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PAIRS OF METRICS AND CODAZZI TENSORS ON MANIFOLDS

BY

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1. Introduction. A pair (I_1, I_2) of Riemannian metrics, on an n-manifold M, is called Codazzi if $(\nabla_X L)Y=(\nabla_Y L)X$, where L is the linear transformation $I_1^{-1}\circ I_2$, ∇ is the Riemannian connection of I_1 and X, Y are any vector fields on M. This definition (in an equivalent form) was given, for $n=2$, by T. Klotz [1]. Using L, one can define, for any integer r, the metrics I_r on M by $I_r(X, Y)=I_1L^{r-1}(X, Y)$. In this paper, using Codazzi's condition for a pair (I_r, I_s), we find sufficient conditions to be the metrics I_1, I_2 analogues. Special cases of our main results are some well known results of [2], [3] and [4].

2. Notations. Let M be a connected n-dimensional manifold equiped with a pair of metrics, namely a pair of symmetric tensor fields of type (0, 2), which are positive definite. We denote them by I_1 (or $<,>$), I_2 (or $<,>_2$) and by ∇, $\overset{2}{\nabla}$ the Riemannian connections of I_1, I_2 respectively. The pair (I_1, I_2) defines a (1, 1) tensor field: the field may be thought as $I_1^{-1}\circ I_2$. In fact at each point $P\in M$, both I_1 and I_2 are symmetric bilinear forms on the tangent space T_PM at P. Hence, these bilinear forms, give nonsingular linear maps from T_PM to the cotangent space $T_P{}^*N$ (in the usual way), which we denote by the same symbol. So composing the inverse of the linear map I_1 with the linear map I_2, we get a linear transformation on T_PM, which we denote by $L=I_1^{-1}\circ I_2(P)$. We also denote the resulting linear transformation field on M by $L=I_1^{-1}\circ I_2$. An immediate consequence of the definition of L is:

$$<X, Y>_2=<LX, Y>=<X, LY) \text{ and } <(\nabla_X L)Y, Z>=<Y, (\nabla_X L)Z>,$$

where

$$(\nabla_X L)Y=\nabla_X(LY)-L\nabla_X Y \tag{1}$$

and X, Y, Z are any vector fields on M.

The pair (I_1, I_2), (or equivalently the pair (I_1, L)) is called Codazzi and it is denoted by Cod(I_1, I_2) (or Cod(I, L)), if

$$(\nabla_X L)Y=(\nabla_Y L)X \tag{2}$$

for any vector fields X, Y on M. It is well known that the class of Riemannian manifolds admiting Codazzi tensors is large.

Using L we define, for any integer r, the metrics $I_r(< >_r)$ by

$$<X, Y>_r=<L^{r-1}X, Y> \tag{3}$$

where $L^k=L...L$ (k-times), L^{-1} is the inverse of L and $L_0=\mathrm{Id}$.

3. Results. The main purpose of this paper is to establish the following

Theorem 1. *Let M be a connected n-dimensional manifold and (I_1, I_2) a pair of metrics on M such that the sectional curvature of M with respect to I_1 is never zero. Then $I_2=cI_1$ (c is a const.$\neq 0$) if and only if* $\mathrm{Cod}(I_1, I_2)$ *and* $\mathrm{Cod}(I_r, I_s)$ *for some pair (r, s) different of $(1, 2)$ and $(3, 2)$.*

Theorem 2. *Let (M, I_1) be a connected n-Riemannian manifold with non zero sectional curvature and I_2 be a non degenerate quadratic form (not necessarily positive definite) on M. If $\nabla L\equiv 0$, then $I_2=cI_1$ (c is const$\neq 0$).*

Remark 1. α) The definitions of section 2 are valid and in the case where I_2 is a quadratic form not necessarily positive definite. β) Theorem 1 is known for $(r, s)=(2, 1)$, see [2] and ([1], Lemma 5 for $n=2$). γ) A special case of Theorem 2 is the following known result of [4, Corollary 2]. "Let M be a hypersurface of Euclidean space R^{n+1} with non degenerate Weingarten map L. If $\nabla L=0$ on M, then M is a piece of a hypersphere of R^{n+1}."

