ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA"
Tomul XXXVI, s. I a, Matematică, 1990, f. 4.

A GLOBAL INVARIANT OF THE SYMPLECTIC MANIFOLDS

BY

V. OPROIU

Les M be a compact symplectic manifold with the fundamental 2-form Ω, i.e. $\Omega^n \neq 0$ where $\dim M = 2n$ and $d\Omega = 0$. The set $\mathcal{A}$ of all associated metrics on M (the Riemannian metrics g on M such that $g(X, JY) = \Omega(X, Y)$ where J is an almost complex structure) is infinite dimensional and is totally geodesic in the set of all Riemannian metrics on M (see [3], [4]). The associated metrics and the corresponding almost complex structures are obtained by the polarization of Ω with respect to arbitary Riemannian metrics on M. All the metrics in $\mathcal{A}$ have the same volume element dv obtained from Ω^n.

It is well known that in the case of the almost Kaehler structures (e. g. defined by the associated metrics on symplectic manifolds) the nontrivial scalar invariants involving the first and second order derivatives of the associated metrics are $\|N\|^2$, $\|\nabla\Omega\|^2$, R, R^* where N is the Nijenhuis tensor field of J, $\nabla\Omega$ is the covariant differential of Ω with respect to g and R, R^* are the scalar and $*$-scalar curvatures of g ([11]). Moreover, these invariants are related by the formulae

$$R - R^* = -\frac{1}{2}\|\nabla\Omega\|^2 = -\frac{1}{8}\|N\|^2$$

so that R, R^* are the only essential invariants defined by an associated metric on a symplectic manifold. In [5] the authors have shown that the associated metric g is a critical point of the integrals $\int R dv$, $\int (R - R^*) dv$ on M if and only if the Ricci tensor field commutes with J.

In the present paper we prove the following result.

Theorem 1. *If M is a compact symplectic manifold then the integral*

$$I = \int_M (R + R^*) dv$$

is a constant on the set of all associated metrics.

Thus I does not depend on the associated Riemannian metric g used to get R, R^*; it is a global invariant of the symplectic manifold (M, Ω). We shall prove the theorem by showing that that the variation of I in $\mathcal{A}$ is zero.

1. Some fundamental formulae. Let $g(t)$ be a smooth path in $\mathcal{A}$ represented locally by $g_{ij}(x^1, \ldots, x^n, t)$ and define the tensor field D_{ij} on $(M, g(t))$ by

$$D_{ij}(x, t) = \frac{\partial g_{ij}(x, t)}{\partial t}. \tag{1}$$

Next, from $g(t)$ we get easily the following tensor field on M:

$$D^h_{ij}=\frac{\partial}{\partial t}\Gamma^h_{ij}=\frac{1}{2}(\Delta_i D^k_j+\Delta_j D^k_i-\Delta^k D_{ij}). \tag{2}$$

where Γ^k_{ij} are the Christoffel symbols of the Levi Civita connection ∇ of $g(t)$. We have used $g(t)$ to raise and lower the indices.

The local coordinate expression of the curvature tensor field of ∇ is

$$R^h_{kij}=\partial_i\Gamma^h_{jk}-\partial_j\Gamma^h_{ik}+\Gamma^h_{il}\Gamma^l_{jk}-\Gamma^h_{jl}\Gamma^l_{ik}$$

and from this expression we get

$$\frac{\partial}{\partial t}R^h_{kij}=\nabla_i D^h_{jk}-\nabla_j D^h_{ik}.$$

Using the definition of g^{ij} we have $\frac{\partial}{\partial t}g^{ij}=-D^{ij}$

Recall also the following formulae obtained from the property of $g(t)$ to be associated

$$\Omega_{ij}=g_{ih}J^h_j=-g_{jh}J^h_i,\quad \Omega^{ij}=J^i_h g^{hj}=-J^j_h g^{hi}\quad g_{ij}=J^h_i\Omega_{hj}=-\Omega_{ih}J^h_j,$$
$$g^{ij}=-J^i_h\Omega^{hj}=-J^j_h\Omega^{hi}=\Omega^{ih}J^j_h\quad J^i_j=\Omega_{hj}g^{hi}=-\Omega_{jh}g^{ih}=g_{jh}\Omega^{ih}=-g_{hj}\Omega^{hi}$$

where

$$J^i_h J^h_j=-\delta^i_j,\quad g_{ih}g^{hj}=\delta^j_i,\quad \Omega_{ih}\Omega^{jh}=\delta^j_i,\quad \Omega^{ij}=g^{ik}\Omega_{kl}g^{lj}.$$

In these formulae, Ω_{ik}, J^i_j denote the local coordinate expressions of Ω and the smooth path $J(t)$ corresponding to $g(t)$.

