The theorem 1 is proved.

If the symplectic manifold (M, Ω) carries a Kaehler metric g then $R^* = R$ and

$$I = 2 \int R dv.$$

Moreover, if $n > 1$ and M is Einstein then R is constant and

$$I = 2R \text{ vol } M.$$

Thus I is not a trivial invariant of M.

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DEFORMATIONS OF 4-MANIFOLDS

BY

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In this paper we outline some properties concerning the topology of 4-manifolds and their deformations from the geometric point of view. An important question in the theory has been: what is the relationship between smooth 4-manifolds theory and the theory of algebraic surfaces? The birational morphisms between two complex projective surfaces are completely understood by means of two basic ones: the blowing up and the blowing down which behaves well from the differential point of view except for a codimension one curve (the exceptional curve). However of great interest should be the study of the almost regular maps i.e. the maps which are diffeomorphism between the complements of two measure zero subsets. Our setup will provide some information concerning continuous deformations of almost regular maps in terms of the topology of the underlying manifolds. First we shall give a brief description of the homotopy theory of 4-manifolds.

1. We are concerned with closed, oriented, simply-connected smooth manifolds. Any such manifold M has an orientation class $[M] \in H_4(M, \mathbf{Z}) \simeq \mathbf{Z}$. Cap product with it induces the Poincaré duality isomorphism $\cap [M] : H^i(M, \mathbf{Z}) \to H_{4-i}(M, \mathbf{Z})$. Because $\pi_1(M) = 0$ we have, $H_1(M, \mathbf{Z}) = 0$. By the Poincaré duality it follows that $H_3(M, \mathbf{Z}) \cong 0$ and that $H_2(M, \mathbf{Z})$ is a free abelian group of finite rank. Furthermore, Poincaré duality induces an intersection pairing $q_M : H_2(M, \mathbf{Z}) \times H_2(M, \mathbf{Z}) \to \mathbf{Z}$ which is symmetric and nonsingular. This then leads to the first invariant associated to a 4-manifold: the algebraic isomorphism class of the pairing $(H_2(M, \mathbf{Z}), q_M)$. The isomorphism classification of such pairing is more or less understood: Let (A, q) be a unimodular, symmetric pairing. Its rank is the rank of the free abelian group A. Its index $I(q)$ is defined as follows. Over $\mathbf{R}$ the pairing can be diagonalized i. e. there is an $\mathbf{R}$-basis $x_1, \ldots, x_k$ for $A \otimes \mathbf{R}$ such that $q(x_i, x_j) = 0$ for $i \neq j$. We set $p(q)$ for the number of x_i for which $q(x_i, x_i) > 0$ and similarly we define $n(q)$ as the number of x_i for which $q(x_i, x_i) < 0$. Since q is nonsingular $p(q) + n(q)$ is equal to the rank. We define $I(q) = p(q) - n(q)$. The pairing is said to be positive (resp. negative definite if $n(q) = 0$ (resp. $p(q) = 0$). Lastly we say that q is even if $q(a, a) \equiv 0 \pmod 2$ for all $a \in A$. Otherwise the pairing is odd. The basic result in the classification of unimodular pairings ([S]) says: 1) if (A, q) is indefinite it is determined up to isomorphism by its rank, index and parity; 2) there are only finitely many definite pairings of a given rank. There are some relations among these invariants: $|I(q)| \leq \text{rank } A$, $I(q) \equiv \text{rank } A \pmod 2$ and for q even $I(q) \equiv 0 (\text{mod. } 8)$. Subject to these restrictions all values

of the rank, index and parity are possible. Returning to the 4-manifolds we set $b^+(M)=p(q_M)$, $b^-(M)=n(q_M)$ and $I(M)=I(q_M)$, calling $I(M)$ the index of M (or signature). It turns out that two simply connected 4-manifolds M and N are homotopy equivalent if and only if their pairings q_M and q_N are isomorphic. Similarly the group of homotopy automorphisms of M is commensurate to the group of automorphisms of the intersection pairing (more precise is an subgroup of finite index in the second). Freedman [F] determined the homeomorphism classification of simply connected 4-manifolds, by showing that every pairing occurs up to isomorphism as the intersection pairing of a simply connected topological 4-manifold (a similar result for simply connected celullar complexes satisfying Poincaré duality has been earlier given by Whitehead). If the pairing is even then all manifolds realizing it are homeomorphic. If the pairing is odd, then there are exactly two homeomorphism clases of manifolds realizing it. Theses are distinguished by the Arf invariant i.e. exactly one is stably smoothable (or equivalent the product with S^1 carries a smooth structure). Donaldson proved ([D 1]) that if $b^+(M)=0$, M is a smooth, oriented 4-manifold then the intersection pairing is equivalent over Z to the standard form $(-1)\oplus\ldots\oplus(-1)$. (Even if M is not simply connected the above result remains valid [D 2]).

