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ON A CLASS OF GROUPS WHOSE LATTICE OF NORMAL SUBGROUPS IS DISTRIBUTIVE

BY

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In this paper we will study the class U of finite groups G in which every normal subgroup is unique of its order among the subgroups of G. The U-groups are a proper subclass of the class of the finite groups for which the lattice of normal subgroups is distributive (DLN-groups); in fact, M. Curzio ([2]) has shown that G is a DLN-group, if in G different normal subgroups have different orders; thus one can regard U as a class of groups that strongly verifies the DLN-property. The DLN-groups have been examined by many authors (see, for instance, the recent paper [7]). The soluble U-groups have been studied in 1958 by M. Curzio ([3]), who provided for them a characterization in terms of normal and Sylow structure. An interesting class of U-groups — groups in which any two subgroups of the same order are conjugate — has been studied in 1972 by F. Gross ([6]).

In this paper, by means of the determination of the soluble radical structure of an U-group, we provide a characterization of the U-groups in terms of group extensions. This characterization reduces the problem of the classification for non soluble U-groups to the case of semisimple U-groups. In this context we show that the structure of the centreless CR-radical of a semisimple U-group is bounded by an arithmetical condition, that, however, is not sufficient to assure that a centreless CR-group, verifying that condition, is the centreless CR-radical of a semisimple U-group. A restriction on the number of semisimple U-groups (non isomorphic), with given centreless CR-radical, suggests a classification of the centreless CR-groups.

All the groups we consider are finite. For the finite simple groups we use the notation of [1].

1. Structure of the soluble radical of an U-group

1.1. *Let G be an U-group and S its soluble radical. Then:*

(i) *for each $p\in\pi(S)$, the Sylow p-subgroups of G are cyclic, elementary abelian or quaternional;*

(ii) *$S_\omega=G_\omega$ is normal in G, where $\omega=\{p\in\pi(S) \mid S_p$ non cyclic$\}$;*

(iii) *every elementary abelian, non cyclic, Sylow subgroup of S is isomorphic to a chief factor of G;*

(iv) *S has a Sylow tower containing S_ω;*

(v) *there exists a supersoluble immersion of S in G, (if and) only if the Sylow subgroups of S are cyclic.*

Proof. We procede by induction on $|G|$.

(*i*) Let p be a prime such that $O_p(G)\neq 1$. Since $O_p(G)$ is unique of its order among the subgroups of every Sylow p-subgroup of G, these ones are cyclic, elementary abelian or quaternional; the inductive hypothesis assures then that so are also the Sylow q-subgroups of G, for each $q\in\pi(S)\setminus\{p\}$.

(*ii*) Let N be a minimal normal subgroup of G, such that $N<O_p(G)$ $(p\in\pi(S))$. The statement follows immediately from the inductive hypothesis in the following two cases: N is not cyclic, and so $N\in \mathrm{Syl}_p(G)$, or $|N|=p\in\omega$, and so $p=2$. Let then $|N|=p\notin\omega$. By inductive hypothesis we get $S_\omega=G_\omega$ and $NS_\omega\lhd G$ and so $G=NN_G(S_\omega)$. Then the statement follows obviously if N is the only minimal normal subgroup of G (contained in S); otherwise, by arguing as previously on a minimal normal subgroup L of G, with $L\neq N$, we obtain $LS_\omega\lhd G$ and so $LS_\omega\cap NS_\omega=S_\omega\lhd G$.

(*iii*) Let N be a minimal normal subgroup of G and $N\leqslant O_p(G)$. The inductive hypothesis assures that every elementary abelian, non cyclic, Sylow q-subgroup of S/N is isomorphic to a chief factor of G/N and so the statement follows obviously, if we recall that, if N is not cyclic, $N\in\mathrm{Syl}_p(G)$.

