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INFINITE PLANAR DAGS

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1. Introduction. The isomorphism between the partially ordered set of dags as terms [1] and the partially ordered set of planar dags as functions is proved. The partial order we introduce permits to define infinite terms as being l.u.b. of directed sets of ideals of directed sets. We show that the set of infinite terms is isomorphic with the set of infinite planar dags as functions.

2. Preliminaries. In this section we give the notations and the definitions used in the paper.

$[\omega]$ denotes the set of neonnegative integers.

$[\omega]^*$ denotes the free monoid generated by $[\omega]$.

if $x \in [\omega]^*$ then $|x|$ denotes the length of x.

if $R \subset X \times X$ is a relation and $A \subset X$, then $R|_A$ denotes the restriction of R to A.

if S is a finite set, then $|S|$ denotes the number of the elements of S.

A magmoid is a partial algebra $< M, \cdot, \otimes, 1, 0 >$ where $M = \cup \langle M^p_q : p, q \in [\omega] \rangle$, $\cdot$ is a partial binary operation called sequential composition, $\otimes$ is a total binary operation called parallel composition, $1 \in M^1_1$ and $0 \in M^0_0$ are two special constants and moreover the following conditions are satisfied :

(M1) $\forall p, p', q, q' \in [\omega] : \langle p, q \rangle \neq \langle p', q' \rangle \Rightarrow M^p_q \cap M^{p'}_{q'} = \Phi$,

(M2a) $\forall p, p', q, q' \in [\omega] \forall a \in M^p_q \ \forall b \in M^{p'}_{q'} : ab \in M \Leftrightarrow q = p' \wedge ab \in M^p_q$.

(M2b) $\cdot$ is associative,

(M3a) $\forall p, p', q, q' \in [\omega] \ \forall a \in M^p_q, \forall b \in M^{p'}_{q'} : a \otimes b \ M^{p+p'}_{q+q'}$,

(M3b) $\otimes$ is associative,

(M4) $\forall a, a', b, b' \in M : (ab) \otimes (a'b') \in M \Rightarrow (ab) \otimes (a'b') = (a \otimes a')(b \otimes b')$,

(M5a) $\forall p, q \in [\omega] \forall a \in M^p_q : 1^{\otimes p} a = a 1^{\otimes q} = a$,

(M5b) $\forall a \in M : a \otimes 0 = 0 \otimes a = a$,

where $a^{\otimes p} = a \otimes \ldots \otimes a$ (p times).

By a^p we denote the element $a^p = a \ldots a$ (p times).

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3. Terms. In this section we recall the magmoid generated by a biranked alphabet and we prove some properties of this magmoid.

Definition 3.1. *A biranked alphabet is formed by a set Σ of symbols and two mappings, arity, coarity: $\Sigma \to [\omega]$. Denote by Σ_q^p = the set $\Sigma_q^p = \{\sigma \in \Sigma / arity(\sigma) = q$ and coarity $(\sigma) = p\}$.*

In this paper we assume that a biranked alphabet contains two special symbols $1 \in \Sigma_1^1$, $0 \in \Sigma_0^0$.

Definition 3.2. *The set of well-formed expressions on Σ is the set $L(\Sigma) = \cup \langle L(\Sigma)_q^p : p, q \in [\omega] \rangle$ recursively defined by*

(i) $\Sigma_q^p \subset L(\Sigma)_q^p$,

(ii) $\forall w \in L(\Sigma)_{q'}^p, \forall w' \in L(\Sigma)_q^{q'} : (ww') \in L(\Sigma)_q^p$,

(iii) $\forall w \in L(\Sigma)_q^p, \forall w' \in L(\Sigma)_{q'}^{p'} : (w \otimes w') \in L(\Sigma)_{q+q'}^{p+p'}$.

Definition 3.3. *The relation $\equiv$ is the smallest equivalence on $L(\Sigma)$ that satisfies the following axioms*

(C1) $(w_1 w_2) w_3 \equiv w_1 (w_2 w_3)$

(C2) $(w_1 \otimes w_2) \otimes w_3 \equiv w_1 \otimes (w_2 \otimes w_3)$

(C3) $(w_1 w_2) \otimes (w_1' w_2') \equiv (w_1 \otimes w_1')(w_2 \otimes w_2')$

(C4) $(1 \otimes \ldots \otimes 1) w \equiv w (1 \otimes \ldots \otimes 1) \equiv w$

(C5) $0 \otimes w \equiv w \otimes 0 \equiv w$

(C6) $w_1 \equiv w_1' \wedge w_2 \equiv w_2' \wedge w_1 w_2 \in L(\Sigma) \Rightarrow w_1 w_2 \equiv w_1' w_2'$

(C7) $w_1 \equiv w_1' \wedge w_2 \equiv w_2' \Rightarrow w_1 \otimes w_2 \equiv w_1' \otimes w_2'$

If $w \in L(\Sigma)$ then $[w]$ denotes the equivalence class that contains w.

Definition 3.4.

(i) if $w_1 w_2 \in L(\Sigma)$ *then* $[w_1][w_2] = [w_1 w_2]$,

(ii) $\forall w_1, w_2 \in L(\Sigma) : [w_1] \otimes [w_2] = [w_1 \otimes w_2]$.

Proposition 3.1. *The algebra $\langle L(\Sigma)/\equiv, ., \otimes, [1], [0] \rangle$ is a magmoid.*

We make the convention to denote $[1]$ by 1 and $[0]$ by 0.

Definition 3.5. *We say that $w \in L(\Sigma)$ is connected if there exist no $w_1, w_2 \in L(\Sigma)$ such that $w \equiv w_1 \otimes w_2$ and $w_1 \neq 0 \neq w_2$. Otherwise we say that w is disconnected.*

Proposition 3.2. *Consider $w \in L(\Sigma)$, w disconnected. Then there exist $w_1, \ldots w_n \in L(\Sigma)$ connected such that $w \equiv w_1 \otimes \ldots \otimes w_n$.*

Proof. We proceed by structural induction on w.

