}}=\mathfrak{A}/\equiv$ of the automaton $\mathfrak{A}$ by the congruence $\equiv$ is defined as $\hat{\mathfrak{A}}=(\hat{S}, I, \Omega, \hat{\delta}, \hat{\beta})$ where $\hat{S}=S/\equiv$, $\hat{\delta}: \hat{S}\times I\to\hat{S}$ is the map such that $\hat{\delta}(\hat{s}, i)=\widehat{\delta(s, i)}$ and $\hat{\beta}: \hat{S}\to\Omega^{I^*}$ is defined by $\hat{\beta}(\hat{s})=\beta(s)$.

A source automaton is a couple $(\mathfrak{A}, s_0)$, where $\mathfrak{A}$ is an automaton and $s_0\in S$ is a *source* or *central state*, which means that every state is reachable from s_0, i.e. $\forall s\in S\,\exists w\in I^*\; s=\delta^*(s_0, w)$. A *source automaton homomorphism* h: from $(\mathfrak{A}, s_0)\to(\mathfrak{A}', s_0')$ is an automaton homomorphism $h: \mathfrak{A}\to\mathfrak{A}'$ such that $h(s_0)=s_0'$.

[1] The latter condition was mistakenly omitted in [7].

We say that two automata are *equivalent*, $\mathfrak{A}\sim\mathfrak{A}'$, provided they have the same set of behaviours, i.e. $\{\beta(s)\mid s\in S\}=\{\beta'(s')\mid s'\in S'\}$. Also, we say that two source automata are *strongly equivalent*, $(\mathfrak{A}, s_0)\sim(\mathfrak{A}', s_0')$, provided $\beta(s_0)=\beta'(s_0')$. The main result in [7] is the construction of an automaton (a source automaton) having the least number of states among the automata equivalent to a given automaton (source automata strongly equivalent to a given source automaton). This minimal automaton is obtained as a quotient automaton and is unique up to an isomorphism. We have thus generalized the well known minimization of Mealy automata and that of Rabin-Scott automata.

2. The synthesis problem. The *synthesis problem* (a term which we borrow from switching theory) *for a function* $\gamma: I^*\to\Omega$ consists in finding a source automaton $(S, I, \Omega, \delta, \beta, s_0)$ which *synthesizes* γ, i.e. such that $\gamma=\beta(s_0)$. If such an automaton exists it will be called a *solution* and we will say that the synthesis problem is *consistent*. In this paper a necessary and sufficient condition for the consistency of the synthesis problem will be found and in the case of a consistent problem a solution will be actually constructed. The results will be then specialized to Mealy and Rabin-Scott automata.

The synthesis problem will be solved under the hypothesis that the function Φ satisfies the identity

(8) $$\Phi(\Phi(x, y), \Phi(x, z))=\Phi(y, z),$$

to the effect that if $\Phi(x, y)$, $\Phi(x, z)$ and one of the two sides of (8) exist, then the other side exists too and the equality holds.

Axiom (8) is fulfilled in the case of Rabin-Scott automata because

$$\Phi(\Phi(x, y), \Phi(x, z))=\Phi(x, z)=z=\Phi(y, z)$$

and it is verified in the case of Mealy automata as well. For suppose $\Phi(x, y)$ and $\Phi(x, z)$ exist, that is $y=xy'$ and $z=xz'$, so that (8) reads $\Phi(y', z')=\Phi(xy', xz')$. Then the existence of either side is equivalent to $z'=y'z''$, in which case both sides equal z''.

Proposition 1. *Suppose axiom* (8) *holds. Then every automaton satisfies identity* (3) *for every* $w\in I^*$.

Proof. If $|w|\leqslant 1$ then (3) holds by the definition of behaviouristic automata. If w verifies (3) and $i\in I$ then applying in turn (1.2), (3) for i and for w, then (8), we get

$$\beta(\delta^*(s, wi))(w')=\beta(\delta^*(\delta^*(s, w), i))(w')=\Phi(\beta(\delta^*(s, w))(i), \beta(\delta^*(s, w))(iw'))=$$

$$=\Phi(\Phi(\beta(s)(w), \beta(s)(wi)), \Phi(\beta(s)(w), \beta(s)(wiw'))=\Phi(\beta(s)(wi), \beta(s)(wiw')). \blacksquare$$

Now associate with γ the following relation $\sim_\gamma$ on I^*:

(9) $$w_1\sim_\gamma w_2\Leftrightarrow\forall w\in I^*\;\gamma(w_1w)=\gamma(w_2w).$$

Proposition 2. *The relation* $\sim_\gamma$ *is the greatest right invariant equivalence included in* $\ker\gamma$.