(*iv*) It follows from (*i*), (*ii*) and (*iii*) (independently of the hypothesis that G is U-group). Proceed by induction on $|G|$. Let N be a minimal normal subgroup of G and $N\leqslant S_\omega$ $(N\leqslant O_p(G)$, with $p\in\omega)$. We get, by (*i*) and (*iii*), that $N=G_p$ or $|N|=2$ and the Sylow 2-subgroups of S are quaternional. In the first case the statement follows obviously from inductive hypothesis. Let then $|N|=2$ and the Sylow 2-subgroups of S, which are the Sylow 2-subgroups of G, as $S_\omega=G_\omega$, be quaternional. If these ones have order >8, then $N_G(G_2)/C_G(G_2)$ is a 2-group; it follows $(S_\omega)_2\leq C_G(G_2)$ and thus we reduce to the previous case. On the other hand, if $G_2\lhd G$, the statement follows from the inductive hypothesis. Let then $Q_8\simeq G_2\lhd G$ and let L/N be a minimal normal subgroup of G/N contained in S_ω/N. If $L/N<O_p(G/N)$, with $p\neq 2$, we are again in the previous case. Let then $L/N\leq O_2(G/N)$ and so L cyclic of order 4; it follows, obviously, $(S_\omega)_2,\lhd G$ and we find again the previous case.

(*v*) Let $1=S_0<S_1<\ldots<S_n=S(|S_{i+1}/S_i|=p_i$ prime) be a supersoluble immersion of S in G. The inductive hypothesis assures that the Sylow subgroups of S/S_1 are cyclic, then so are also the Sylow subgroups of S, if we recall that the Sylow p_1-subgroups of G are cyclic or quaternional.

1.2. *Let G be an U-group. If G is not an A-group* [1] *, then S_ω is 2-nilpotent.*

Proof. We proceed by induction on $|G|$. By (*iv*) of 1.1 there exists $G_p\lhd G$, with $p\in\omega$. If $p\neq 2$, the statement follows from the inductive hypothesis. Let then $p=2$. By Feit-Thompson theorem ([4]) G is soluble and so, if G_2 is abelian, G is an A-group, which contradicts the hypothesis. We get then that $G_2\simeq Q_8$ ([3]). We obtain obviously the statement, if $C_{S_{2'}}(G_2)=1$; otherwise, with N minimal normal subgroup of G and $N<G_{2'}$, the inductive hypothesis assures that NS_ω/N is 2-nilpotent, then so is S_ω, too.

1.3. Theorem. *The soluble U-groups are the following ones:*

(*I*) *groups with cyclic Sylow subgroups;*

[1] An A-group is a soluble group with abelian Sylow subgroups

(*II*) *semidirect products $K\rhd\!\!\langle_\Phi\, Q$, where: $Q\simeq Q_8$; K is a group of odd order divisible by 3, with cyclic Sylow subgroups; $\Phi: K\to \mathrm{Aut}\, Q$ is the product of the canonical homomorphism $K\to K/M$, with $|K/M|=3$, and a monomorphism $K/M\to\mathrm{Aut}\, Q$;*

(*III*) *semidirect products $U\rhd\!\!\langle_\Phi\, P$, where: P is an elementary abelian p-group (of order p^n, p prime, $n>1$); U is a soluble U-group with $p\nmid|U|$; $\Phi: U\to GL(n, p)$ is an irreducible representation of U on GF(p) (with the identification Aut $P=GL(n, p)$).*

Proof. Let G be a soluble U-group. If G is supersoluble, then, by (*i*) of 1.1, the Sylow subgroups of G are cyclic. Let then G be non supersoluble and so (1.1) $G_\omega\neq 1$. If G has no elementary abelian, non cyclic, Sylow subgroup, then ([3]) $G_\omega=G_2\simeq Q_8$ and so, as $G_\omega\lhd G$, G is obviously of type (*II*). Let then G possess an elementary abelian, non cyclic, Sylow subgroup. If G is an A-group, by (*iv*) of 1.1, there exists a Sylow subgroup G_p minimal normal in G, with $p\in\omega$. If G is not A-group, by 1.2 G is 2-nilpotent and so, as $G_\omega\lhd G$, we obtain again, by (*iv*) of 1.1, a Sylow subgroup G_p minimal normal in G, with $p\in\omega$. That being stated, it follows, obviously, that G is of type (*III*), if we put $P=G_p$ and $U=G_{p'}$. Vice versa, it is easy to verify that the groups (*I*), (*II*) and (*III*) are U-groups.