Basis. If $w = (\sigma_1 \otimes \sigma_2)$, $\sigma_1, \sigma_2 \in \Sigma$, then $w_1 = \sigma_1$, $w_2 = \sigma_2$.

Induction. Because w is disconnected, there exist $w', w'' \in L(\Sigma)$ such that $w \equiv w' \otimes w''$. By inductive hypothesis there exist $w_1', \ldots, w_n', w_1'', \ldots, w_n'' \in L(\Sigma)$ connected such that $w \equiv (w_1' \otimes \ldots \otimes w_n') \otimes (w_1'' \otimes \ldots \otimes w_n'')$. Now the conclusion follows applying the associativity.

Remark. The decomposition given by Proposition 3.2 is uniquely determined up to associativity.

Definition 3.6. *The function depth : $L(\Sigma) \to [\omega]$ is defined as follows*

(i) *depth* (0) = *depth*$(1) = 0$,

(ii) $\forall \sigma \in \Sigma - \{0, 1\}$: *depth* $(\sigma) = 1$ and

(iii) $\forall w \in L(\Sigma)$: if w is connected then *depth* $(w) = \min\{$depth (w_1) + *depth* $(w_2) / w \equiv w_1 w_2\}$ else *depth* $(w) = \max\{$*depth* $(w_i) / 1 \leq i \leq n\}$ where $w \equiv w_1 \otimes \ldots \otimes w_n$ (by Proposition 3.2).

Proposition 3.3. *Consider $w_1, w_2 \in L(\Sigma)$. If $w_1 \equiv w_2$ then depth (w_1) = depth (w_2).*

The proof follows immediately from Definition 3.6.

Proposition 3.4. *Let $w \in L(\Sigma)$ with depth $(w) > 1$. There exist $w_1, w_2 \in L(\Sigma)$ such that $w \equiv w_1 w_2$ and depth $(w_i) <$ depth (w), $i = 1, 2$.*

Proof. If w is connected then there exist w_1, w_2 such that $w = w_1 w_2$ and *depth* (w) = *depth* (w_1) + *depth* (w_2). If depth $(w_i) > 0$, $i = 1, 2$, then the conclusion follows. If there is i such that *depth* (w) = *depth* (w_i), then we repeat the above reasoning for w_i. If w is disconnected then there exist $v_1, \ldots, v_n$ connected such that $w = v_1 \otimes \ldots \otimes v_n$, *depth* (w) = *depth* (v_i) for some i and *depth* $(v_j) \leq$ *depth* (w), $1 \leq j \neq i \leq n$. v_i connected implies there exist v_i', v_i'' such that $v_i \equiv v_i' v_i''$ and *depth* (v_i'), *depth* $(v_i'') <$ *depth* (v_i). Now $w \equiv v_1 \otimes \ldots \otimes v_i \otimes \ldots \otimes v_n \equiv v_1 \otimes \ldots \otimes v_i' v_i'' \otimes \ldots \otimes v_n \equiv (v_1 \otimes \ldots \otimes v_i' \otimes \ldots \otimes v_n)(1 \otimes \ldots \otimes v_i'' \otimes \ldots \otimes 1)$. If i is the only subscript such that *depth* (v_i) = *depth* (w) then the conclusion follows. Otherwise we repeat the above argument for all subscripts for which we have equality.

Example. Consider $\sigma_1 \in \Sigma_1^1$, $\sigma_2 \in \Sigma_2^2$, $w = (\sigma_1 \otimes 1 \otimes 1)(((\sigma_1 \otimes \sigma_1)\sigma_2) \otimes \sigma_1)$. Then $w \equiv (\sigma_1 \otimes 1 \otimes 1)(\sigma_1 \otimes \sigma_1 \otimes 1)(\sigma_2 \otimes \sigma_1) \equiv ((\sigma_1 \otimes 1)(\sigma_1 \otimes \sigma_1)\sigma_2) \otimes \sigma_1$ and *depth*$(\sigma_1 \otimes 1 \otimes 1) = 1$, *depth*$(((\sigma_1 \otimes \sigma_1)\sigma_2) \otimes \sigma_1) = 2$, *depth*$(w)$ = *depth*$(((\sigma_1 \otimes 1)(\sigma_1 \otimes \sigma_1)\sigma_2)) = 3$.

Every equivalence class of $L(\Sigma)/\equiv$ can be specified by a cannonical element (normal form) which is computed by means of two rewriting systems. The first system transforms a given expression into a sequential composition of subexpressions with depth ≤ 1. We call these subexpressions elementary expressions. The second system eliminates the elementary subexpressions with depth $= 0$ and imposes a certain structure for the elementary subexpressions with depth $= 1$.

From now on we assume that the operation . has the application priority greater than $\otimes$. We also assume that the expressions of $L(\Sigma)$ have the number of the brakets maximum reduced by the associativity and the priority rules. For example we write $w_1 w_2 w_3 \otimes w_4$ instead of $((w_1 w_2) w_3) \otimes w_4$.

Definition 3.7. *The rewriting system (S) contains the following rules:*

(i) $\forall w_1, w_2, w_1', w_2' \in L(\Sigma) : w_1 w_2 \otimes w_1' w_2' \in L(\Sigma) \Rightarrow$
$$w_1 w_2 \otimes w_1' w_2' \xrightarrow[(S)]{} (w_1 \otimes w_1')(w_2 \otimes w_2')$$

(ii) $\forall w_1, w_2 \in L(\Sigma)$, $\forall w \in L(\Sigma)_q^p : w_1 w_2 \in L(\Sigma) \Rightarrow$
$$w \otimes w_1 w_2 \xrightarrow[(S)]{} (w \otimes w_1)(1^{\otimes q} \otimes w_2),$$
$$w_1 w_2 \otimes w \xrightarrow[(S)]{} (w_1 \otimes w)(w_2 \otimes 1^{\otimes q})$$

Proposition 3.5. [1] *(S) is noetherian.*

Proof. Define the function $nbp : L(\Sigma) \to [\omega]$ as follows

(i) $\forall \sigma \in \Sigma : nbp(\sigma) = 0$

(ii) $\forall w_1, w_2 \in L(\Sigma) : nbp(w_1 w_2) = nbp(w_1) + nbp(w_2) + 1$
$$nbp((w_1)(w_2)) = nbp(w_1 + nbp(w_2)$$
$$nbp(w_1 \otimes w_2) = nbp(w_1) + nbp(w_2).$$

The conclusions follows from the fact that $w \xrightarrow[(S)]{} w'$ implies $nbp(w) > nbp(w') \geq 0$.