To prove the theorems we need some preliminary results.

Lemma 1. *If $I_2=fI_1$ for some smooth function f on M and* $\mathrm{Cod}(I_1, I_2)$, *then f is a constant.*

Proof. Since $L=I_1^{-1}\circ I_2$ and $I_2=fI_1$ we take $L=f\mathrm{Id}$. So from (2) we get $(Xf)Y=(Yf)X$. Because X, Y are arbitrary, f must be constant

Lemma 2. *If* $\mathrm{Cod}(I_r, I_{r+p})$ *for some integers r, p, ($p\neq 0$), then*

i) $$2L^p(\overset{r+p}{\nabla}_X Y-\overset{r}{\nabla}_X Y)=(\overset{r}{\nabla}_X L^p)Y,$$

ii) $$L^p(\overset{r+2p}{\nabla}_X Y-\overset{r}{\nabla}_X Y)=(\overset{r}{\nabla}_X L^p)Y.$$

In particular, if $\mathrm{Cod}(I_1, I_2)$, *then*

(iii) $$2(\overset{r}{\nabla}_X Y-\nabla_X Y)=\begin{cases}\displaystyle\sum_{\mu=1}^{r-1}\{L^{-\mu}(\nabla_X L)L^{\mu-1}Y+L^{-\mu}(\nabla_Y L)L^{\mu-1}X-\\ \qquad -L^{1-r}(\nabla_{L^{\mu-1}X}L)\,L^{r-\mu-1}Y\},\ \text{if } r>1,\\ \displaystyle-\sum_{\mu=1}^{-r+1}\{L^{\mu-1}(\nabla_X L)L^{-\mu}Y+L^{\mu-1}(\nabla_{L^{-\mu}X})Y-\\ \qquad -L^{1-r}(\nabla_{L^{-\mu}X}L)L^{r+\mu-2}Y\},\ \text{if } r\geqq 0\end{cases}$$

Proof. i) Differentiating the relation

$$<X, Y>_{r+p}=<L^pX, Y>_r$$

with respect to Z and using (1) (for $\overset{r}{\nabla}$), we easily get

(4) $$<\overset{r+p}{\nabla}_Z X-\overset{r}{\nabla}_Z X,\ Y>_{r+p}+<\overset{r+p}{\nabla}_Z Y-\overset{r}{\nabla}_Z Y,\ X>_{r+p}=<(\overset{r}{\nabla}_Z L^p)X,\ Y>_r.$$

Replacing X, Y, Z by Y, Z, X and Z, X, Y one is led to following pair of relations

(5) $$<\overset{r+p}{\nabla}_X Y-\overset{r}{\nabla}_X Y,\ Z>_{r+p}+<\overset{r+p}{\nabla}_X Z-\overset{r}{\nabla}_X Z,\ Y>_{r+p}=<(\overset{r}{\nabla}_X L^p)Y,\ Z>_r$$

(6) $$<\overset{r+p}{\nabla}_Y Z-\overset{r}{\nabla}_Y Z,\ X>_{r+p}+<\overset{r+p}{\nabla}_Y X-\overset{r}{\nabla}_Y X,\ Z>_{r+p}=<(\overset{r}{\nabla}_Y L^p)Z,\ X>_r.$$

Because $\overset{r+p}{\nabla}_X Y-\overset{r}{\nabla}_X Y=\overset{r+p}{\nabla}_Y X-\overset{r}{\nabla}_Y X$, adding (5), (6) and substracting (4) we take

$$2<\overset{r+p}{\nabla}_X Y-\overset{r}{\nabla}_X Y,\ Z>_{r+p}=<(\overset{r}{\nabla}_X L^p)Y,\ Z>_r+<(\overset{r}{\nabla}_Y L^p)Z,\ X>_r-$$
$$-<(\overset{r}{\nabla}_Z L^p)X,\ Y>_r.$$