1.4. *Let G be a non soluble U-group and S its soluble radical. Then the order of S is even, (if and) only if the Sylow 2-subgroups of G are quaternional. In this case $|S/O_{2'}(G)|=2$.*

Proof. Let $|S|$ be even. Since G is not soluble, the Sylow 2-subgroups of G are not cyclic; on the contrary, are cyclic the Sylow 2-subgroups of S: otherwise, by (*ii*) of 1.1, the Sylow 2-subgroups of G are contained in S and so, by Feit-Thompson theorem, G is soluble, which contradicts the hypothesis. Thus S is 2-nilpotent and so the Sylow 2-subgroups of G are not elementary abelian. We have then ((*i*)) of 1.1) that the Sylow 2-subgroups of G are quaternional. The sufficient condition and the last part of the proposition are well known results, independent of the hypothesis that G is an U-group.

1.5. Theorem *The non soluble U-groups are the following ones:*

(*I'*) *semisimple U-groups;*

(*II'*) *non split group extensions of a group of prime order p by a non soluble U-group U, such that the Sylow p-subgroups of U are cyclic, if $p\neq 2$, dihedral (in case $\simeq V_4$), if $p=2$;*

(*III'*) *semidirect products $U\rhd\!\!\langle_\Phi\, P$, where: P is an elementary abelian p-group (of order p^n, p prime, $n\geq 1$); U is a non soluble U-group, with $p\nmid|U|$; $\Phi: U\to GL(n, p)$ is an irreducible representation of U on GF(p) (with the identification Aut $P=GL(n, p)$).*

Proof. Let G be a non soluble U-group and G non semisimple; let S be its soluble radical. If G has a normal subgroup G_p, in particular if $S_\omega\neq 1$ ((*iv*) of 1.1), G is obviously of type (*III'*) (G_p is minimal normal in G). Let then G have no normal Sylow subgroup and let P be a minimal normal subgroup of G contained in S. Let $|P|=p$ (p prime), we have that the Sylow p-subgroups of G/P are cyclic, if $p\neq 2$, whereas, if $p=2$, are dihedral: thus G is of type (*II'*). Vice versa it is easy to verify that the groups (*II'*) and (*III'*) are U-groups.

1.6. Corollary. *Let U be a semisimple group with dihedral Sylow subgroups. Then every non split group extension of a group of order 2 by U is an U-group.*

Proof. It follows immediately from Theorem 1.5, if we recall the classification of groups with dihedral Sylow 2-subgroups ([5]).

2. Some remarks on the soluble U-groups.

From the Theorem 1.3 easily follows that there is no restriction for the nilpotent lenght of a soluble U-group. Nevertheless, if a soluble U-group G is not an A-group, that is, if G involves Q_8, its nilpotent lenght is at most $n+1$, if G is n-dual minimal non A-group $(n \geq 1)$ [2] ; to get this, it is enough to notice that the Theorem 1.3 assures that the only soluble U-group, dual minimal non A-group, is, up to isomorphisms, $C_3 \triangleright\langle Q_8 \simeq SL(2,3)$, whose nilpotent length is 2. From this remark and from Theorem 1.3 we can easily obtain the classification of the soluble U-groups, 2-dual minimal non A-groups :

2. Theorem. *The soluble U-groups, 2-dual minimal non A-groups, are, up to isomorphisms, the following ones :*

(a) $C_9 \triangleright\langle_\Phi Q_8$, where Φ is the product of the canonical homomorphism $C_9 \to C_9/\Phi(C_9) \simeq C_3$ and a monomorphism $C_3 \to \mathrm{Aut}\, Q_8$;

(b) semidirect products $SL(2,3) \triangleright\langle_\Phi P$, where: P is an elementary abelian p-group, p prime≥ 5 and Φ is an irreducible representation of $SL(2,3)$ on $GF(p)$.

3. Semisimple U-groups

In order to investigate the structure of the centreless CR-radical of a semisimple U-group, we provide the following lemma, whose proof is obvious.

3.1. Lemma. *Let $G = G_1 \times G_2$. If H is a subgroup of G, there exists a subgroup H_1 of G_1 and a subgroup H_2 of G_2, such that $|H| = |H_1 \times H_2|$.*

By Lemma 3.1 we have easily the following proposition.