2. We define now an intermediary relation of equivalence for 4-manifolds. If $f:M\to N$ is a map providing an diffeomorphism between $M-A$ and $N-B$ where A, B are measure zero closed subsets then in general nothing can be said about the topology of M and N. E. g. if M is a Riemann surface then choose a canonical basis $\{a_1,\ldots,a_g,b_1,\ldots,b_g\}$ in the homology $H_1(M,\mathbf{Z})$ i.e. $a_i\cdot a_j=0$, $b_i\cdot b_j=0$, $a_i\cdot b_j=\delta_{ij}$, where the product is the cup product $H_1(M,\mathbf{Z})\times H_1(M,\mathbf{Z})\to H_2(M,\mathbf{Z})\simeq Z$; then $M-\{a_1,\ldots,a_g,b_1,\ldots,b_g\}$ is diffeomorphic to $\mathbf{R}^2$. However in dimension four it is known (see [F, Q] that all compact topological 4-manifolds are smoothable in the complement of a single point and the homeomorphisms between two of them are isotopic to almost diffeomorphisms (named flexomorphisms after Freedmann) i.e. we can choose A, B of codimension two. On the other hand every blowing up $M\to N$ between compact complex surfaces identifies up to a diffeomorphism M and $N\#\mathbf{CP}^2$, hence the behaviour of birational maps is given, topologically by the connected sum with several $\overline{\mathbf{CP}^2}$ (i.e. $\mathbf{CP}^2$ with the reverse orientation).

Definition. *For M, N two smooth 4-manifolds we say that M and N have the same deformation type, $M\underset{d}{\simeq}N$ if there exist a compact conex manifold B (the base, or parameter space) and a family M_t, $t\in B$ of compact smooth manifolds, a collection of closed, subsets of measure zero $A_{t,s}\subset M_t$ such that the following conditions are fulfilled;*

1. the set of the pairs (t,s), where $A_{t,s}$ is defined, is a conex neighborhood F of the diagonal $\Delta\subset B\times B$;

2. $M_t-A_{t,s}$ are smooth 4-manifolds, and for some t_0, $t_1\in B$ $M_{t_0}=M$, $M_{t_1}=N$.

3. there are the maps $h_{t,s}:M_t-A_{t,s}\to M_s-A_{s,t}$ such that

(i) $$h_{s,s}=\mathrm{id}$$

(ii) $$h_{s,t}\in C^\infty$$

$\eta=(h_{s,t},M_{t,s},F)$ is called a deformation between M and N.

The regularity of the deformation η is given by its type, namely we set $\eta\in D(p,k,a,b)$ if

4. the map $\rho:F\to\mathbf{R}_+$ given by

$$\rho(s,t)=\left\|I_{s,t}\right\|_{L^b_a(M_s-A_{s,t})}$$

lies in $L^k_p(F)$. Here $I_{s,t}$ represents the determinant of the jacobian of $h_{s,t}$, and $L^r_u(X)$ is the Sobolev space of functions φ having $\int\sum_{|\alpha|\leq u}\|\partial^\alpha\varphi\|^p<\infty$ (This is defined first locally then use a partition of unity argument as in [T, FU]). If $L^b_a\subset L^1_0$, $L^k_p\subset C^0$ (there are sufficient conditions given by Sobolev imbedding theorems), we say that η is a continuous deformation. Our primary question is: what is the relationship between the various deformation types and the classical ones: topological, C^∞ or complex deformation type (as defined in [FM]) ? Our main result can be stated

Theorem 1. *Suppose that there exists a continuous deformation between the compact smooth manifolds M and N, and one of the following conditions is satisfied*

1) $H_1(M)$, $H_1(N)$ are finite and q_M is negative definite.

2) $H_1(M)$, $H_1(N)$ are finite, has no 2-torsion and q_M is of type $(1,n)$, $n\geq 2$.

Then M is homeomorphic to N.