The tensor field D_{ij} satisfies the following relation

$$D_{ij}J^i_k J^j_l=-D_{kl} \tag{3}$$

because of the property of $g(t)$ to be associated. We have also

$$D^{ij}J^k_i J^l_j=-D^{kl}. \tag{3'}$$

Remark the properties $g^{ij}D_{ij}=0$, $D^h_{ih}=0$. Then we get

$$\frac{d}{dt}J^i_j=-D^i_h J^h_j=J^i_h D^h_j \tag{4}$$

From the property of Ω to be closed we have (see [11], [5])

$$J^h_i\nabla_h J^k_l=\nabla_i J^k_h J^h_l,\quad \nabla_l J^l_k=0 \tag{5}$$

so that the local coordinate expression of the Nijenhuis tensor field of J becomes

$$N^k_{ij}=-2J^k_h(\nabla_i J^h_j-\nabla_j J^h_i).$$

The local coordinate expressions of the Ricci tensor field and the scalar curvature are

$$R_{ij}=R^h_{ihj}=g^{hk}R_{hikj},\quad R=g^{ij}R_{ij}.$$

Finally the $*$-Ricci tensor field and the $*$-scalar curvature are given by

$$R^*_{ij}=R_{ihkj}\Omega^{hk}J^l_j=-\frac{1}{2}R_{ithk}\Omega^{hk}J^t_j,\qquad R^*=R^*_{ij}g^{ij}=-\frac{1}{2}R^i_{thk}\Omega^{hk}J^t_i.$$

Remark the following formulae

$$R^i_{kt}J^k_i J^t_j=R^*_{ji},\quad R_{ijkl}\Omega^{kl}=2R^*_{ih}J^h_j,\quad R_{ikl}\Omega^{kl}=-R^*_{ih}J^h_j. \tag{6}$$

2. The variation formulae. From the definition of the scalr curvature we get

$$\frac{dR}{dt}=-D^{ij}R_{ij}+g^{ij}(\nabla_h D^h_{ij}-\nabla_j D^h_{ih})=-D^{ij}R_{ij}+\nabla_h(g^{ij}D^h_{ij}).$$

so that

$$\frac{d}{dt}\int R\,dv=-\int R_{ij}D^{ij}dv \tag{7}$$

since the last term in the expression of dR/dt is a divergence and contribute nothing to the integral.

Due to (3'), (6) we have $R^*_{ij}D^{ij}=0$ so that

$$\frac{dR^*}{dt}=-\frac{1}{2}R^i_{thk}\Omega^{hk}\frac{d}{dt}J^t_i-\frac{1}{2}J^t_j\Omega^{hk}\frac{d}{dt}R^j_{thk}=-R^*_{ji}D^{ij}-\nabla_h D^j_{kl}\Omega^{hk}J^l_j-$$
$$-\nabla_h\nabla_l D^{jk}J^h_k J^l_j=\nabla_l D^{jk}J^h_k\nabla_h J^l_j-\nabla_h(\nabla_l D^{jk}J^h_k J^l_j).$$

Then, using the Green theorem we get

$$\frac{d}{dt}\int R^*dv=\int\nabla_l D^{jk}J^h_k\nabla_h J^l_j dv=\int(-D^{jk}\nabla_l J^h_k\nabla_h J^l_j-D^{jk}J^h_k\nabla_l\nabla_h J^l_j)\,dv.$$

The first term in the last integral vanishes since, using (3'), (5) we have

$$D^{jk}\nabla_l J^h_k\nabla_h J^l_j=-D^{ab}J^j_a J^k_b\nabla_l J^h_k\nabla_h J^l_j=D^{ab}J^j_a\nabla_l J^k_b J^h_k\nabla_h J^l_j=$$
$$=D^{ab}J^j_a\nabla_l J^k_b\nabla_k J^l_h J^h_j=-D^{hb}\nabla_l J^k_b\nabla_k J^l_h.$$