3.2. *Let $G = G_1 \times G_2$. The following conditions are equivalent :*

(i) G_i is unique of its order among the subgroups of G $(i = 1, 2)$;

(ii) G_i has no proper subgroup of index equal to the order of a subgroup of G_j $(i \neq j = 1, 2)$;

(iii) G_j has no non trivial subgroup of order equal to the index $(in\, G_i)$ of a subgroup of G_i $(i \neq j = 1, 2)$.

Let $G = H_1 \times \ldots \times H_l$ $(l \geq 1)$ be a direct decomposition of G. We will say that it verifies the „order-index" condition, if $l = 1$ or : for every non empty proper subset J of $[1, l]$, with $G_1 = \underset{i \in J}{\times} H_i$ and $G_2 = \underset{i \notin J}{\times} H_i$, G_1 and G_2 verify one of the equivalent conditions *(ii)* or *(iii)* of the proposition 3.2.

Let R be a centreless CR-group and write $R = R_1 \times \ldots \times R_l$ $(l \geq 1)$, $(l \geq 1)$, where $R_1, \ldots, R_l$ are the characteristically simple characteristic subgroup of R. If $R_1 \times \ldots \times R_l$ verifies the „order-index" condition, we will say that R verifies the „o.-i" condition.

3.3. *Let G be a semisimple U-group and R its centreless CR-radical. Let $R_1, \ldots, R_l$ be the characteristically simple characteristic subgroups of G. Then :*

(i) R_i is minimal normal in G, for each $i = 1, \ldots, l$;

(ii) R verifies the „o.-i" condition.

Proof. Lf L is a minimal normal subgroup of G, we have $L < R_i$, for some $i = 1, \ldots, l$. If $L \neq R_i$, we have $R_i = \overset{m}{\underset{j=1}{\times}} U_{i,j}$ and $L = \overset{n}{\underset{j=1}{\times}} U_{i,j}$, with $n < m$ and

(2) If P is a class of groups, the class $\hat{P}_n$ of the n dual minimal non P-groups, is defined, inductively, as follows :

$$\hat{P}_0 = P \ ; \ G \in \hat{P}_n \ \text{ iff } \ G \notin \bigcup_{i=0}^{n-1} \hat{P}_i \ \text{ and } \ G/N \in \bigcup_{i=0}^{n-1} \hat{P}_i \ \text{ for each } \ 1 \neq N < G.$$

$U_{i,j} \simeq U_i$, where U_i is the simple constituent of R_i. It follows that $1 \neq \overset{m}{\underset{j=n+1}{\times}} U_{i,j} = C_{R_i}(L) \triangleleft G$ and so one of the subgroups L or $C_{R_i}(L)$ is not unique of its order among the subgroups of G, which contradicts the hypothesis ; thus *(i)* is proved. The *(ii)* follows immediately from proposition 3.2.

Remark. It is immediate to observe that, if a centreless CR-group verifies the „o.-i." condition, then so does the direct product of its non isomorphic simple constituents, whereas the converse is not true. On the other hand, it is easy to verify that there exist simple groups (for instance A_5) that, when constituents of a centreless CR-group, verifying the „o.-i." condition, force it to be characteristically simple.

Problem. Can the arithmetical condition „o.-i.", for a centreless CR-group, be somehow characterized in terms of structure ?

The author has not found a satisfactory answer.

From the proposition 3.3 and the well known classification of finite semisimple groups (see, for instance, [8], 3.3.18) we obtain the following theorem.

3.4. Theorem. *Let G be a semisimple U-group. Then, up to isomorphisms, $\mathrm{Inn}\, R < G < \mathrm{Aut}\, R$, where :*

(i) R is a centreless CR-group that verifies the „o.-i." condition ;

(ii) $G/\mathrm{Inn}\, R$ acts transitively (by means of conjugation) on the (non trivial) normal simple subgroups of each characteristically simple characteristic subgroup of R.

As it is well known (see, for instance, [8], 3.3.19), the number of the non isomorphic semisimple groups, whose centreless CR-radical is isomorphic to a given centreless CR-group R, is equal to the number of the conjugacy classes of subgroups of $\mathrm{Out}\, R$. From this result and from Theorem 3.4 we immediately get the following proposition.

