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ON $\overset{P}{D}$-CONNECTIONS ON TM

BY

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Introduction. The notions used in ours paper are those given by R. Miron ([1]) and by R. Miron and M. Anastasiei ([2]). Let $\xi=(E, p, M)$ be a vector bundle with the base manifold M of dimension n, the type fibre $\mathbf{F}=\mathbf{R}^m$, N a non-linear connection on the total space E of ξ coresponding to an horizontal distribution $H: u\in E\to H_uE$, such that $E_u=H_uE\oplus V_uE$, where $V: u\in E\to V_uE$ is the vertical distribution, and h, v are the horizontal projector and the vertical projector, respectively. The direct descomposition $E_u=H_uE\oplus V_uE$ implies $X_u=hX_u+vX_u (X_u\in E_u)$ and a vector field $X\in\mathscr{X}(E)$ can be written uniquely in the form $X=hX+vX$.

The geometry of the total space E of ξ is now studied with intens interest, first of all, because of its applications in several branches of mathematics and of theoretical peysics. The monograph [2] written by R. Miron and M. Anastasiei is enlightened in this sense.

A linear conection D on E which preserves H and V, by parallelism, is called, by R. Miron, a d-linear connection on E.

The local coefficients of D are d-geometrical object fields, in a adapted basis, with a simple transformation law ([2]).

By using Miron's d-linear connection in [3] are obtained some results. In the special case of tangent bundle $E=TM$ and $\mathbf{F}=\mathbf{R}^n$, we consider a general isomorphism $\varphi: V\to H$ $(\varphi: vX\to\varphi(vX)=hX)$.
In a local adapted basis $\{\delta/\delta x^i, \partial/\partial y^i\}$ $i=1 \dots n$ ([2]) it is posible to consider $\varphi\equiv\psi$, where $\psi: \partial/\partial y^i\to\psi(\partial/\partial y^i)=\delta/\delta x^i$; $i=1 \dots n$. In Section 1, the $\overset{P}{D}$-connections are definied and studied. In Section 2 these are used in order to derive the most general transformations of normal connections.

1. $\overset{P}{\mathbf{D}}$-connections on the total space E=TM.

Definition 1.1. A linear connection D on TM is called almost normal connection if

$$\varphi(vD_{hX}vY)=hD_{hX}\varphi(vY)\ ;\ \varphi(vD_{vX}vY)=hD_{vX}\varphi(vY)\ ;\ \forall X, Y\in\mathscr{X}(TM). \tag{1.1}$$

Definition 1.2. *A d-linear connection D on TM is called a normal connection if D is an almost normal connection.*

Since $hD_XvY=0$ and $vD_XhY=0$, $\forall X, Y\in\mathscr{X}(TM)$, if D is a d-linear connection, the relations (1.1) are given by

$$\varphi(D_{hX}vY)=hD_{hX}\varphi(vY)\ ;\ \varphi(D_{vX}vY)=D_{vX}\Phi(vY)\ ;\ \forall X, Y \tag{1.2}$$

if D is a normal connection.

Definition 1.3. *Let D be a normal connection on TM and T the torsion of D,* $T(X, Y)=D_XY-D_YX-[X, Y]$.

A $\overset{P}{D}$-*connection on* TM *is defined by*

(1.3) $$\overset{P}{D}_X Y = D_X Y - \varphi(vT(hX, \varphi^{-1}(hY)) - vT(vX, hY) - hT(hX, hY) - \varphi^{-1}(hT(vX, \varphi(vY)) \quad \forall X, Y \in \mathcal{X}(TM).$$

Theorem 1.1. *A* $\overset{P}{D}$-*connection on* TM *is a normal connection.*

Proof. From (1.3) we have: $h\overset{P}{D}_X vY = 0$; $v\overset{P}{D}_X hY = 0$ and

(1.4) $$\varphi(v\overset{P}{D}_{hX} vY) = \varphi(vD_{hX}vY) - \varphi(vT(hX, vY)); \quad h\overset{P}{D}_{hX}\varphi(vY) = hD_{hX}\varphi(vY) - \varphi(vT(hX, vY));$$