The last term may be transformed by using (5) and the Ricci identity

$$-D^{jk}J^h_k\nabla_l\nabla_h J^l_j=-D^{jk}J^h_k(\nabla_l\nabla_h-\nabla_h\nabla_l)J^l_j=-D^{jk}J^h_k(R^l_{ilh}J^i_j-R^i_{jlh}J^l_i)=$$
$$=-R_{ih}(D^{jk}J^i_j J^h_k)+D^{jk}R_{ijlh}\Omega^{li}J^h_k=R_{ih}D^{ih}+D^{jk}R^*_{jk}=R_{ih}D^{ih}.$$

Thus we get

$$\frac{d}{dt}\int R^*dv=\int R_{ih}D^{ih}\,dv, \tag{8}$$

so that

$$\frac{dI}{dt}=0.$$

The theorem 1 is proved.

If the symplectic manifold (M, Ω) carries a Kaehler metric g then $R^* = R$ and

$$I = 2\int R dv.$$

Moreover, if $n > 1$ and M is Einstein then R is constant and

$$I = 2R \operatorname{vol} M.$$

Thus I is not a trivial invariant of M.

Received 29.VIII.1989

Faculty of Mathematics
University of Iași, România

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Tomul XXXV, s. I a. Matematică, 1990, f. 4.

DEFORMATIONS OF 4-MANIFOLDS

BY

VALENTIN BOJIU and LOUIS FUNAR

In this paper we outline some properties concerning the topology of 4-manifolds and their deformations from the geometric point of view. An important question in the theory has been: what is the relationship between smooth 4-manifolds theory and the theory of algebraic surfaces? The birational morphisms between two complex projective surfaces are completely understood by means of two basic ones: the blowing up and the blowing down which behaves well from the differential point of view except for a codimension one curve (the exceptional curve). However of great interest should be the study of the almost regular maps i.e. the maps which are diffeomorphism between the complements of two measure zero subsets. Our setup will provide some information concerning continuous deformations of almost regular maps in terms of the topology of the underlying manifolds. First we shall give a brief description of the homotopy theory of 4-manifolds.

I. We are concerned with closed, oriented, simply-connected smooth manifolds. Any such manifold M has an orientation class $[M] \in H_4(M, \mathbf{Z}) \simeq \mathbf{Z}$. Cap product with it induces the Poincaré duality isomorphism $\cap [M] : H^i(M, \mathbf{Z}) \to H_{4-i}(M, \mathbf{Z})$. Because $\pi_1(M) = 0$ we have, $H_1(M, \mathbf{Z}) = 0$. By the Poincaré duality it follows that $H_3(M, \mathbf{Z}) \cong 0$ and that $H_2(M, \mathbf{Z})$ is a free abelian group of finite rank. Furthermore, Poincaré duality induces an intersection pairing $q_M : H_2(M, \mathbf{Z}) \times H_2(M, \mathbf{Z}) \to \mathbf{Z}$ which is symmetric and nonsingular. This then leads to the first invariant associated to a 4-manifold: the algebraic isomorphism class of the pairing $(H_2(M, \mathbf{Z}), q_M)$. The isomorphism classification of such pairing is more or less understood: Let (A, q) be a unimodular, symmetric pairing. Its rank is the rank of the free abelian group A. Its index $I(q)$ is defined as follows. Over $\mathbf{R}$ the pairing can be diagonalized i. e. there is an $\mathbf{R}$-basis $x_1, \dots, x_k$ for $A \otimes \mathbf{R}$ such that $q(x_i, x_j) = 0$ for $i \neq j$. We set $p(q)$ for the number of x_i for which $q(x_i, x_i) > 0$ and similarly we define $n(q)$ as the number of x_i for which $q(x_i, x_i) < 0$. Since q is nonsingular $p(q) + n(q)$ is equal to the rank. We define $I(q) = p(q) - n(q)$. The pairing is said to be positive (resp. negative definite if $n(q) = 0$ (resp. $p(q) = 0$). Lastly we say that q is even if $q(a, a) \equiv 0 \pmod 2$ for all $a \in A$. Otherwise the pairing is odd. The basic result in the classification of unimodular pairings ([S]) says: 1) if (A, q) is indefinite it is determined up to isomorphism by its rank, index and parity; 2) there are only finitely many definite pairings of a given rank. There are some relations among these invariants: $|I(q)| \leq \operatorname{rank} A$, $I(q) \equiv \operatorname{rank} A \pmod 2$ and for q even $I(q) \equiv 0 (\text{mod. } 8)$. Subject to these restrictions all values