Definition 3.8. *Consider $u, v \in L(\Sigma)$. We say that v is a pseudo-normal form of u if $u \xrightarrow[(S)]{*} v$ and does not exist $w \in L(\Sigma)$ such that $v \xrightarrow[(S)]{} w$.*

Proposition 3.6. *Consider* $u_1, u_2, v \in L(\Sigma)$ *such that* $u_1u_2 \in L(\Sigma)$.
Then v is a pseudo-normal form of u_1u_2 iff there exist v_1, v_2 pseudo-normal forms of u_1, respectively u_2, such that $v = v_1v_2$.

Proof. We note that a rule $\alpha \xrightarrow[(S)]{} \beta$ can be applied to u_1u_2 iff α is a subexpression of u_1 or u_2.

Definition 3.9. *The rewriting system* (S') *contains the following rules:*

(i) $\forall u, v, u' \in L(\Sigma)$ *pseudo-normal forms*: $uv \xrightarrow[(S)]{} u'v'$ iff $u = u_1 \otimes 1^{\otimes p} \otimes u_2$, $v = v_1 \otimes \sigma \otimes v_2 (\sigma \in \Sigma_q^p)$, $u' = u_1 \otimes \sigma \otimes u_2$ and $v' = v_1 \otimes 1^{\otimes q} \otimes v_2$.

(ii) $\forall u \in L(\Sigma)_q^p$ *pseudo-normal form*: $u1^{\otimes q} \xrightarrow[(S')]{} u$

(iii) $\forall u \in L(\Sigma) : u \otimes O \xrightarrow[(S')]{} u$ *and* $O \otimes u \xrightarrow[(S')]{} u$.

Proposition 3.7. (S') *is noetherian.*
The *proof* is given in [1].

Definition 3.10. *Let* $v, w, \in L(\Sigma)$ *where* v *is in pseudo-normal form. We say that* w *is a normal form of* v *iff* $v \xrightarrow[(S')]{*} w$ *and does not exist* w' *such that* $w \xrightarrow[(S')]{} w'$.

Example. Consider $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_1^3$. Then

$$(\sigma_1 \otimes 1 \otimes \sigma_1)(\sigma_1 1 \otimes \sigma_1)(\sigma_1 \otimes \sigma_1 \otimes \sigma_1)\sigma_2 \xrightarrow[(S')]{}$$

$$(\sigma_1 \otimes 1 \otimes \sigma_1)(\sigma_1 \otimes \sigma_1 \otimes \sigma_1)(\sigma_1 \otimes 1 \otimes \sigma_1)\sigma_2 \xrightarrow[(S')]{}$$

$$(\sigma_1 \otimes \sigma_1 \otimes \sigma_1)(\sigma_1 \otimes 1 \otimes \sigma_1)(\sigma_1 \otimes 1 \otimes \sigma_1)\sigma_2.$$

Theorem 3.1. [1] *If* $u \xrightarrow[(S)]{*} v \xrightarrow[(S')]{*} w$, $u' \xrightarrow[(S)]{*} v' \xrightarrow[(S')]{*} w'$, $u \equiv u'$, v *and* v' *are pseudo-normal forms,* w *and* w' *are normal forms then* $w = w'$.

The *dual normal form* is defined by replacing in Definition 3.9 $uv \to u'v'$ with $u'v' \to uv$ and $u1^{\otimes q} \to u$ with $1^{\otimes p}u \to u$. In [1] is proved that the dual normal form has the same properties as the normal form. For example, $(\sigma_1 \otimes 1 \otimes \sigma_1)(\sigma_1 \otimes \sigma_1 \otimes \sigma_1)\sigma_2$, $\sigma_1, \sigma_2 \in \Sigma$, is in dual normal form.

Proposition 3.8. *Consider* $u, v \in L(\Sigma)$. *If* v *is a pseudo-normal form of* u *then* $v = v_1 \ldots v_n$ *where* $v_i \in L(\Sigma)$, $depth\,(v_i) \leq 1$, $1 \leq i \leq n$.

Proof. We proceed by induction on $depth\,(u)$. The basis of induction, $depth\,(u) = 0$ or $depth\,(u) = 1$, is obvious. Assume now that $depth\,(u) > 1$. By Proposition 3.4. there exist u_1, u_2 such that $u \equiv u_1u_2$ and $depth\,(u_i) < depth(u)$, $i = 1, 2$. Now we apply the inductive hypothesis and Proposition 3.6.

If $u \in L(\Sigma)$, then $NF(u)$ denotes the normal form of u.

Theorem 3.2. *Consider* $u \in L(\Sigma)$. *Then* $NF(u) = u_1 \ldots u_n$ *where* $depth\,(u_i) = 1$, $1 \leq i \leq n$.

Proof. We apply Proposition 3.8 and Definition 3.9.

Corollary. If $NF(u) = u_1 \ldots u_n$ *then* $depth\,(u) = n$.

Proposition 3.9. *Consider* $u, v, \in L(\Sigma)$. *Assume that* $NF(u) = u_1 \ldots u_m$, $NF(v) = v_1 \ldots v_n$, $m \leq n$. *Then* $NF(u \otimes v) = (u_1' \otimes v_1) \ldots (u_n' \otimes v_n)$ *where* $\forall 1 \leq i \leq m : u_i' = u_i$ *and* $\forall m < i \leq n : u_i' = 1 \otimes \ldots \otimes 1$.