The pair (I_r, I_{r+p}) defines the linear transformation L^p. So, because $\mathrm{Cod}(I_r, I_{r+p})$, i.e. $(\overset{r}{\nabla}_X L^p)Y=(\overset{r}{\nabla}_Y L^p)X$, the last equation can be written as

$$2<\overset{r+p}{\nabla}_X Y-\overset{r}{\nabla}_X Y,\ Z>_{r+p}=<(\overset{r}{\nabla}_X L^p)Y,\ Z>_r$$

or

$$2<L^p(\overset{r+p}{\nabla}_X Y-\overset{r}{\nabla}_X Y),\ Z>_r=<(\overset{r}{\nabla}_X L^p)Y,\ Z>_r.$$

So

$$2L^p(\overset{r+p}{\nabla}_X Y-\overset{r}{\nabla}_X Y)=(\overset{r}{\nabla}_X L^p)Y.$$

Differentiating $<X, Y>_{r+2p}=<L^pX, L^pY>_r$. (ii) follows similarly.

iii) Differentiating $<X, Y>_r=<L^{r-1}X, Y>$ and working as in case (i) we obtain

(7) $$\begin{aligned}2<\overset{r}{\nabla}_X Y-\nabla_X Y,\ Z>_r&=<(\nabla_X L^{r-1})Y,\ Z>+<(\nabla_Y L^{r-1})Z,\ X>-\\ &-<(\nabla_Z L^{r-1})X,\ Y>.\end{aligned}$$

But, because $(\nabla_X L^r)Y=L^{r-1}(\nabla_X L)Y+(\nabla_X L^{r-1})LY$, it is easy to prove by induction, that

(8) $$(\nabla_X L^r)Y=\begin{cases}\displaystyle\sum_{\mu=1}^{r}L^{r-\mu}(\nabla_X L)L^{\mu-1}Y,\ \text{if } r\geqq 1,\\ \displaystyle-\sum_{\mu=1}^{-r}L^{r+\mu-1}(\nabla_X L)L^{-\mu}Y,\ \text{if } r\leqq -1.\end{cases}$$

The rest of the proof is an immediate consequence of (7) and (8).

Lemma 3. *For any integers r and p:*
$\mathrm{Cod}(I_r, I_{r+p})$ *if and only fi* $\mathrm{Cod}(I_{r+2p}, I_{r+p})$.

Proof. α) We assume $\mathrm{Cod}(I_r, I_{r+p})$. Substituting Y with $L^{-p}Y$ in (ii) of Lemma 2 and using (1) we get.

$$L^p((\overset{r+2p}{\nabla}_X L^{-p})Y+L^{-p}\overset{r+2p}{\nabla}_X Y-(\overset{r}{\nabla}_X L^{-p})Y-L^{-p}\overset{r}{\nabla}_X Y)=(\overset{r}{\nabla}_X L^p)L^{-p}Y$$

or

$$L^p(\overset{r+2p}{\nabla}_X L^{-p})Y+(\overset{r+2p}{\nabla}_X Y-\overset{r}{\nabla}_X Y)-L^p(\overset{r}{\nabla}_X L^{-p})Y=(\overset{r}{\nabla}_X L^p)L^{-p}Y.$$

So using (ii) and $(\overset{r}{\nabla}_X L^p)L^{-p}Y=-L^p(\overset{r}{\nabla}_X L^{-p})Y$ we obtain

$$L^p(\overset{r+2p}{\nabla}_X L^{-p})Y=-L^{-p}(\overset{r}{\nabla}_X L^p)Y.$$

Because

$$(\overset{r}{\nabla}_X L^p)Y=(\overset{r}{\nabla}_Y L^p)X$$

we get

$$(\overset{r+2p}{\nabla}_X L^{-r})Y=(\overset{r+2p}{\nabla}_Y L^{-r})X.$$

Consequently the pair $(I_{r+2p},\ I_{r+2p})$ is Codazzi.