(1.5) $$\varphi(v\overset{P}{D}_{vX} vY) = \varphi(vD_{vX}vY) - hT(vX, \varphi(vY)); \quad h\overset{P}{D}_{vX}\varphi(vY) = hD_{vX}\varphi(vY) - hT(vX, \varphi(vY)).$$

Since D is a normal connection, from (1.4) and (1.5) we obtain

(1.6) $$\varphi(v\overset{P}{D}_{hX}vY) = h\overset{P}{D}_{hX}\varphi(vY); \quad \varphi(v\overset{P}{D}_{vX}vY) = h\overset{P}{D}_{vX}\varphi(vY); \quad \forall X, Y \in \mathcal{X}(TM)$$

and $\overset{P}{D}$ is an almost normal connection. Since $\overset{P}{D}$ is a d-linear connection, it is a normal connection on TM.

The v-torsion τ of non-linear connection N is given by ([3] [4])

(1.7) $$\tau(U, Z) = \varphi^{-1}(h[\varphi(vU), \varphi(vZ)]) - v[\varphi(vU), vZ] + v[\varphi(vZ), vU]; \quad \forall U, \quad \forall U, Z \in \mathcal{X}(TM),$$

and the h-torsion $\tilde{N}$ of non-linear connection N is given by

(1.8) $$\tilde{N}(U, Z) = h[vU, \varphi(vZ)] - h[vZ, \varphi(vU)] - \varphi(v[vU, vZ]); \ \forall U, Z \in \mathcal{X}(TM).$$

If the d-tensors $\underset{1}{\mathcal{T}}$ and $\underset{2}{\mathcal{T}}$, are defined by ([3] [4])

(1.9) $$\underset{1}{\mathcal{T}}(X, Y) = hT(hX, hY) - \varphi(vT(hX, \varphi^{-1}(hY)) + \varphi(vT(hY, \varphi^{-1}(hX))$$

(1.10) $$\underset{2}{\mathcal{T}}(X, Y) = \varphi(vT(vX, vY)) - hT(vX, \varphi(vY)) + hT(vY, \varphi(vX)) \quad \forall X, Y \in \mathcal{X}(TM),$$

since a normal connection D is a particular case of a semi-normal connection ([3] [4]), we obtain

Proposition 1.1. *For every normal connection* D *with the torsion* T, *we have*

(1.11) $$\underset{1}{\mathcal{T}}(X, Y) = -\varphi(\tau(\varphi^{-1}(hX), \varphi^{-1}(hY)); \quad \underset{2}{\mathcal{T}}(X, Y) = \tilde{N}(X, Y); \quad \forall X, Y \in \mathcal{X}(TM).$$

From (1.3) and (1.11) we obtain

Theorem 1.2. *The torsion* $\overset{P}{T}$ *of a* $\overset{P}{D}$-*connection is given by*

(1.12) $$\overset{P}{T}(X, Y) = -\varphi(\tau(\varphi^{-1}(hX), \varphi^{-1}(hY)) + \varphi^{-1}(\tilde{N}(X, Y)) + \Omega(X, Y) \quad \forall X, Y \in \mathcal{X}(TM),$$

where Ω *is the curvature of the non-linear connection* N.

The torsion $\overset{P}{T}$ *has the intrinsec character for a fixed non-linear connection* N.

2. Transformations. The general linear connections transformation $\bar{\tau}: D \to \bar{D}$ is given by

(2.1) $$\bar{D}_X Y = D_X Y + \bar{\tau}(X, Y) \quad \forall X, Y \in \mathcal{X}(TM)$$

where $\bar{\tau}$ is an arbitrary tensor field of type (1, 2) on TM.