Proof. We have $u \otimes v \equiv u_1 \ldots u_m \otimes v_1 \ldots v_n \equiv (u_1 \otimes v_1)(v_2 \ldots u_m \otimes v_2 \ldots v_n) \equiv \ldots \equiv (u_1 \otimes v_1) \ldots (u_n' \otimes v_n) = w$. Obviously w is in normal form.

Remark. A similar result can ba obtained if $m > n$.

Definition 3.11. *The set of terms defined on* Σ *is* $T(\Sigma) = L(\Sigma)/\equiv$.

4. **The set** $\langle T(\Sigma), \subseteq \rangle$.

Definition 4.1. *Consider* $a, b, \in T(\Sigma)$. *We say that* $a \subseteq b$ *iff* $\exists b', b'' \in T(\Sigma)$: $b = ab'$ *or* $b = b' \otimes a \otimes b''$ *or* $(b = b'b''$ *and* $a \subseteq b')$ *or* $(b = b' \otimes b''$ *and* $(a \subseteq b'$ *or* $a \subseteq b''))$.

Theorem 4.1. *Consider* $a, b \in T(\Sigma)$. *Then* $a \subseteq b$ *iff* $\exists b_1, b_2, b_3 \in T(\Sigma) : b = (b_1 \otimes a \otimes b_2)b_3$.

Proof. $\Rightarrow$. Assume that $a \subseteq b$. We proceed by structural induction on b

Basis. If $b = ab'$ then $b_1 = b_2 = 0$, $b_3 = b'$. If $b = b' \otimes a \otimes b''$ then $b_1 = b'$, $b_2 = b''$ and $b_3 = 1 \otimes \ldots \otimes 1$.

Induction. If $b = b'b''$ or $b = b' \otimes b''$ and $a \subseteq b'$ then by inductive hypothesis there exist b_1', b_2', b_3' such that $b' = (b_1' \otimes a \otimes b_2')b_3'$. If $b = b'b''$ then we get $b_1 = b_1'$, $b_2 = b_2'$, $b_3 = b_3'b''$ and if $b = b' \otimes b''$ then we get $b_1 = b_1'$, $b_2 = b_2' \otimes b''$, $b_3 = b_3' \otimes 1 \otimes \ldots \otimes 1$.

The *proof* of $\Leftarrow$ follows immediately.

Corollary. If $a \subseteq b$ *then* $depth\,(a) \leq depth\,(b)$.

Proposition 4.1. $\langle T(\Sigma), \subseteq \rangle$ *is a directed partially ordered set.*

Proof. $\subseteq$ is reflexive. $\forall a \in T((\Sigma)_q^p : a = a1^{\otimes q} \Rightarrow a \subseteq a$.

$\subseteq$ is antisymmetric. If $a \subseteq b$ and $b \subseteq a$ then by Theorem 4.1 we have $a = (a_1 \otimes b \otimes a_2)a_3 = (a_1 \otimes (b_1 \otimes a \otimes b_2)b_3 \otimes a_2)a_3 = (a_1 \otimes b_1 \otimes a \otimes b_2 \otimes a_2)(1 \otimes \ldots \otimes b_3 \otimes \ldots \otimes 1)a_3$. It follows $a_1 = a_2 = b_1 = b_2 = O$ and $a_3 = 1 \otimes \ldots \otimes 1$ and $b_3 = 1 \otimes \ldots \otimes 1$.

$\subseteq$ is transitive. If $a \subseteq b \subseteq c$ then by Theorem 4.1. $c = (c_1 \otimes b \otimes c_2)c_3) = (c_1 \otimes (b_1 \otimes a \otimes b_2)b_3 \otimes c_2)c_3 = (c_1 \otimes b_1 \otimes a \otimes b_2 \otimes c_2)c_3$ and hence $a \subseteq c$.

$T(\Sigma)$ is directed. If $a, b, \in T(\Sigma)$ then $a \otimes b \in T(\Sigma)$ and moreover $a \subseteq a \otimes b$ and $b \subseteq a \otimes b$.

Proposition 4.2. *Let* $a, b \in T(\Sigma)$, $u, v \in L(\Sigma)$ *such that* $a = [u]$, $b = [v]$. *If* $a \subseteq b$ *then*

(i) $NF(u) = u_1 \ldots u_m$, $NF(v) = v_1 \ldots v_n$ *and* $m \leq n$ *and*

(ii) $\forall 1 \leq i \leq m : u_i \subseteq v_i$.

Proof. If $a \subseteq b$ there exist w_1, w_2, w_3 such that $v \equiv (w_1 \otimes u \otimes w_2)w_3$. Assume that $NF(w_1) = v_1' \ldots v_{n'}'$, and $NF(w_2) = v_1'' \ldots v_{n''}''$. By Proposition 3.9 we have $NF(w_1 \otimes u \otimes w_2) = (v_1' \otimes u_1 \otimes v_1'')(v_2' \otimes u_2 \otimes v_2'') \ldots$. If we denote by $k = \max\,\{m, n', n''\}$ then by the uniqueness of the normal form we have $NF(w_1 \otimes u \otimes w_2) = v_1 \ldots v_k$ and $v_1 = v_1' \otimes u_1 \otimes v_1$, $v_2 = v_2' \otimes u_2 \otimes v_2'', \ldots$. Now $m \leq k \leq n$ and $u_i \subseteq v_i$, $1 \subseteq i \subseteq m$.

Proposition 4.3. *Let* $a \in T(\Sigma)$ *such that coarity* $(a) = p$. *Then*

(i) $0 \subseteq a$ *and*

(ii) $\forall 1 \leq k \leq p : 1^{\otimes k} \subseteq a$.

The *proof* is obvious.

Definition 4.2. $T(\Sigma)^{\#}$ *is the completion by ideals (of directed sets) of* $T(\Sigma)$.

By Theorem 5.13 of [3] $T(\Sigma)^{\#}$ is algebraic (inductive) and complete.

5. **Planar dags.** We give a functional definition for dags (directed acyclic graphs). Then we define a partial order on the set of dags and we show that this set is isomorphic with the set of terms defined in Sections 4.3.