Proof. Clearly $\sim_\gamma$ is an equivalence and taking $w=\lambda$ in (9) it follows that $\sim_\gamma\subseteq\ker\gamma$. *Right invariance*, i.e.

(10) $$w_1\sim_\gamma w_2\Rightarrow\forall w_0\in I^*\;w_1w_0\sim w_2w_0,$$

follows also easily. Finally if $\equiv$ is a right invariant equivalence included in $\ker\gamma$ and $w_1 \equiv w_2$ then for every w we have $w_1 w \equiv w_2 w$ hence $w_1 w \ \ker\gamma \ w_2 w$, that is $\gamma(w_1 w) = \gamma(w_2 w)$: this proves $w_1 \sim_\gamma w_2$. ∎

Theorem. *Suppose axiom (8) is fulfilled and let $\gamma : I^* \to \Omega$. Then :*

(i) *The synthesis problem for γ is consistent if and only if the following conditions hold*

$$\Phi(\gamma(w), \gamma(ww')) \text{ exists for every } w, w' \in I^*, \tag{11}$$

$$\Phi(\gamma(\lambda), \gamma(w)) = \gamma(w) \text{ for every } w \in I^*. \tag{12}$$

(ii) *When this is the case, a solution $(\hat{\mathfrak{A}}, \hat{\lambda})$ is obtained as follows: $\hat{\mathfrak{A}} = (S, I, \Omega, \hat{\delta}, \hat{\beta})$, where $S = I^*/\sim_\gamma$, $\hat{\delta}^* : S \times I^* \to S$ is defined by*

$$\hat{\delta}^*(\hat{w}, w') = \widehat{ww'} \tag{13}$$

and $\hat{\beta} : S \to \Omega^{I^*}$ is defined by

$$\hat{\beta}(\hat{w})(w') = \Phi(\gamma(w), \gamma(ww')). \tag{14}$$

Proof. Suppose first that $(S, I, \Omega, \delta, \beta, s_0)$ is a solution of the synthesis problem for γ. Then it follows from Proposition 1 that

$$\beta(\delta^*(s_0, w))(w') = \Phi(\beta(s_0)(w), \beta(s_0)(ww')) = \Phi(\gamma(w), \gamma(ww'))$$

exists, i.e. (11) holds. Then (12) follows from

$$\gamma(w) = \beta(s_0)(w) = \beta(\delta^*(s_0, \lambda))(w) = \Phi(\beta(s_0)(\lambda), \beta(s_0)(w)) = \Phi(\gamma(\lambda), \gamma(w)).$$

Conversely, suppose γ satisfies (11), (12) and define $\hat{\mathfrak{A}}$ as indicated in the statement of the theorem.

If $\hat{w'} = \hat{w''}$ then $\widehat{w'w} = \widehat{w''w}$ by Proposition 2, hence $\hat{\delta}^*$ is well defined. Moreover, $\hat{\delta}^*(\hat{w}, \lambda) = \widehat{w\lambda} = \hat{w}$ and

$$\hat{\delta}^*(\hat{w}, w_1 w_2) = \widehat{ww_1 w_2} = \hat{\delta}^*(\widehat{ww_1}, w_2) = \hat{\delta}^*(\hat{\delta}^*(\hat{w}, w_1), w_2),$$

therefore $(S, I, \hat{\delta})$ is a semiautomaton.