Proof. Denote by Ω^j the $C^\infty(M)$-module of differential j-forms over M, d the differentiation operator on forms, d^* its adjoint with respect to the usual inner product on the space of forms. Then $d+d^*$ is an elliptic first order operator whose square is the Laplace operator Δ of the Hodge theory. Is defined an involution $\tau:\Omega^j\to\Omega^{4-j}$ by $\tau(\alpha)=-(\sqrt{-1})^{j(j-1)}*\alpha$, $\alpha\in\Omega^j$ where $*:\Omega^j\to\Omega^{4-j}$ is the duality operator defined by the metric. Denoting by $\Omega_+\subset\subset\bigoplus_{j=0}^{4}\Omega^j$ the ± 1 — eigenspaces of τ one can easily check that $d+d^*$ interchanges Ω_+ and Ω_-. The signature operator S is the restriction of $d+d^*$ as an operator from Ω_+ to Ω. Because S is an elliptic operator ker S coincides with ker S^* $S=(\ker\Delta)\cap\Omega_+$ that is with the space H_+ of harmonic forms ω such that $\tau(\omega)=\omega$. Similarly coker $S\simeq$ ker S^* coincides with the space H_- of harmonic forms ω such that $\tau(\omega)=-\omega$. Now index $S=\dim H_+-\dim H_-=\dim H^2_+-\dim H^2_-$ (the other contributions from H^j and H^{-j} will cancel). Since

$$\alpha\to\int\alpha\wedge*\alpha$$

is positive definite for α a real 2-form it follows that

$$\alpha\to\int\alpha\wedge\alpha$$

is positive definite on H^2_+ and negative definite on H^2_-. Since every harmonic form ω lies in a unique de Rham cohomology class $[\omega]\in H^2(M,\mathbf{R})$ and by Poincaré duality we see therefore that

$$\dim H^2_+=p(q_M),\ \dim H^2_-=p(q_M)$$

because $q_M : H^2(M, \mathbf{Z})/\text{Torsion} \to \mathbf{Z}$, and $\text{Torsion}\, H^2(M, Z) \simeq H_1(M, \mathbf{Z})$ by the universal coefficient theorem ($H_1(M, \mathbf{Z})$ is finite from hypothesis). Then

$$I(M) = \text{index } S.$$

From Atiyah-Singer theorem [ABP] index S can be expressed as

$$\text{index } S = \int_M \omega_M$$

where ω_M is a canonically associated 4-form, in fact a multiple of p_1, the Pontrjagin form on M. We wish first to prove that $I(M)$ is constant under a continuous deformation. We mention that if $h : M \to N$ is a difeomorphism, $h^*\omega_M = \omega_N$. Recalling the notations made earlier we have

$$I(M_t) = \int_{M_t} \omega_{M_t} = \int_{M_t - A_{t,s}} \omega_{M_t} = \int_{M_s - A_{s,t}} I_{s,t}\omega_{M_s}$$

hence

$$I(M_t) - I(M_s) = \int_{M_s - A_{s,t}} (I_{s,t} - 1)\omega_{M_s}.$$

Since we have a continuous deformation $\rho : F \to \mathbf{R}_+$ is uniformly continuous and there exists $\eta(\varepsilon) > 0$ such that in the apropiate metric on $F \subset B \times B$ for dist $(s, t) < \eta(\varepsilon)$ it follows

$$\|I_{s,t} - 1\|_{L^1(M_s - A_{s,t})} < \varepsilon.$$

Therefore

$$|I(M_s) - I(M_t)| < \varepsilon I(M_s). \tag{$*$}$$

Now set $\varepsilon_0 < 1/I(M_{t_0})$, $M = M_{t_0}$ and $A = \{s \in F;\ I(M_s) = I(M)\}$, $B_n = \{t \in F,$ dist$(t_0, t) < n\eta(\varepsilon_0)\}$. By ($*$) because $I(M_s)$ is an integer $B_1 \subset A$. By induction it follows $B_n \subset A$ for every $n > 0$. Furthermore F being compact and conex $B_m = F$ for some $m > 0$, hence $I(M_t)$ is constant. The same argument holds for the Euler-Poincaré characteristic which also has an integral expression of same sort. So

$$I(M) = I(N) \text{ and } \mathcal{X}(M) = \mathcal{X}(N).$$

But N is a smooth compact 4-manifld having q_N of same signature as M hence diagonalizable over integers.

The rank of q_N can be obtained from $\mathcal{X}(M) = 2 + \text{rank } q_M$ and the topological classification due to Freedmann asserts that M and N are homeomorphic. Our claim is proved.

Let now consider that $h_{t,s} : M_t \dots M_s$ are birational morphisms between algebraic surfaces, for $(t, s) \in F$, $A_{t,s}$ will be the singular locus of $h_{t,s}$. In thus case the deformation we called algebraic. Since two nonsingular algebraic surfaces birational isomorphic and homeomorphic should be difeomorphic we have from above:

Theorem 2. *If there exists a continuous algebraic deformation between two nonsingular algebraic surfaces M, N satisfying one of the conditions listed in Theorem 1 then M and N are diffeomorphic.*

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