Theorem 2.1. *The transformation (2.1) preserves the class of almost normal connections, if and only if* $\bar{\tau}$ *is a solution of the equations*

(2.2) $$\varphi(v\bar{\tau}(hX, vY)) = h\bar{\tau}(hX, \varphi(vY)); \quad \varphi(v\bar{\tau}(vX, vY)) = h\bar{\tau}(vX, \varphi(vY)) \quad \forall X, Y \in \mathcal{X}(TM).$$

Proof. From (2.1) we obtain:

(2.3) $$\varphi(v\bar{D}_{hX}vY) = \varphi(vD_{hX}vY) + \varphi(v\bar{\tau}(hX, vY)); \quad h\bar{D}_{hX}\varphi(vY) = hD_{hX}\varphi(vY) + h\bar{\tau}(hX, \varphi(vY))$$

(2.4) $$\varphi(v\bar{D}_{vX}vY) = \varphi(vD_{vX}vY) + \varphi(v\bar{\tau}(vX, vY)); \quad h\bar{D}_{vX}\varphi(vY) = hD_{vX}\varphi(vY) + h\bar{\tau}(vX, \varphi(vY)) \quad \forall X, Y \in \mathcal{X}(TM).$$

From (1.1), (2.3), 2.4) we obtain (2.2), if D and $\bar{D}$ are almost normal connections on TM. Conversely, from (2.2), (2.3) and (2.4) we have

$$\varphi(v\bar{D}_{hX}vY) = h\bar{D}_{hX}\varphi(vY); \quad \varphi(v\bar{D}_{vX}vY) = h\bar{D}_{vX}\varphi(vY), \quad \forall X, Y$$

if D is a fixed almost normal connection.

From (2.2) and from the definition of normal connections we have:

Theorem 2.2. *The trnasformations (2.1), preserves the class of normal connection on* TM, *if and only if,* $\bar{\tau}$ *is a solution of tensorial system equations (2.2) and*

(2.5) $$h\bar{\tau}(X, vY) = 0; \quad v\bar{\tau}(X, hY) = 0 \quad \forall X, Y \in \mathcal{X}(TM).$$

The purpose of the present section is to determine the general solution of system (2.2), (2.5).

If $\bar{\tau}: D \to \bar{D}$ is a general transformation of normal connections then if $\overset{P}{\bar{D}}$ is associated to $\bar{D}$ and $\overset{P}{D}$ is associated to D, we have $\overset{P}{\bar{T}} = \overset{P}{T}$. Since $\overset{P}{\bar{T}} = \overset{P}{T}$ and $\overset{P}{\bar{D}}$, $\overset{P}{D}$ are two normal connections, we obtain.

Proposition 2.1. *For every two normal connections* D *and* $\bar{D}$, *we have* $\overset{P}{\bar{D}} = \overset{P}{D}$.

From the Proposition 2.1, we obtain

Theorem 2.3. *For every two normal connection* D *and* $\bar{D}$ *we have an invariant*

$$\bar{D}_X Y - \varphi(v\bar{T}(hX, \varphi^{-1}(hY)) - v\bar{T}(hX, vY) - h\bar{T}(vX, hY) - \varphi^{-1}(h\bar{T}(vX, \varphi(vY)) = \tag{2.6}$$
$$= D_X Y - \varphi(vT(hX, \varphi^{-1}(hY)) - vT(hX, vY) - hT(vX, hY) -$$
$$- \varphi^{-1}(hT(vX, \varphi(vY)). \quad \forall X, Y \in \mathcal{X}(TM).$$

Let D be a fixed normal connection and $\bar{D}$ an arbitrary normal connection. From (2.6) we obtain

Theorem 2.4. *The general solution of system (2.2) (2.5) is given by*

$$\tau(X, Y) = \varphi(v\underset{1}{\mathcal{B}}(hX, \varphi^{-1}(hY)) + v\underset{1}{\mathcal{B}}(hX, vY) + h\underset{2}{\mathcal{B}}(vX, hY) + \tag{2.7}$$
$$+ \varphi^{-1}(h\underset{2}{\mathcal{B}}(vX, \varphi(vY)) \quad \forall X, Y \in \mathcal{X}(TM),$$

where $\underset{1}{\mathcal{B}}$ and $\underset{2}{\mathcal{B}}$ are arbitrary d-tensor fields.