The map $\hat{\beta}$ is well defined because the right side of (14) exists by (11) and if $\hat{w}_1 = \hat{w}_2$ then $\gamma(w_1) = \gamma(w_2)$ by Proposition 2 and $\gamma(w_1 w) = \gamma(w_2 w)$ by the very definition of $\sim_\gamma$. Moreover, it follows from (13), (14) and (8) that

$$\hat{\beta}(\hat{\delta}^*(\hat{w}_0, w))(w') = \hat{\beta}(\widehat{w_0 w})(w') = \Phi(\gamma(w_0 w), \gamma(w_0 ww')) = \Phi(\Phi(\gamma(w_0), \gamma(w_0 w)),$$

$$\Phi(\gamma(w_0), \gamma(w_0 ww'))) = \Phi(\hat{\beta}(\hat{w}_0)(w), \hat{\beta}(\hat{w}_0)(ww')).$$

Thus $\hat{\mathfrak{A}}$ is an automaton and besides, $\hat{\lambda}$ is a central state because $\hat{\delta}^*(\hat{\lambda}, w) = \widehat{\lambda w} = \hat{w}$ for every $\hat{w} \in S$. Finally $\hat{\beta}(\hat{\lambda})(w) = \Phi(\gamma(\lambda), \gamma(w)) = \gamma(w)$ by (12). ∎

In the case of Rabin-Scott automata the function $\gamma : I^* \to \{0, 1\}$ can be identified with the language $L \subseteq I^*$ such that $w \in L \Leftrightarrow \gamma(w) = 1$. Since the

language $L(\mathfrak{A}, s_0)$ accepted by an initial-state automaton $(\mathfrak{A}, s_0) = (S, I, \delta, s_0, F)$ is characterized by

$$w \in L(\mathfrak{A}, s_0) \Leftrightarrow \delta^*(s_0, w) \in F \Leftrightarrow \beta(s_0)(w) = 1 \Leftrightarrow \gamma(w) = 1,$$

it follows that the synthesis problem for γ becomes the problem of finding an automaton $(\mathfrak{A}, s_0)$ with $L(\mathfrak{A}, s_0) = L$. We now recapture the following well known result :

Proposition 3. (i) *A language $L \subseteq I^*$ is accepted by an initial-state Rabin-Scott automaton if and only if the equivalence $\sim_L$ on I^* defined by*

$$w_1 \sim_L w_2 \Leftrightarrow \forall w \in I^* (w_1 w \in L \Leftrightarrow w_2 w \in L) \tag{15}$$

is of finite index.

(ii) *When this is the case, the minimal automaton $(\hat{\mathfrak{A}}, \hat{\lambda}) = (S, I, \hat{\delta}, \hat{\lambda}, F)$ which accepts L is constructed as follows : $S = I^*/\sim_L$, $\hat{\delta}^*(\hat{w}, w') = \widehat{ww'}$ $\forall \hat{w} \in S$, $\forall w' \in I^*$, $F = \{\hat{w} \mid w \in L\}$.*

Proof. Note first that (15) is nothing but the equivalence (9) expressed in terms of L. Furthermore, conditions (11) and (12) in the Theorem are trivially fulfilled for $\Phi(x, y) = y$, so that the behaviouristic automaton described in the Theorem synthesizes γ. Moreover, since (14) reduces to $\hat{\beta}(w)(w') = \gamma(ww')$, it follows that for every $w_1, w_2 \in I^*$,

$$\hat{w}_1 \ker\hat{\beta} \ \hat{w}_2 \Leftrightarrow \gamma(w_1 w) = \gamma(w_2 w) \forall w \in I^* \Leftrightarrow w_1 \sim_\gamma w_2 \Leftrightarrow \hat{w}_1 = \hat{w}_2,$$

therefore $\hat{\mathfrak{A}}/\ker\hat{\beta} = \hat{\mathfrak{A}}$, which shows that $\hat{\mathfrak{A}}$ is minimal by Theorem 2 in [7].

Now if $\sim_L$, alias $\sim_\gamma$, is of finite index, then S is finite, so that $(\hat{\mathfrak{A}}, \hat{\lambda})$ becomes an initial-state Rabin-Scott automaton, where

$$F = \{\hat{\delta}^*(\hat{\lambda}, w) \mid w \in L(\hat{\mathfrak{A}}, \hat{\lambda})\} = \{\hat{w} \mid w \in L\}.$$

Conversely, if L is accepted by a Rabin-Scott automaton $(\mathfrak{B}, s_0)$, then since $\mathfrak{B}$ (viewed as a behaviouristic automaton) has finitely many states, it follows that the set S of states of the minimal automaton $(\hat{\mathfrak{A}}, \hat{\lambda})$ which synthesizes γ is a fortiori finite, i.e. $\sim_\gamma$, alias $\sim_L$, is of finite index.