Definition 5.1. *An equivalence relation* $\sim \subseteq ([\omega]^*)^2$ *is called length-preserving right congruence (lpr-congruence) iff*

(LPRC1) $\forall x, y, z \in [\omega]^* : x \sim y \Rightarrow xz \sim yz$ ($\sim$ *is a right congruence*)

(LPRC2) $\forall i, j \in [\omega] : i \sim j \Rightarrow i = j$,

(LPRC3) $\forall x, y \in [\omega]^* : x \sim y \Rightarrow |x| = |y|$

Definition 5.2. $\langle [\omega]^*, \subseteq \rangle$ *is the set of string on* $[\omega]$ *lexicographically ordered.*

Definition 5.3. *Consider* $\sim$ *a lpr-congruence and* $[x], [y] \in [\omega]^*/\sim$. *We say that* $[x] \subseteq [y]$ *iff* $|x| = |y|$ *and* $\max [x] \subseteq \max [y]$.

Remark. $\langle [\omega]^*/\sim, \subseteq \rangle$ is a partially ordered set.

Definition 5.4. *A planar dag domain is a pair* $\langle D, \sim \rangle$ *where* $D \subseteq [\omega]^+/\sim$ *is a lpr-congruence and moreover the following conditions are satisfied :*

(PDDO) $\forall i \in [\omega] : i \in D \Rightarrow i > 0$,

(PDD1) $\forall i, j \in [\omega] : 1 \leqslant i \leqslant j$ and $j \in D \Rightarrow i \in D$,

(PDD2) $\forall x \in [\omega]^+ \forall i \in [\omega] : xi \in D \Rightarrow x \in D$ (D is prefix closed)

(PDD3) a) $\forall x \in D \, \exists r^+(x) \in [\omega] \, \forall i \in [\omega] : xi \in D \Leftrightarrow 1 \leqslant i \leqslant r^+(x)$

b) $\forall x, y \in D : x \sim y \Rightarrow r^+(x) = r^+(y)$

(PDD4) $\forall x, y \in D \, \forall 1 \leqslant i \leqslant r^+(x) \, \forall 1 \leqslant j \leqslant r^+(y) : [xi] \subseteq [yj]$

(PDD5) $\forall x \in D \, \forall 1 \leqslant i < j \leqslant r^+(x) : [xi] \subseteq [xj]$

Definition 5.5. *Consider* $\langle D, \sim \rangle$ *a planar dag domain,* $x \in D$ *and* $n \in [\omega]$.

(i) $slice_n(D) = \{y \in D / \, |y| = n\}$

(ii) $depth\,(D) = \sup \, \{|y| / y \in D\}$.

Note that a planar dag domain $\langle D, \sim \rangle$ defines a graph $\langle V, E \rangle$ in the following way : $V = D/\sim$ and $E = \{\langle [x], [xi] \rangle / x \in D, \, 1 \leqslant i \leqslant r^+(x)\}$. It is evidently that $\langle V, E \rangle$ is a directed acyclic graph with multiple arcs and this motivates Definition 5.4. Now we can define $r^-(x)$ as being the indegree of $[x]$.

Proposition 5.1. Let $\langle D, \sim \rangle$ *be a planar dag domain. Then for any* $n \in [\omega]$ $slice_n(D)/\sim$ *is a maximal chain.*

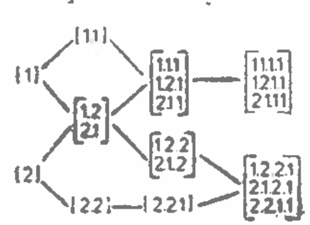
Proof. We proceed by induction on n.

Basis. If $n = 1$ then $[1] \subseteq [2] \subseteq \ldots$.

Induction. Assume that $slice_n(D)/\sim = \{[x_1], [x_2], \ldots\}$ and $[x_1] \subseteq [x_2] \subseteq \ldots$ is a maximal chain.

By (PDD3), (PDD4) and (PDD5) we have $[x_1 1] \subseteq \ldots \subseteq [x_1 r^+(x_1)] \subseteq [x_2 1] \subseteq \ldots \subseteq [x_2 r^+(x_2)] \subseteq \ldots$ that implies $slice_{n+1}(D)$ is a maximal chain.

Example. Consider $D/\sim = \{\{1\}, \{2\}, \{1,1\}, \{1.2, 2.1\}, \{2.2\}, \{1.1.1, 1.2.1, 2.1.1\}, \{1.2.2, 2.1.2\}, \{2.2.1\}, \{1.1.1.1, 1.2.1.1, 2.1.1.1\}, \{1.2.2.1, 2.1.2.1, 2.2.1.1\}\}$, Then $[1] \subseteq [2]$, $[1.1] \subseteq [2.1] \subseteq [2.2]$, $[2.2.1] \subseteq [2.1.2] \subseteq [2.2.1]$, $[2.1.1.1] \subseteq [2.2.1.1]$, Pictorially we represent $\langle D, \sim \rangle$ by



Definition 5.6. *Let* Σ *be a biranked alphabet. A* Σ*-planar dag is a triple* $\delta = \langle D, \sim, d \rangle$ *where* $\langle D, \sim \rangle$ *is a planar dag domain and* d *is a function* $d : D \to \Sigma$ *which satisfies*

(PD1) $\forall x, y \in D : x \sim y \Rightarrow d(x) = d(y)$,

(PD2) $\forall x \in D : d(x) \in \Sigma^p_q$ and $|x| > 1 \Rightarrow r^-(x) = p$,

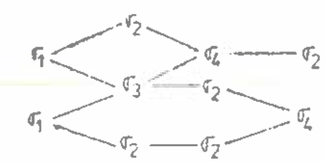
(PD3) $\forall x \in D : d(x) \in \Sigma^p_q$ and $slice_{|x|+1}(x) \neq \Phi \Rightarrow r^+(x) = q$

(PD4) $\forall x \in D : (\forall yi \in [x] : d(y) = 1) \Rightarrow d(x) = 1$

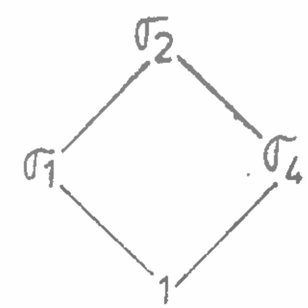
(PD5) $\forall n \geqslant 1 : slice_n(D) = [x_1] \cup \ldots \cup [x_k] \wedge$ depth $(D) > 1 \Rightarrow$ non $d(x_1) = \ldots = d(x_k) = 1$.