3.5. *Let R be a centreless CR-group and let l_R be the number of the conjugacy classes of subgroups of $\mathrm{Out}\, R$, that act transitively on the normal simple subgroups of each characteristically simple characteristic subgroup of R. Then the number u_R of non isomorphic semisimple U-groups, whose centreless CR-radical is isomorphic to R, is at most l_R.*

The Theorem 3.4 does not characterize the semisimple U-groups. In fact, for a centreless CR-group, that verifies the „o.-i." condition, each of the following occurrences is possible.

1st : Every automorphism group of R, that verifies *(ii)* of Theorem 3.4, is an U-group ; equivalently, with the notation of 3.5, $u_R = l_R$;

2nd : R is centreless CR-radical of some semisimple U-group, but there exist automorphism groups of R, that verify *(ii)* of Theorem 3.4 and are not U-groups ; equivalently, $0 < u_R < l_R$;

3rd : R is not centreless CR-radical of any semisimple U-group ; equivalently, $u_R = 0$.

Throughout this paper a centreless CR-group will be said of 1st, 2nd or 3rd kind, if for it the 1st, 2nd or 3rd occurrence holds.

The following example shows the existence of centreless CR-groups of 3rd kind, that verify the „o.-i." condition.

Example. $R = L_2(8)^{17} \times L_2(16)$ *is a centreless CR-group of 3rd kind.*

Let be Inn $R \leq G \leq \mathrm{Aut}\, R$ and $G/\mathrm{Inn}\, R$ acts trasitively on normal simple subgroups of $L_2(8)^{17}$. Then G contains a 17-cycle $\gamma \in S_{17} \simeq \mathrm{Aut}\ L_2(16) \wr \mathrm{Aut}\, L_2(8) \sim S_{17} \simeq \mathrm{Aut}\, R$. If N is the normalizer in Inn $L_2(16)$ of one of its Sylow 2-subgroups, whe have:

$$|<\gamma, N>| = 17|N| = 17 \cdot 2^4 \cdot 3 \cdot 5 = |L_2(16)|$$

and so G is not an U-group.

One can directly verify that in R the „o.-i." condition holds.

The following result shares into centreless CR-groups of 1st or 2nd kind the simple (non abelian) groups.

3.6. *A simple (non abelian) group is a centreless CR-group of 1st kind, if and only if the Sylow subgroups of its outer automorphism group are cyclic.*

Proof. If R is a simple group of 1st kind, every subgroup of Out R is an U-group and so, obviously, the Sylow subgroups of Out R are cyclic.

Vice versa, let R be simple and the Sylow subgroups of Out R be cyclic. If G is an automorphism group of R, containing Inn R, and G is not an U-group, there exist different subgroups H and K of G, such that $H \lhd G$ and $|H| = |K|$. Suppose, without loss of generality, $G = HK$ and assume $R = \mathrm{Inn}\, R$. Since $|H/R|$ divides $|KR/R|$ and the Sylow subgroups of KR/R are cyclic, as those of G/R, we have $H/R < KR/R$ and so $G = KR$. It follows $1 \neq |R : K \cap R| = |G : K| = |G : H| \,|\, |\mathrm{Out}\, R|$. Thus R has a proper subgroup whose index divides the order of its outer automorphism group: but the classification of finite simple groups excludes this occurrence.

3.6.1. Corollary. *The alternating groups of degree ≥ 5 and $\neq 6$ are centreless CR-groups of 1st kind.*

Proof. It follows immediately from 3.6. if we recall that, for each $n \geq 5$ and $n \neq 6$, Aut $A_n \simeq S_n$.

3.6.2. Corollary. *A simple group, that is not of Lie type, is a centreless CR-group of 1st kind.*

Proof. It follows at once from 3.6 and 3.6.1, by using the classification of finite simple groups.

The next propositions provide some necessary or sufficient conditions for a centreless CR-group to be of 1st or 2nd kind.