In the case of Mealy automata the synthesis of a function $\gamma : I^* \to O^*$ consists in finding an initial-state Mealy automaton $(S, I, O, \delta, \mu, s_0)$ such that $\gamma(w) = \mu(s_0, w) \forall w \in I^*$. We need some prerequisites.

Lemma 1. *Let $\Phi: O^* \times O^* \to O^*$ be defined by $\Phi(\omega_1, \omega_1\omega_2) = \omega_2$ $\forall \omega_1, \omega_2 \in O^*$. Let $\gamma : I^* \to O^*$. Then*

1° *The following conditions are equivalent*

(i) (11) *and* (12) ;

(ii) γ *satisfies*

$$\gamma(\lambda) = \lambda. \tag{16}$$

and there is a function $z : I^* \times I^* \to O^*$ such that

$$\gamma(ww') = \gamma(w) z(w') \quad w, w' \in I^* ; \tag{17}$$

(iii) (16) *holds and there is a function* $\varphi : I^* \times I \to O^*$ *such that*

$$\gamma(wi) = \gamma(w)\varphi(w, i) \quad \forall w \in I^* \quad \forall i \in I. \tag{18}$$

2° *When this is the case the following identities hold:*

$$\varepsilon(w, \lambda) = \lambda \quad \forall w \in I^*, \tag{19}$$

$$\varepsilon(w, i) = \varphi(w, i) \quad \forall w \in I^* \quad \forall i \in I, \tag{20}$$

$$\varepsilon(w, i_1 \dots i_n) = \varepsilon(w, i_1)\varepsilon(wi_1, i_2) \dots \varepsilon(wi_1 \dots i_{n-1}, i_n) \quad \forall w \in I^* \quad \forall i_1, \dots, i_n \in I \quad \forall n \in \mathbb{N},\ n \geqslant 2, \tag{21}$$

$$\gamma(w) = \varepsilon(\lambda, w) \quad \forall w \in I^*. \tag{22}$$

Comment. Note that (17) is equivalent to

$$\Phi(\gamma(w), \gamma(ww')) = \varepsilon(w, w') \quad \forall w, w' \in I^*. \tag{17'}$$

Proof. 1°. (i)$\Leftrightarrow$(ii): Clearly (17) is a paraphrase of (11) while, (16) implies (12) because

$$\Phi(\gamma(\lambda), \gamma(w)) = \Phi(\lambda, \lambda\gamma(w)) = \gamma(w).$$

Conversely, suppose (11) and (12). Then (17) holds, as remarked above, while (16) follows from (17), (17') and (12):

$$\gamma(w) = \gamma(\lambda w) = \gamma(\lambda)\varepsilon(\lambda, w) = \gamma(\lambda)\Phi(\gamma(\lambda), \gamma(w)) = \gamma(\lambda)\gamma(w),$$

therefore $\gamma(\lambda) = \lambda$.

(ii)$\Leftrightarrow$(iii): Suppose (ii) and define φ by (20). Then (18) holds. Conversely, suppose (iii) and define ε by (19)–(21). Then (17) follows by induction on w':

$$\gamma(w\lambda) = \gamma(w)\lambda = \gamma(w)\varepsilon(w, \lambda),\ \gamma(wi) = \gamma(w)\varphi(w, i) = \gamma(w)\varepsilon(w, i),$$

$$\gamma(wi_1 \dots i_n) = \gamma(wi_1 \dots i_{n-1})\varphi(wi_1 \dots i_{n-1}, i_n) = \gamma(wi_1 \dots i_{n-2})\varphi(wi \dots i_{n-2}, i_{n-1})\varphi(wi_1 \dots i_{n-1}, i_n)$$
$$= \dots = \gamma(w)\varphi(w, i_1)\varphi(wi_1, i_2) \dots \varphi(wi_1 \dots i_{n-1}, i_i) = \gamma(w)\varepsilon(w, i_1)\varepsilon(wi_1, i_2) \dots$$
$$\dots \varepsilon(wi_1 \dots i_{n-1}, i_n) = \gamma(w)\varepsilon(w, i_1 \dots i_n).$$