If D is finite then we say that δ is a finite Σ-planar dag.

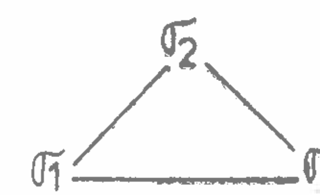
Example. Consider $\langle D, \sim \rangle$ defined in the above example and d given as follows : $d(1) = d(2) = \sigma_1 \in \Sigma^1_2$, $d(1.1) = d(2.2) = d(2.2.1) = d(1.2.2) = d(1.1.1.1) = \sigma_2 \in \Sigma^1_1$, $d(1.2) = d(1.2.2.1) = \sigma_3 \in \Sigma^2_2$, $d(1.1.1) = \sigma_4 \in \Sigma^2_1$. Now we represent $\langle D, \sim, d \rangle$ as



We make the convention to represent —— instead of — 1 —. For example, the dag



will be represented by



In what follows we use the following denotations. PDAG (Σ) denotes the set of all Σ-planar dags and FPDAG (Σ) denotes the set of all finite Σ-planar dags. If $\delta = \langle D, \sim, d \rangle$ is a Σ-planar dag then $dom\,(\delta) = D$, $con\,(\delta) = \sim$ and $fun\,(\delta) = d$.

Definition 5.7. *For any integer* k *the function* $t_k : [\omega]^* \to [\omega]^*$ *is defined as follows*

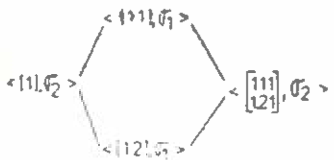
(i) $\forall i \in [\omega] : \ t_k(i) = i + k$

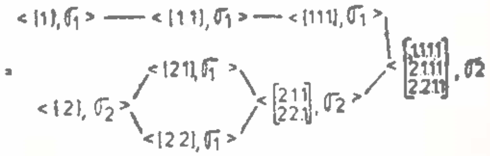
(ii) $\forall xi \in [\omega]^* \in : x \neq \varepsilon \Rightarrow t_k(xi) = t_k(x)i$

Definition 5.8. *Consider* $\delta_1 = \langle D_1, \sim_1, d_1 \rangle$, $\delta_2 = \langle D_2, \sim_2, d_2 \rangle \in PDAG(\Sigma)$. *We say that* $\delta_1 \subseteq \delta_2$ *iff there exists* $k \in [\omega]$ *such that*

(i) $\forall x \in [\omega]^* : x \in D_1 \Rightarrow t_k(x) \in D_2$,
(ii) $\forall x, y \in D_1 : x \sim_1 y \Leftrightarrow t_k(x) \sim_2 t_k(y)$,
(iii) $\forall x \in D_1 : d_1(x) = d_2(t_k(x))$.

Example. If

$\delta_1 =$ 

$\delta_2 =$ 

then $\delta_1 \subseteq \delta_2$ with $k=1$.

Proposition 5.2. *$\langle PDAG(\Sigma), \subseteq \rangle$ is a partially ordered set.*

In order to prove the following theorems we need a trick. Denote by PDAG'(Σ) the set of objects obtained by replacing (PDD1) with

(PDD1') $\exists n \in [\omega] \forall i, j \in [\omega]$: $n \leqslant i \leqslant j$ and $j \in D \Rightarrow i \in D$.

We also denote by $\subseteq'$ the relation obtained by considering in Definition 5.8 objects of PDAG'(Σ). Now $\subseteq'$ is a preorder on PDAG'(Σ). Moreover $\delta_1' \subseteq' \delta_2'$ iff there is δ_3' such that $\delta_1' \equiv' \delta_3'$ and $dom(\delta_3') \subseteq dom(\delta_2')$, $con(\sim_3') \subseteq con(\sim_2')$, $graph(fun(\delta_3')) \subseteq graph(fun(\delta_2'))$ where $\equiv'$ is the equivalence generated by $\subseteq'$. We denote $embedding(\delta_1', \delta_2')$ with δ_3'. The Σ-planar dags are "canonical elements" in the equivalence classes of PDAG'(Σ)/$\equiv'$. If S is a directed subset of PDAG(Σ) then we define $'S = \{'\delta \in \text{PDAG}'(\Sigma) / \delta \in S\}$ that satisfies

(i) $\forall \delta \in S : '\delta \equiv \delta$

(ii) $\forall \delta_1, \delta_2 \in S$: $\delta_1 \subseteq \delta_2 \Leftrightarrow dom('\delta_1) \subseteq dom('\delta_2)$, $con('\delta_1) \subseteq con('\delta_2)$, $graph(fun('\delta_1)) \subseteq graph(fun('\delta_2))$. Denote by $\square S$ the canonical element of the class defined of δ' where $dom(\delta') = \cup \langle dom(\delta_1') : \delta_1' \in 'S \rangle$, $con(\delta') = \cup \langle con(\delta_1') : \delta_1' \in 'S \rangle$, $graph(fun(\delta')) = \cup \langle graph(fun(\delta_1')) : \delta_1' \in 'S \rangle$. Note that the above unions are well defined by (ii) and S is directed.

Theorem 5.1. *$\langle$PDAG(Σ), $\subseteq \rangle$ is complete.*

Proof. Let S be a directed subset of PDAG(Σ). We prove that $\square S = l.u.b.(S)$. If $\delta \in S$ then $'\delta = embedding(\delta, \square S)$ and hence $\delta \subseteq \square S$. This implies that $\square S$ is an upper bound of S.