β) We now assume $\mathrm{Cod}(I_{r+2p},\ I_{r+p})$. Writing $(r+2p,\ r+p)=(r+2p,\ r+2p-p)$ and using (α) we have that the pair $(I_{r+2p+2(-p)},\ I_{r+2p-p})=(I_r,\ I_{r+p})$ is Codazzi. This completes the proof of the lemma.

Remark 2. α) According to Lemma 3, $\mathrm{Cod}(I_1,\ I_2)$ if and only if $\mathrm{Cod}\,(I_3,\ I_2)$. So if we put $A_r=I_{4-r}$, then $\mathrm{Cod}\,(I_1,\ I_2)$ iff $\mathrm{Cod}\,(A_1,\ A_2)$, and the following diagram is valid

$$\begin{array}{ccc} \mathrm{Cod}\,(I_r,\ I_s) & \Leftrightarrow & \mathrm{Cod}\,(I_{2s-r},\ I_s) \\ \Updownarrow & & \Updownarrow \\ \mathrm{Cod}\,(A_{4-r},\ A_{4-s}) & \Leftrightarrow & \mathrm{Cod}\,(A_{r-2s+4},\ A_{4-s}). \end{array}$$

We denote that the pair $(A_1,\ A_2)$ defines the linear transformation field L^{-1}.

β) Lemma 3 is known for $n=2$ [1, *Lemma* 2].

γ)(i) and (ii) of Lemma 2 and Lemma 3 are also known for $r=p=1$. For this and for the duality which is established between L and L^{-1}, see for example [6].

Proof of Theorem 1. If $I_2=cI_1$, then $L=cId$ and $\overset{r}{\nabla}L_s\equiv 0$, for any integers r and s. So the pair $(I_r,\ I_{r+s})$ is Codazzi.

We now suppose $\mathrm{Cod}(I_1,\ I_2)$ and $\mathrm{Cod}(I_r,\ I_s)$. To prove that $I_2=cI_1$ we distinguish the following cases: α) $0\leqq r<s$, (β) $0\leqq s<r$, (γ) $s<0<r$, (δ) $r<0<s$, (ε) $r<s\leqq 0$, and (ζ) $s<r\leqq 0$.

Case α. Let $0\leqq r<s$ and so $q=s-r>0$.

We assume that theorem is not valid, that is there exists a point $P\in M$ such that $I_2(P)\neq cI_1(P)$. Then we have $I_2\neq cI_1$ in a neighbourhood U of P. It is well known [5, Theorem 2] that there exists an open, dense subset A of M where the eigenvalues $\lambda_1,\ \ldots,\ \lambda_n$ of L are smooth. We choose, in a neighbourhood W of $U\cap A$, a basis of smooth vector fields $X_1,\ \ldots,\ X_n$ (eigenvectors of L) which simultaneously orthonormalizes I_1 and orthogonalizes I_2. Since $\mathrm{Cod}(I_1,\ I_2)$, using (iii) of Lemma 2 we calculate

$$(\overset{r}{\nabla}_{X_i}L^q)X_j=\overset{r}{\nabla}_{X_i}(L^qX_j)-L^q\overset{r}{\nabla}_{X_i}X_j=\nabla_{X_i}(L^qX_j)-L^q\overset{r}{\nabla}_{X_i}X_j+$$
$$+\frac{1}{2}\sum_{\mu=1}^{r-1}\{L^{-\mu}(\nabla_{X_i}L)L^{q+\mu-1}X_j+L^{-\mu}(\nabla_{L^qX_i}L)\,L^{\mu-1}X_i-L^{1-r}(\nabla_{L^{\mu-1}X_i}L)L^{q+r-\mu-1}X_j\}.$$

But, using (1) and (8) we obtain

$$\nabla_{X_i}(L^qX_j)-L^q\overset{r}{\nabla}_{X_i}X_j=(\nabla_{X_i}L^q)X_j+L^q\nabla_{X_i}X_j-L^q\overset{r}{\nabla}_{X_i}X_j=$$
$$=\sum_{\mu=1}^{q}L^{q-\mu}(\nabla_{X_i}L)L^{\mu-1}X_j+L^q(\nabla_{X_i}X_j-\overset{r}{\nabla}_{X_i}X_j).$$