2°. Suppose (16)–(18). Then taking in turn $w' = \lambda$ and $w' = i$ in (17) we obtain (19) and (20) (via (18)), respectively. Then from

$$\gamma(wi_1 \dots i_n) = \gamma(w)\varepsilon(w, i_1 \dots i_n)$$

and

$$\gamma(wi_1 \dots i_n) = \gamma(wi_1 \dots i_{n-1})\varepsilon(wi_1 \dots i_{n-1}, i_n) = \gamma(wi_1 \dots i_{n-2})\varepsilon(wi_1 \dots i_{n-2}, i_{n-1})\varepsilon(wi_1 \dots i_{n-1}, i_n)$$
$$= \dots = \gamma(w)\varepsilon(w, i_1)\varepsilon(w, i_1 i_2) \dots \varepsilon(wi_1 \dots i_{n-1}, i_n)$$

we infer (21). Finally (22) follows from (17) with $w = \lambda$, via (16).

Lemma 2. *If the hypothesis and the equivalent conditions in Lemma 1 are fulfilled, then a minimal source behaviouristic automaton* $(\check{\mathfrak{A}}, \check{\lambda}) = (S, I, O^*, \check{\delta}, \check{\beta}, \check{\lambda})$ *which synthesizes* γ *is obtained as follows: define the equivalence* $\ker_1 \varepsilon$ *on* I^* *by*

$$w' \ker_1 \varepsilon\, w'' \Leftrightarrow \forall w \in I^*\ \varepsilon(w', w) = \varepsilon(w'', w) \tag{23}$$

and let $\check{w}$ *denote the class of an element* $w \in I^*$; *then*

$$S = I^* / \ker_1 \varepsilon., \tag{24}$$

$$\check{\delta}^*(\check{w}, w') = \widecheck{ww'} \quad \forall \check{w} \in S \quad \forall w' \in I^*, \tag{25}$$

$$\check{\beta}(\check{w})(w') = \varepsilon(w, w') \quad \forall \check{w} \in S \quad \forall w' \in I^*. \tag{26}$$

Proof. It follows from Lemma 1 and the Theorem that the behaviouristic automaton $(\hat{\mathfrak{A}}, \hat{\lambda}) = (I^*/\sim_\gamma, I, O^*, \hat{\delta}, \hat{\beta}, \hat{\lambda})$ constructed in the Theorem synthesizes γ. We now apply Theorem 2 in [7] to $(\hat{\mathfrak{A}}, \hat{\lambda})$ to obtain the minimal behaviouristic source automaton $(\mathfrak{A}, [\hat{\lambda}])$ strongly equivalent to $(\hat{\mathfrak{A}}, \hat{\lambda})$. Let $S_0 = (I^*/\sim_\gamma)/\ker \hat{\beta}$ and denote by $[\hat{w}] \in S_0$ the class of an element $w \in I^*/\sim_\gamma$, where $w \in I^*$. Then $(\mathfrak{A}_0, [\hat{\lambda}]) = (S_0, I, O^*, \delta_0, \beta_0, [\hat{\lambda}])$, where

$$\delta_0^*([\hat{w}], w') = [\hat{\delta}^*(\hat{w}, w')] = [\widehat{ww'}],$$

$$\beta_0([\hat{w}])(w') = \hat{\beta}(\hat{w})(w') = \Phi(\gamma(w), \gamma(ww')) = \varepsilon(w, w').$$

Futher we note that

$$\check{w}' = \check{w}'' \Leftrightarrow [\hat{w}'] = [\hat{w}''] \tag{27}$$

because

$$\check{w}' = \check{w}'' \Leftrightarrow w' \ker_1 \varepsilon\, w'' \Leftrightarrow \forall w \in I^*\ \varepsilon(w', w) = \varepsilon(w'', w) \Leftrightarrow \forall w \in I^* \quad \hat{\beta}(\hat{w}')(w) =$$
$$= \hat{\beta}(\hat{w}'')(w) \Leftrightarrow \hat{\beta}(\hat{w}') = \hat{\beta}(\hat{w}'') \Leftrightarrow \hat{w}' \ker \hat{\beta}\ \hat{w}'' \Leftrightarrow [\hat{w}'] = [\hat{w}''].$$