Let δ be an upper bound of S and denote by S_1 the set $S_1 = \{\delta_1 \in \text{PDAG}(\Sigma) / \delta_1 \subseteq \delta\}$. Because δ is an upper bound of S it follows $S \subseteq S_1$. This implies $'S \subseteq 'S_1$ and hence $\square S \subseteq \square S_1 = \delta$.

Theorem 5.2. FPDAG(Σ) *coincides with the set of the isolated elements (compacts) of PDAG(Σ).*

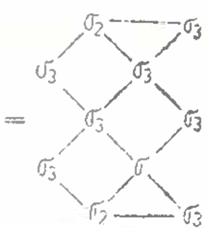
Proof. We first introduce a denotation. If $\delta \in$ PDAG(Σ) then the n-th approximation of δ, $trunc_n(\delta) = \langle D^{(n)}, \sim^{(n)}, d^{(n)} \rangle$, is defined by the following algorithm:

```
begin
k : =1 ;
ADD : = . TRUE. ;
D(n) : ={1,..., n} ∩ D ;
while k<n and ADD do
```



```
    begin
    ADD : = . FALSE. ;
    E : ={xi ∈ D/x ∈ slice_k(D(n)), i ∈ [ω]} ;
    if |E/~| ≤ n then
        begin
        D(n) : =D(n) ∪ E ;
        k = k + 1 ;
        ADD : = . TRUE.
        end
end.
end.
```

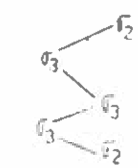
For example, if

$\delta =$ 

then

$trunc_1(\delta) = \sigma_3$

$trunc_2(\delta) = \begin{matrix}\sigma_3\\ \sigma_3\end{matrix}$

$trunc_3(\delta) =$ 

$trunc_4(\delta) =$ 

$trunc_n(\delta) = \delta$, $n \geqslant 5$.

Denote by $I(\delta)$ the set $I(\delta) = \{trunc_n(\delta) / n \in [\omega]\}$. It is evidently that $I(\delta)$ is directed and by Theorem 5.1 $\delta = $l.u.b.($I(\delta)$). Assume first that δ is isolated. Then $\delta \subseteq$ l.u.b.($I(\delta)$) implies there is $\delta_1 \in I(\delta)$ such that $\delta \subseteq \delta_1$. But δ_1 is finite and hence δ is finite. Conversely, it is obvious that any finite Σ-planar dag is isolated.

Corollary 1. *$\langle$PDAG(Σ), $\subseteq \rangle$ is algebraic (inductive).*

Corollary 2. PDAG(Σ) *is isomorphic with* FPDAG(Σ)# *where* FPDAG(Σ)# *is the completion by ideals of* FPDAG(Σ).

Theorem 5.3. FPDAG(Σ) *is isomorphic with* $T(\Sigma)$.

Proof. Define Φ : FPDAG(Σ) $\rightarrow T(\Sigma)$ as follows. Let $\delta = \langle D, \sim, d \rangle \in$ FPDAG(Σ). If $depth(D) = n$ then $\Phi(\delta) = [u_1 \ldots u_n]$ where $slice_k(D) = [x_1] \cup \ldots \cup [x_m]$, $[x_1] \subseteq \ldots \subseteq [x_m]$ and $u_k = d(x_1) \otimes \ldots \otimes d(x_m)$, $1 \leqslant k \leqslant n$. By (PD4) and (PD5) $u_1 \ldots u_n$ is in normal form.

Φ is monotone. Consider $\delta_i = \langle D_i, \sim_i, d_i \rangle \in$ FPDAG(Σ), $i = 1, 2$, $\delta_1 \subseteq \delta_2$. Assume $\Phi(\delta_1) = [u_1 \ldots u_m]$, $\Phi(\delta_2) = [v_1 \ldots v_n]$. Denote $\delta_1' = \langle D_1', \sim_1', d_1' \rangle = embedding(\delta_1, \delta_2)$. Then $slice_k(D_1')$ is a subchain of $slice_k(D_2)$ and hence $[u_k] \subseteq [v_k]$, $1 \leqslant k \leqslant m$. By Thereorem 4.1, $v_k = v_k' \otimes u_k \otimes v_k''$. Now $\Phi(\delta_2) = [(v_1' \otimes u_1 \otimes v_1'') \ldots (v_m' \otimes u_m \otimes v_m'') \ldots v_n] = [(v_1' \ldots v_m' \otimes u_1 \ldots u_m \otimes v_1'' \ldots v_m'') \ldots v_n]$ that implies $\Phi(\delta_1) \subseteq \Phi(\delta_2)$.

Note that a Σ-planar dag is applied by Φ in the normal form of a term and hence Φ is injective.

In order to construct $\Phi^{-1}(a)$, $a \in T(\Sigma)$, we proceed by induction on $depth(a)$. If $depth(a) = 1$ then $a = [\sigma_1 \otimes \ldots \otimes \sigma_m]$ and we put $D = \{1, \ldots, m\}$, $\sim =$ id and

$d(i)=\sigma_i$, $i=1,\dots,m$. If $a=[u_1\dots u_n u_{n+1}]$, $u_1\dots u_{n+1}$ in normal form, then $\Phi^{-1}(a)$ is computed by means of $\Phi^{-1}(u_1\dots u_n)=\langle D', \sim', d'\rangle$ of the following algorithm :

begin
{Let $u_{n+1}=\sigma_1\otimes\dots\otimes\sigma_m$; $slice_n(D)=[y_1]\cup\dots\cup[y_p]$, $[y_1]\subseteq\dots\subseteq[y_p]$}
$D=D'$; $\sim=\sim'$; $d=d'$;
for $i:=1$ to m do $D_i=\Phi$;
$i:=1$; $ic:=1$;
for $k:=1$ to p do
    for $j:=1$ to $arity(d'(y_k))$ do
        begin
        if $ic>coarity(\sigma_i)$ then
            begin $i:=i+1$; $ic:=1$; end
        $D_i:=D_i\cup\{xj/x\in[y_k]\}$;
        $ic:=ic+1$;
        end ;
    for $i:=1$ to m do
        begin
        $D:=D\cup D_i$;
        for each $\langle x, y\rangle\in D_i\times D_i$ do $x\sim y$;
        for each $x\in D_i$ do $d(x)=\sigma_i$;
        end
end.