Also, we observe that if, $\underset{1}{\mathcal{U}}$ and $\underset{2}{\mathcal{U}}$ are defined by

$$\underset{1}{\mathcal{U}}(X, Y) = v\underset{1}{\mathcal{B}}(hX, vY); \quad \underset{2}{\mathcal{U}}(X, Y) = h\underset{2}{\mathcal{B}}(vY, hX) \quad \forall X, Y \in \mathcal{X}(TM), \tag{2.8}$$

we obtain

$$\tau(X, Y) = \varphi(v\underset{1}{\mathcal{U}}(X, \varphi^{-1}(hY)) + \underset{1}{\mathcal{U}}(X, Y) + \underset{2}{\mathcal{U}}(Y, X) + \varphi^{-1}(\underset{2}{\mathcal{U}}(\varphi(vY), X). \tag{2.9}$$

We have

Theorem 2.5. *The group $G(\varphi)$ of the general transformations which preserve the class of normal connections, on TM, is given by the transformations (2.1), where τ is given by (2.7) or (2.9).*

Theorem 2.6. *The group $\underset{p}{G}(\varphi)$ admit the invariants $\underset{1}{\mathcal{T}}$ and $\underset{2}{\mathcal{T}}$.*

Since a D connection on TM, is the unique normal connection with the intrinsec torsion T given by (1.12) we obtain.

Theorem 2.7. *The group $G(\varphi)$ is generated by $\overset{p}{D}$.*

We observe that locally, in adapted basis $\{\delta/\delta x^i, \partial/\partial y^i\}$ the invariant $\tilde{\mathcal{T}} = \mathcal{T}$ is trivial, since $\mathcal{T} = \tilde{N}$ and we have $\tilde{N} = 0$.

In another note we shall give a generalisation of this to almost seminormal connection on TM.

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A NOTE ON THE PROJECTIVE CURVATURE TENSOR OF A SEMISYMMETRIC RECURRENT METRIC CONNECTION ON A RIEMANNIAN MANIFOLD

BY

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0. Introduction. Let M be an n-dimensional Riemannian manifold with Riemannian metric g and L.C. connection ∇.

An affine connection $\bar{\nabla}$ on a Riemannian manifold is called a recurrent metric connection [1] if there exists a differentiable oneform μ on M such that

$$(\bar{\nabla}_X g)(Y, Z) = \mu(X) g(Y, Z) \tag{0.1}$$

holds for all differentiable vectorfields $X, Y, Z, \ldots$ on M. μ is called the one-form of recurrence. If, further, the torsion tensor $T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y]$ of the connection $\bar{\nabla}$ is of the form

$$T(X, Y) = \pi(y) X - \pi(X) Y \tag{0.2}$$

where π is a differentiable one-form on M, then $\bar{\nabla}$ is called a semisymmetric recurrent metric connection [2] on M.

Ray and Barua [2] have shown that such a connection is of the form

$$\bar{\nabla}_X Y = \nabla_X Y + \alpha(Y) X - g(X, Y) A - \frac{1}{2}\mu(X) Y \tag{0.3}$$

where α is a differentiable one-form given by

$$\alpha(X) = \pi(X) - \frac{1}{2}\mu(X) \tag{0.4}$$

and A is a differentiable vectorfield satisfying

$$g(A, X) = \alpha(X) \tag{0.5}$$

In the present note we have considered M to be a Reimannian manifold admitting a semisymmetric recurrent metric connection.

The aim of the present note is to see under what conditions the projective curvature tensors of the two connections are equal.

1. Projective curvature tensors. Let $\bar{R}(X, Y)Z$ and $\bar{S}(Y, Z)$ denote the curvature tensor and the Ricci tensor of the connection $\bar{\nabla}$. From (0.3) we can see that if $R(X, Y)Z$ is the Riemann curvature tensor then