Now we claim that $(\check{\mathfrak{A}}, \check{\lambda})$ constructed in the present Lemma is a behaviouristic source automaton isomorphic to $(\mathfrak{A}_0, [\hat{\lambda}])$. For set $h([\hat{w}]) = \check{w}\ \forall [\hat{w}] \in S_0$. It follows from (27) that the map $h : S_0 \to S$ is well defined and injective. Clearly h is also a surjection and besides,

$$h([\hat{\lambda}]) = \check{\lambda}, \tag{28}$$

$$\delta^*(h([\hat{w}]), w') = \check{\delta}^*(\check{w}, w') = \widecheck{ww'} = h([\widehat{ww'}]) = h(\hat{\delta}^*([\hat{w}], w')), \tag{29}$$

$$\check{\beta}(h([\hat{w}]))(w') = \check{\beta}(\check{w})(w') = \varepsilon(w, w') = \beta_0([\hat{w}])(w'),$$

therefore

$$\check{\beta}(h([\hat{w}])) = \beta_0([\hat{w}]). \tag{20}$$

The bijectivity of h and properties (27)–(30) imply that the maps $\check{\delta}^*$ and $\check{\beta}$ are well defined and $(\check{\mathfrak{A}}, \check{\lambda})$ is actually an automaton isomorphic to $(\mathfrak{A}_0, [\hat{\lambda}])$. This isomorphism shows that $(\check{\mathfrak{A}}, \check{\lambda})$ shares the property of being a minimal source behaviouristic automaton strongly equivalent to $(\hat{\mathfrak{A}}, \hat{\lambda})$. But the class of automata strongly equivalent to $(\check{\mathfrak{A}}, \check{\lambda})$ is precisely the class of all automata which synthesize γ. Therefore $(\check{\mathfrak{A}}, \check{\lambda})$ is a minimal automaton which synthesizes γ. ∎

Lemma 3. *If the function γ is synthesized by an initial-state Mealy automaton, then γ satisfies the equivalent conditions in Lemma 1 and*

$$\varepsilon(w, i) \in O \quad \forall w \in I^* \quad \forall i \in I. \tag{31}$$

Proof. The first statement follows from the Theorem and Lemma 1. Besides, for every $w' \in I^*$,

$$\gamma(w') = \beta(s_0)(w') = \mu^*(s_0, w'),$$

hence for every $w \in I^*$ and $i \in I$,

$$\gamma(w)\varepsilon(w, i) = \gamma(wi) = \mu^*(s_0, wi) = \mu^*(s_0, w)\mu(\delta^*(s_0, w), i) = \gamma(w)\mu(\delta^*(s_0, w), i)$$

therefore $\varepsilon(w, i) = \mu(\delta^*(s_0, w), i) \in O$. ∎

Proposition 4. (i) *A function $\gamma : I^* \to O^*$ is synthesized by an initial-state Mealy automaton if and only if it satisfies the equivalent conditions in Lemma 1, (31) and the equivalence* $\ker_1 \varepsilon$ *defined by (23) is of finite index.*

(ii) *When this is the case, the unique (up to an isomorphism) minimal initial-state Mealy automaton which synthesizes γ is $(\mathfrak{A}_m, \check{\lambda}) = (S, I, O, \check{\delta}, \check{\mu}, \check{\lambda})$, where $S = I^*/\ker_1\varepsilon$ and the functions $\check{\delta}^*$, $\check{\mu}^*$ are defined by $\check{\delta}^*(\check{w}, w') = \widecheck{ww'}$ and $\check{\mu}^*(\check{w}, w') = \varepsilon(w, w')$.*

Proof. Suppose γ satisfies the conditions in Lemma 1, (31) and $\ker_1 \varepsilon$ is of finite index. Then the automaton $(\check{\mathfrak{A}}, \check{\lambda})$ constructed in Lemma 2 is a minimal behaviouristic automaton which synthesizes γ. Furthermore $S = I^*/\ker_1\varepsilon$ is finite and since

$$\check{w}' \ker \check{\beta}\check{w}'' \Leftrightarrow w' \ker_1\varepsilon w'' \Leftrightarrow \check{w}' = \check{w}'',$$

it follows that the minimal automaton $(\check{\mathfrak{A}}, \check{\lambda})/\ker \check{\beta} = (\check{\mathfrak{A}}, \check{\lambda})$ is unique up to an isomorphism by Theorem 2 in [7]. On the other hand

$$\check{\beta}(\check{w})(i) = \varepsilon(w, i) \in O \quad \forall \check{w} \in S \quad \forall i \in I,$$

$$\check{\beta}(\check{w})(w') = \varepsilon(w, w') = \check{\mu}^*(\check{w}, w') \quad \forall \check{w} \in S \quad \forall w' \in I^*,$$

therefore $(\check{\mathfrak{A}}, \check{\lambda})$ is associated with the Mealy automaton $(\mathfrak{A}_m, \check{\lambda})$. It follows easily that the latter is the minimal Mealy automaton which synthesizes γ and is unique up to an isomorphism, as well (a homomorphism of Mealy automata is defined by (4) and $\mu'^*(h(s), w) = \mu^*(s, w)$ instead of (5), plus $h(s_0) = s_0'$ for source automata).