So

$$(9)\qquad (\overset{r}{\nabla}_{X_i}L^q)X_j=\sum_{\mu=1}^{q}L^{q-\mu}(\nabla_XL)L^{\mu-1}X_j+L^q(\nabla_{X_i}X_j-\overset{r}{\nabla}_{X_i}X_j)+$$
$$+\frac{1}{2}\sum_{\mu=1}^{r-1}\{L^{-\mu}(\nabla_{X_i}L)L^{q+\mu-1}X_j+L^{-\mu}(\nabla_{L^qX_j}L)L^{\mu-1}X_i-L^{1-r}(\nabla_{L^{\mu-1}X_i}L)L^{q+r-1}X_j)$$

$\left(\text{Especially for } r=1,\ (9) \text{ is}: (\nabla_{X_i}L^q)X_j=\sum_{\mu=1}^{q}L^{q-\mu}(\nabla_{X_i}L)L^{\mu-1}X_j\right).$

Because $\mathrm{Cod}(I_1,\ I_2)$ and $\mathrm{Cod}(I_r,\ I_s)$ that is

$$(\nabla_{X_i}L)X_j=(\nabla_{X_j}L)X_i \quad\text{and}\quad (\overset{r}{\nabla}_{X_i}L^q)X_j=(\overset{r}{\nabla}_{X_j}L^q)X_i,$$

we easily obtain, using (9)

$$(10)\qquad \Lambda_{ij}(L^{1-r}(\nabla_{X_i}L)X_j)=0$$

where Λ_{ij} is the linear transformation

$$(11)\qquad \Lambda_{ij}=\sum_{\mu=1}^{q}(\lambda_j^{\mu-1}-\lambda_i^{\mu-1})L^{S-\mu-1}+\frac{1}{2}\sum_{\mu=1}^{r-1}\{(\lambda_j^q-\lambda_i^q)(\lambda_i^{\mu-1}+\lambda_j^{\mu-1})\,L^{r-\mu-1}+$$
$$+(\lambda_i^{S-\mu-1}\lambda_j^{\mu-1}-\lambda_i^{\mu-1}\ \lambda_j^{S-\mu-1})Id\}.$$

We define the functions Λ_{ij}^{ρ} and h_{ij}^{ρ} on W, by the relations $\Lambda_{ij}X_\rho=\Lambda_{ij}^{\rho}X_\rho$ ($\rho=1,\ \ldots,\ n$) and $(\nabla_{X_i}L)X_j=\sum_{\rho=1}^{n}h_{ij}^{\rho}X_\rho$. The functions h_{ij}^{ρ} are symmetric with respect to $i,\ j,\ p$ and

$$\Lambda_{ij}^{\rho}=\sum_{\mu=1}^{q}(\lambda_j^{\mu-1}-\lambda_j^{\mu-1})\lambda_\rho^{S-\mu-1}+\frac{1}{2}\sum_{\mu=1}^{r-1}\{(\lambda_j^q-\lambda_i^q)(\lambda_i^{\mu-1}+\lambda_j^{\mu-1})\lambda_\rho^{r-\mu-1}+$$
$$+(\lambda_i^{s-\mu-1}\lambda_j^{\mu-1}-\lambda_i^{\rho-1}\lambda_j^{S-\mu-1}\}.$$

So, (10) can be written

$$\sum_{\rho=1}^{n}h_{ii}^{\rho}\lambda_\rho^{1-r}\Lambda_{ii}^{\rho}X_\rho=0.$$

But, because $X_i (i=1,\ \ldots,\ n)$ are linearly independent and $\lambda_i>0$ we have0

$$(12)\qquad h_{ij}^{\rho}\Lambda_{ij}^{\rho}=0 \text{ for any } i,\ j,\ \rho=1,\ \ldots,\ n\ (i\neq j).$$

Now, let $0<\lambda_1\leqq\lambda_2\leqq\ldots\leqq\lambda_n$. We distinguish two cases: α_1) $\lambda_i\neq\lambda_j$ for all $i\neq j$ and α_2) at least two of the λ_i (but not all) are equal.