Theorem 5.4. *$PDAG(\Sigma)$ is isomorphic with $T(\Sigma)\#$.*

Proof. By Theorem 5.16 of [3] there exist $\Phi\#$ and $\Phi^{-1}\#$ such that the following diagrams comute :

$$\begin{tikzcd} FPDAG(\Sigma) \arrow[r, "\wr"] \arrow[dr, "\Phi"'] & PDAG(\Sigma)\simeq FPDAG(\Sigma)\# \arrow[d, "\Phi\#"] \\ & T(\Sigma)\# \end{tikzcd} \qquad \begin{tikzcd} T(\Sigma) \arrow[r, "\wr"] \arrow[dr, "\Phi^{-1}"'] & T(\Sigma)\# \arrow[d, "\Phi^{-1}\#"] \\ & PDAG(\Sigma) \end{tikzcd}$$

Because Φ is bijective if follows that $(\Phi\#)^{-1}=\Phi^{-1}\#$.

6. Conclusions. The results proved in this paper transfer the fundamental properties of partial trees [2], [3] to dags. By Theorem 5.3 the set $FPDAG(\Sigma)$ is structured as a magmoid if we are setting $\delta_1.\delta_2=\Phi^{-1}(\Phi(\delta_1)\,.\,\Phi(\delta_2))$, $\delta_1\otimes\delta_2=\Phi^{-1}(\Phi(\delta_1)\otimes\Phi(\delta_2))$, $1=\langle\{1\}, \{[1]\}, \{\langle 1,1\rangle\}\rangle$, $0=\langle\Phi,\Phi,\Phi\rangle$. Now Φ is an isomorphism of ordered magmoids. This isomorphism can be obtained also in other way : we first structure $FPDAG(\Sigma)$ as a magmoid by defining the operation . and $\otimes$, and then we find the isomorphism between the two magmoids. This way we consider more incovenient.

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A HIERARCHY OF UNARY PRIMITIVE RECURSIVE STRING-FUNCTIONS

BY

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1. Introduction. Let N be the set of naturals i.e. $N=\{0, 1, 2,\dots\}$. Consider a fixed alphabet $A=\{a_1, a_2,\dots,a_r\}$, $r\geq 2$ and denote by A^* the free monoid generated by A under concatenation (with e the null string). The elements of A^* are called strings ; if reffering to strings, $<$ denotes the lexicographical order induced by $a_1<a_2<\dots<a_r$.

Denote by Fnc (respectively Fnc_A) the set of all unary number-theoretical (respectively, string) functions. By I, Succ, C_m, Pd we denote the following number-theoretical functions : $I(x)=x$, $\mathrm{Succ}\,(x)=x+1$, $C(x)=m$, $\mathrm{Pd}\,(x)=x\dot{-}1$, where $x\dot{-}y=\max\,(x-y, 0)$. By I^A, Succ_i^A, C_u^A, σ^m, π, we denote the following string-functions: $I^A(w)=w$, $\mathrm{Succ}_i^A(w)=wa_i$ $(1\leq i\leq r)$, $C_u^A(w)=u$, $\sigma(e)=a_1$, $\sigma(wa_i)=wa_{i+1}$ if $1\leq i<r$ and $\sigma(wa_r)=\sigma(w)a_1$, $\pi(e)=e$, $\pi(\sigma(w))=w$. Furthermore one uses the primitive recursive bijections $c: A^*\to N$, $\bar{c}: N\to A^*$ given by $c(e)=0$, $c(wa_i)=rc(w)+i$, $1\leq i\leq r$, and $\bar{c}(0)=e$, $\bar{c}(m+1)=\sigma(\bar{c}(m))$.

To each f in Fnc one associates the string-function $s(f)\in\mathrm{Fnc}_A$ defined by $s(f)\,(w)=\bar{c}(f(c(w)))$ and for each g in Fnc one associates the number-theoretical function $n(g)$ defined by $n(g)\,(x)=c(g(\bar{c}(x)))$. It is easily seen that for every string-function g, $s(n(g))=g$ and for every number-theoretical function f, $n(s(f))=f$. For example, $s(\mathrm{Succ})=\sigma$, $n(I^A)=I$, $s(\mathrm{Pd})=\pi$.

A mapping from Fnc^n to Fnc is called an operator in Fnc, and analogously for Fnc_A. We consider the following operators in Fnc and Fnc_A :

$\mathrm{sub}\,(f, g)=h$ iff $f, g, h\in\mathrm{Fnc}$ and $f(g(x))=h(x)$;

$\mathrm{diff}\,(f, g)=h$ iff $f, g, h\in\mathrm{Fnc}$ and $h(x)=f(x)\dot{-}g(x)$;

$\mathrm{it}_x(f)=h$ iff $f, h\in\mathrm{Fnc}$ and $h(0)=x$, $h(y+1)=f(h(y))$;

$\mathrm{sub}_A(f, g)=h$ iff $f, g, h\in\mathrm{Fnc}_A$ and $f(g(w))=h(w)$;

$\sigma-\mathrm{it}_{A,w}\,(f)=h$ iff $f, h\in\mathrm{Fnc}_A$ and $h(e)=w$, $h(\sigma(u))=f(h(u))$.

To every operator φ in Fnc, $s(\varphi)\,(f)=s(\varphi(n(f)))$, for every $f\in\mathrm{Fnc}_A$; analogously, for every operator θ in Fnc_A, $n(\theta)\,(g)=n(\theta(s(g)))$, for every $g\in\mathrm{Fnc}$. For example, $s\,(\mathrm{it}_x)=\sigma-\mathrm{it}_{A,\bar{c}(x)}$, $n(\sigma-\mathrm{it}_{A,w})=\mathrm{it}_{c(w)}$.

2. Ackermann-Peter string-function. The primitive recursive string-functions were introduced by A s s e r [1] and studied by various authors (see [4], [6], [8]).