Conversely, suppose γ is synthesized by an initial-state Mealy automaton $(\mathfrak{B}, s_0)$. Then γ satisfies the equivalent conditions in Lemma 1 and (31), by

Lemma 3. On the other hand $(\mathfrak{B}, s_0)$ is strongly equivalent to the minimal automaton $(\check{\mathfrak{A}}, \check{\lambda})$ constructed in Lemma 2 and since $(\mathfrak{B}, s_0)$ has finitely many states, the minimality of $(\check{\mathfrak{A}}, \check{\lambda})$ implies that $S = I^*/\ker_1 \varepsilon$ is a fortiori finite, i.e. $\ker_1 \varepsilon$ is of finite index. ∎

An example. Let us construct a Mealy automaton which should operate as follows. As long as an input signal is maintained, an output signal is produced every four time units; in the absence of the imput signal there is no output signal (the rest position).

Thus $I = O = \{0,1\}$ and the function γ is defined by

$$\gamma(0^{p_0}1^{4q_1+r_1}0^{p_1}1^{4q_2+r_2}0^{p_2}\ldots 1^{4q_n+r_n}0^{p_n}) = 0^{p_0}(0001)^{q_1}0^{r_1+p_1}(00001)^{q_2}0^{r_2+p_2}\ldots$$
$$\ldots(0001)^{q_n}0^{r_n+p_n} \tag{32}$$

for every $p_0, p_1, \ldots, p_n, q_1, \ldots, q_n \in \mathbb{N}$, $r_1, \ldots, r_n \in \{0, 1, 2, 3\}$, $n \in \mathbb{N}$ which satisfy $q_i + r_i > 0$ for $i = 1, \ldots, n$ and $p_j > 0$ for $j = 1, \ldots, n-1$ whenever $n > 0$ (note that $w^0 = \lambda$ for every $w \in \{0,1\}^*$).

It follows easily from (32) that γ satisfies (16), (17) and (31). Besides, for each $r \in \{0, 1, 2, 3\}$, $q \in \mathbb{N}$ and $w, w' \in \{0,1\}^*$,

$$\gamma(w01^{4q+r}w') = \gamma(w)0(0001)^q\gamma(1^r w') = \gamma(w)0(0001)^q\gamma(1^r)\varepsilon(1^r, w') =$$
$$= \gamma(w)0(0001)^q 0^r \varepsilon(1^r, w') = \gamma(w01^{4q+r})\varepsilon(1^r, w'),$$

therefore $\varepsilon(w01^{4q+r}, w') = \varepsilon(1^r, w')$ and similarly $\varepsilon(1^{4q+r}, w') = \varepsilon(1^r, w')$. This shows that $\{0,1\}^*/\ker_1 \varepsilon$ consists of four classes:

$$s_r = \{1^{4q+r} \mid q \in \mathbb{N}\} \cup \{w01^{4q+r} \mid w \in \{0,1\}^*,\ q \in \mathbb{N}\} (r = 0, 1, 2, 3).$$

Now Proposition 4 yields the minimal Mealy automaton, which synthesizes the above function γ: $(S, \{0, 1\}, \{0, 1\}, \check{\delta}, \check{\mu}, \check{\lambda})$, where $S = \{s_0, s_1, s_2, s_3\}$ and it remains to construct $\check{\delta}$ and $\check{\mu}$. Since $1^r \in s_r$ and $w0 \in s_0$ (take $q = r = 0$) we obtain

$$\check{\delta}(s_r, 0) = 1^r0 = s_0,$$

$$\check{\delta}(s_r, 1) = 1^r1 = s_{r+1} \text{ modulo } 4,$$

and since

$$\gamma(1^r i) = \begin{cases} 0001 = \gamma(1^3)1 & \text{if } r = 3 \text{ and } i = 1, \\ 0^{r+1} = \gamma(1^r)0 & \text{otherwise,} \end{cases}$$

it follows that

$$\check{\mu}(s_r, 0) = \varepsilon(1^r, 0) = 0,$$

$$\check{\mu}(s, 1) = \varepsilon(1^r, 1) = \begin{cases} 1 \text{ if } r = 3, \\ 0 \text{ if } r \neq 3. \end{cases}$$ ∎

This paper was much influenced by the work of Gr. C. Moisil in switching theory. It would be interesting to relate the present study to the theory of exact and normal programmes in [6].