α_1) Let $\lambda_i\neq\lambda_j$ for all $i\neq j$. If $i<j$, then for all $\rho\geqq i$ we have

$$\Lambda_{ij}^{\rho}\geqq\sum_{\mu=1}^{q}(\lambda_j^{\mu-1}-\lambda_i^{\mu-1})\lambda_\rho^{s-\mu-1}+\frac{1}{2}\sum_{\mu=1}^{r-1}\{(\lambda_j^q-\lambda_i^q)(\lambda_i^{\mu-1}+\lambda_j^{\mu-1})\lambda_i^{r-\mu-1}+$$
$$+\lambda_i^{s-\mu-1}\lambda_j^{\mu-1}-\lambda_i^{\mu-1}\lambda_j^{s-\mu-1}\}=\sum_{\mu=1}^{q}(\lambda_j^{\mu-1}-\lambda_i^{\mu-1})\lambda_\rho^{s-\mu-1}+\frac{1}{2}\sum_{\mu=1}^{r+1}\lambda_i^{r-2}(\lambda_j^q-\lambda_i^q)>0.$$

(If $i>j$, then for all $\rho\geqq j$ we find $\Lambda_{ij}^{\rho}<0$).

So, because the functions h_{ij}^{ρ} are symmetric with respect to $i,\ j,\ \rho$ we get, from (12), $h_{ij}^{\rho}=0$ for all $i,\ j,\ \rho$ with $i\neq j$.

Hence $(\nabla_{X_i}L)X_j=0$, for all $i\neq j$. But, for any k, we have

$$0=<(\nabla_{X_i}L)X_j,\ X_k>=<(\nabla_{X_k}L)X_i,\ X_j>=<(X_k\lambda_i)X_i+$$
$$+\lambda_i\nabla_{X_k}X_i-\lambda_j\nabla_{X_k}X_i,\ X_j>.$$

So

(13) $$(\lambda_i-\lambda_j)<\nabla_{X_k}X_i,\ X_j>=0 \text{ for all } k \text{ and } i\neq j$$

and thus $<\nabla_{X_k}X_i,\ X_j>=0$ for all k and $i\neq j$. From this, we deduce that the sectional curvature K_{ij} of $(M,\ I_1)$ with respect to the plane $X_i,\ X_j$ is zero. In fact,

$$K_{ij}=<\nabla_{X_i}\nabla_{X_j}X_j,\ X_i>-<\nabla_{X_j}\nabla_{X_i}X_j,\ X_i>-<\nabla_{[X_i,\ X_j]}X_j,\ X_i>=$$
$$=X_i<\nabla_{X_j}X_j,\ X_i>-<\nabla_{X_j}X_j,\ \nabla_{X_i}X_i>-X_j<\nabla_{X_i}X_j,\ X_i>+$$
$$+<\nabla_{X_i}X_j,\ \nabla_{X_j}X_i>=-<\nabla_{X_j}X_i,\ \nabla_{X_i}X_i>+<\nabla_{X_i}X_j,\ \nabla_{X_j}X_i>=$$
$$=\sum_k\{-<\nabla_{X_j}X_j,\ X_k><\nabla_{X_i}X_i,\ X_k>+<\nabla_{X_i}X_j,\ X_k><\nabla_{X_j}X_i,\ X_k>\}=0.$$

So, we have a contradiction.

α_2) We now assume that $\lambda_1=\ldots=\lambda_m$ for $1<m<n$ and $\lambda_i\neq\lambda_j$ for $1\leqq i\leqq m$ and $m+1\leqq j\leqq n$. For a such pair $(i,\ j)$ we also get $(\nabla_{X_i}L)X_j=0$; so from (13) we take $<\nabla_{X_k}X_i,\ X_j>=0$, for k, $1\leqq i\leqq m$, $m+1\leqq j\leqq n$. So, we also get $K_{ij}=0$, wich is a contradiction. Hence the assumption that M contains a point P such that $I_2(P)\neq cI_1(P)$ gives a contradiction. This yields $\lambda_1=\ldots=\lambda_n=\lambda$ on M which means $L=\lambda\mathrm{Id}$ or equivalently $I_2=\lambda I_1$. But, by Lemma 1, λ must be a constant.