3.7. *A (non abelian) characteristically simple group is a centreless CR-group of 1st or 2nd kind.*

Proof. Let R be a characteristically simple group and then $R = \times_{i=1}^{n} U_i$ $(n \geq 1)$, where $U_i \simeq U$ simple (non abelian) $(i = 1, \ldots, n)$. R trivially verifies the „o.-i." condition and the wreath product $U \sim C_n$ (C_n cyclic of order n) is a semisimple group whose centreless CR-radical is isomorphic to R. Thus it is sufficient to prove that $U \sim C_n$ is an U-group. Assume (with obvious identification) $G = C_n R$, let $1 \neq H \lhd G$ and $K \leq G$, such that $|H| = |K|$. We have $KH = KR = (C_n \cap KR)R$ and so $|C_n \cap KR / C_n \cap H| = |R : K \cap R|$, so that $|R : K \cap R|$ divides n. On the other hand, by Lemma 2.1, for each $i = 1, \ldots, n$ there exists a subgroup $\bar{U}_i$ of U_i, such that $|K \cap R| = |\times_{i=1}^{n} \bar{U}_i|$ and we can assume $\bar{U}_1 = K \cap U_1$.

We have then, with $|U_i : \bar{U}_i| = r_i$ $(i = 1, \ldots, n)$, that $\prod_{i=1}^{n} r_i$ divides n and so $r_i = 1$,

for some $i = 1, \ldots, n$. Without loss of generality, we can assume $r_1 = 1$, that is $U_1 \leq K$. We have then, with $C_n \cap KR = <\gamma>$ and $o(\gamma) = m$, that $\times_{j=0}^{m-1} U_1^{\gamma^j} \leq K$;
thus, if $m = n$, we have $R \leq K$. That being stated, proceeding by induction on n/m, we easily obtain that $R \leq K$. It follows, obviously, that $H = K$ and so G is an U-group.

3.8. *Let $R = U^n$ $(n > 1)$ be a characteristically simple group of 1st kind (U non abelian simple group). Then:*

(i) Out $U = 1$,

(ii) n is a prime not greater than the smallest index of a proper subgroup of U.

Proof. (i) Since $R = U^n$ is of 1st kind, we have that Out $R \simeq \mathrm{Out}\, U \sim S_n$ is an U-group. It follows, obviously, as Out U is soluble (see, for instance, [1]) that Out $U = 1$.

(ii) Since R is of 1st kind, we have that N, the normalizer of an n-cycle in $S_n \simeq \mathrm{Out}\, R$, is an U-group, and so, as N is isomorphic to the complete holomorph of a cyclic group of order n, n is prime. If n is greater than the smallest index m of a subgroup of U, let $\bar{U}$ be a subgroup of A_m isomorphic to U. With $R = \times_{i=1}^{n} U_i$ $(U_i \simeq U,\ i = 1, \ldots, n)$ and denoted by $G_1 \simeq S_{n-1}$ the stabilizer of 1 in $S_n \underset{\simeq}{\leq} \mathrm{Aut}\, R$, as $m < n$, we can assume $\bar{U} < G_1$. We have then:

$$|<\bar{U}, \times_{i=2}^{n} \mathrm{Inn}\, U_i>| = |\bar{U}| \cdot |U^{n-1}| = |R|$$

and so Aut R is not an U-group, which contradicts the hypothesis.

It is easy verify that the converse of 3.8 does not hold. Consider, for instance, M_{11}^5.

3.9. *Let R be a non characteristically simple centreless CR-group of 1st kind. Then, denoted by $U_1, \ldots, U_t$ $(t > 1)$ its non isomorphic simple constituents and by $n_1, \ldots, n_t$ the respective multiplicities, we have:*

(i) $n_i = 1$, for each $i = 1, \ldots, t$;

(ii) the Sylow subgroups of Out U_i are cyclic $(i = 1, \ldots, t)$ and $|\mathrm{Out}\, U_1|, \ldots, |\mathrm{Out}\, U_t|$ are pairwise coprime;

(iii) R verifies the „o.-i." condition.