REFERENCES

1. Birkhoff, G., Lipson, J. D. — *Heterogeneous algebras*. J. Combinatorial Theory 8 (1970), 115–133.
2. Birkhoff, G., Lipson, J. D. — *Universal algebra and automata*. Proceedings of the Tarski Symposium. Proc. Symp. Pure Math., XXV, Univ. of Calif., Berkeley 1971, pp. 41–51.
3. Goguen, J. A. — *Realization is universal*. Math. Syst. Theory 6 (1973), 350–374.
4. Goguen, J. A., Thatcher, W., Wagner, E. G., Wright, J. B. — *A junction between computer science and category theory. Basic concepts and examples. Part I*. IBM Research RC 4526, September 11, 1973.
5. Harrison, M. A. — *Introduction to Switching*. McGraw Hill, New York 1965.
6. Moisil, Gr. C. — *The Algebraic Theory of Switching Circuits*, Pergamon Press, Oxford / New York & Editura Tehnică, Bucharest 1969.
7. Rudeanu, S. — *Behaviouristic automata. I. Minimal automata*. An. Univ. Bucureşti, Mat., 38 (1989), No. 2, (Mat.-Inf.), 64–71.

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INFINITE PLANAR DAGS

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1. Introduction. The isomorphism between the partially ordered set of dags as terms [1] and the partially ordered set of planar dags as functions is proved. The partial order we introduce permits to define infinite terms as being l.u.b. of directed sets of ideals of directed sets. We show that the set of infinite terms is isomorphic with the set of infinite planar dags as functions.

2. Preliminaries. In this section we give the notations and the definitions used in the paper.

$[\omega]$ denotes the set of neonnegative integers.

$[\omega]^*$ denotes the free monoid generated by $[\omega]$.

if $x \in [\omega]^*$ then $|x|$ denotes the length of x.

if $R \subset X \times X$ is a relation and $A \subset X$, then $R|_A$ denotes the restriction of R to A.

if S is a finite set, then $|S|$ denotes the number of the elements of S.

A magmoid is a partial algebra $< M, \cdot, \otimes, 1, 0 >$ where $M = \cup \langle M_q^p : p, q \in [\omega] \rangle$, . is a partial binary operation called sequential composition, $\otimes$ is a total binary operation called parallel composition, $1 \in M_1^1$ and $0 \in M_0^0$ are two special constants and moreover the following conditions are satisfied:

(M1) $\forall p, p', q, q' \in [\omega] : \langle p, q \rangle \neq \langle p', q' \rangle \Rightarrow M_q^p \cap M_{q'}^{p'} = \Phi$,

(M2a) $\forall p, p', q, q' \in [\omega] \forall a \in M_q^p \ \forall b \in M_{q'}^{p'}: ab \in M \Leftrightarrow q = p' \wedge ab \in M_q^p$.

(M2b) $\cdot$ is associative,

(M3a) $\forall p, p', q, q' \in [\omega] \ \forall a \in M_q^p, \forall b \in M_{q'}^{p'}: a \otimes b \ M_{q+q'}^{p+p'}$,

(M3b) $\otimes$ is associative,

(M4) $\forall a, a', b, b' \in M : (ab) \otimes (a'b') \in M \Rightarrow (ab) \otimes (a'b') = (a \otimes a')(b \otimes b')$,

(M5a) $\forall p, q \in [\omega] \forall a \in M_q^p : 1^{\otimes p} a = a 1^{\otimes q} = a$,

(M5b) $\forall a \in M : a \otimes 0 = 0 \otimes a = a$,

where $a^{\otimes p} = a \otimes \ldots \otimes a$ (p times).

By a^p we denote the element $a^p = a \ldots a$ (p times).