Case β. Let $0\leqq s<r$ and so $q=s-r<0$. The procedure of the proof is similar with the proof of the case α. We only denote that the expression of Λ_{ij} in this case is

$$\Lambda_{ij}=-\sum_{\mu=1}^{-q}(\lambda_j^{-\mu}-\lambda_i^{-\mu})L^{s+\mu-1}+\frac{1}{2}\sum_{\mu=1}^{r-1}\{(\lambda_i^q+\lambda_j^q)(\lambda_j^{\mu-1}-\lambda_i^{\mu-1})L^{r-\mu-1}+$$
$$+(\lambda_i^{s-\mu-1}\lambda_j^{\mu-1}-\lambda_i^{\mu-1}\lambda_j^{s-\mu-1})\mathrm{Id}\}$$

Case γ. Let $s<0<r$ and so $q=s-r<0$. According to diagram of the Remark 2, the assumption $\mathrm{Cod}(I_1,\ I_2)$ and $\mathrm{Cod}\ (I_r,\ I_s)$ is equivalent with the assumption $\mathrm{Cod}(A_1,\ A_2)$ and $\mathrm{Cod}(A_{r-2s+4},\ A_{4-s})$. But $r'=r-2s+4>0$ and $s'=4-s<r'$. So this case is essentially the case β.

Analogously, the proofs of the cases (δ), (ε), (ζ) are refered to the proofs of (β), (β) and (α) respectively. This completes the proof of Theorem 1.

Proof of Theorem 2. Supposing that the theorem is not valid, we choose the fields X_i as in Theorem 1. Because $(\nabla_{X_i}L)X_j=0$, we immediately get (13). So, distinguishing two cases for λ_i, as in Theorem 1, and working similarly we also get $K_{ij}=0$. This contradiction and Lemma 1 complete the proof of the theorem.

4. Applications. Two immediate consequences of Theorem 1 are presented in the following corollaries.

Corollary 1. *Let M be an n-dimensional connected, orientable Riemannian manifold with metric tensor I_1 and non zero sectional curvature. Let $f: M\to N$ be an isometric immersion of M into an $(n+1)$-dimensional space form such that the second fundamental form I_2 defines a metric on M. If $\mathrm{Cod}(I_r,\ I_s)$ for some pair $(r,\ s)\neq(1,\ 2)$ and $(3,\ 2)$, then f is totally umbilical.*

Let $(M,\ I_1)$ be an n-dimensional Riemannian manifold and L be the field of symmetric endomorphisms corresponding to the Ricci tensor $Ric=I_2$, that is,

$$I_1(LX,\ Y)=I_2(X,\ Y)=\sum_{i=1}^n I_1(R(X_i,\ X)Y,\ X_i)$$

where R is the curvature tensor of $(M,\ I_1)$ and X_i $(i=1,\ \ldots,\ n)$ orthonormal vector fields. $I_2(X,\ X)$ is called Ricci curvature of M with respect to X. When $I_2(X,\ X)>0$ for any $X\neq0$, then I_2 defines a metric on M.

Corollary 2. *Let $(M,\ I_1)$ be an n-dimensional Riemannian manifold whose Ricci curvature is positive everywhere. If $\mathrm{Cod}\ (I_1,\ I_2)$ and $\mathrm{Cod}(I_r,\ I_s)$ for some $(r,\ s)\neq(1,\ 2)$ and $(3,\ 2)$ then M is an Eistein manifold.*

Remark 3. Corollary 1 is known for $n=2$ [3]. Also Corollary 2 is known for $(r,\ s)=(2,\ 1)$ [2].

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