Proof. Put $n_1 \leq \ldots < n_t$, since R is of 1st kind, we have that $S_{n_1} \times \ldots \times S_{n_t} \simeq\leq \mathrm{Aut}\, R$ is an U-group; it follows, obviously, $n_i = 1$, for each $i = 1, \ldots, t-1$. If $n = n_t > 1$, we have, as R is of 1st kind, that $\mathrm{Aut}\, R \simeq \mathrm{Aut}\, U_1 \times \ldots \times \mathrm{Aut}\, U_{t-1} \times \mathrm{Aut}\, U_t \sim S_n$ is an U-group and so therefore is $\mathrm{Aut}\, R/\mathrm{Aut}\, U_t \sim A_n \simeq \mathrm{Aut}\, U_1 \times \ldots \times \mathrm{Aut}\ U_{t-1} \sim C_2$; but this is false, because $2 \,\|\, |\mathrm{Aut}\, U_i|$ (Feit-Thompson theorem).

(ii) Since R is of 1st kind, we have that the subgroups of Out $R \simeq$ Out $U_1 \times \ldots \times$ Out U_t are U-groups; it follows, obviously, the statement.

(iii) It follows from Theorem 3.4.

It is easy to verify that the converse of 3.9 does not hold. Consider, for instance, $L_2(8) \times L_2(32)$.

An obvious partial converse of 3.9 is the following proposition.

3.10. *Let R be a direct product of non abelian simple groups pairwise non isomorphic. If R verifies the „o.-i." condition, then R is a centreless CR-group of 1st or 2nd kind.*

Proof. Obvious, if we notice that R is an U-group.

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ON Γ-DIVISION RING

BY

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1. Introduction. N. Nobusawa [13] in the year 1964 defined Γ-ring as follows: Let M and Γ be two additive abelian groups. If there exist mappings from $M\times\Gamma\times M$ to M (images to be denoted by $a\alpha b$, $a, b\in M$, $\alpha\in\Gamma$) and from $\Gamma\times M\times\Gamma\to\Gamma$ (images to be denoted by $\alpha a\beta$, $\alpha, \beta\in\Gamma$, $a\in M$), satisfying, for all $a, b, c\in M$ and $\alpha, \beta, \gamma\in\Gamma$:

$$N_1 : a\alpha(b\beta c)=a(\alpha b\beta)c=(a\alpha b)\beta c\,;$$

$$N_2 : (a+b)\alpha c=a\alpha c+b\alpha c,\quad a(\alpha+\beta)b=a\alpha b+a\beta b,\quad a\alpha(b+c)=a\alpha b+a\alpha c\,;$$

N_3 : $a\gamma b=0$ for all $a, b\in M$ implies that $\gamma=0$

then M is called a Γ-ring.

Later in the year 1966 Barnes [1] slightly weakened the defining conditions for Nobusawa's Γ-ring and defined Γ-ring as follows:

Let M and Γ be two additive abelian groups. If there exists a mapping from $M\times\Gamma\times M$ to M (images to be denoted by $a\alpha b$, $a, b\in M$, $\alpha\in\Gamma$), satisfying, for all $a, b, c\in M$ and $\alpha, \beta\in\Gamma$

$$B_1 : a\alpha(b\beta c)=(a\alpha b)\beta c\,;$$

$$B_2 : \text{same as } N_2,$$

then M is called a Γ-ring.

Usually, the former one is called Nobusawa's Γ-ring and the latter one is called Barnes' Γ-ring. The class of Γ-rings contains not only all rings but also Hestenes' ternary ring [7]. Coppage [3], Luh [11], [12], Kyuno [8], [9], [10] studied the structure of Γ-ring and obtained various generalizations of the corresponding parts in ring theory. The first author [4], [5], [6] also studied some properties of Barnes' Γ-ring and introduced the notion of Γ-field.

In this paper we have studied further some properties of Γ-field. Also we have introduced the notion of Γ-division ring (using Barnes' Γ-ring) and studied some properties of it. We have obtained some necessary and sufficient conditions for a Γ-ring to be a Γ-division ring or to be a Γ-field. Also we have established a relation between the Γ-field and the operator ring of a Γ-ring.

2. Preliminaries. A Γ-ring M is said to be commutative if $a\gamma b=b\gamma a$ for all $a, b\in M$ and for all $\gamma\in\Gamma$.

A Γ-ring M is said to be a Γ-ring with divisor of zero if there exist elements $a(\neq 0)$, $b(\neq 0)$ in M and an element $\gamma(\neq 0)$ in Γ such that $a\gamma b=0$.