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FINSLERIAN QUASI-CONNECTIONS

BY

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1. Introduction. Let M be an n-dimensional differentiable manifold of class C^∞ and let (x^i, y^i) be the canonical coordinates of a point $y \in T(M)$, where $T(M)$ is the tangent bundle of M [3] and let π be the canonical projection. The natural basis of $T(M)$ with respect to canonical coordinates is $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right)$ and the mapping $N : y \in T(M) \to N_y \subset T(M)_y$ is a non-linear connection on $T(M)$, such that: $T(M)_y = N_y \oplus T(M)_y^v$. Let $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^k \frac{\partial}{\partial y^k}$ be a local basis of the n-dimensional local distribution N, where $N_k^i(x, y)$ are called the coefficients of the non-linear connection N. The notions and the notations of [2] and [3] are used. See also [1, 6].

Let $F\Gamma = (N, F, C)$ be a Finsler connection with the coefficients $(N_j^i, F_{jk}^i, C_{jk}^i)$. We consider the h- and v-Finsler torsion tensors

$$T_{jk}^i = F_{jk}^i - F_{kj}^i, \quad S_{jk}^i = C_{jk}^i - C_{kj}^i. \tag{1.1}$$

In general, for a Finsler connection $F\Gamma = (N, F, C)$ we have the following three curvature tensors [3]:

$$\begin{aligned} S_{jkh}^i &= \frac{\partial C_{jk}^i}{\partial y^h} - \frac{\partial C_{jh}^i}{\partial y^k} + C_{jk}^r C_{rh}^i - C_{jh}^r C_{jk}^i, \\ R_{jkh}^i &= \frac{\delta F_{jk}^i}{\delta x^h} - \frac{\delta F_{jh}^i}{\delta x^k} + F_{jk}^r F_{rh}^i - F_{jh}^r F_{rk}^i + C_{jr}^i R_{kh}^r, \\ P_{jkh}^i &= \frac{\partial F_{jk}^i}{\partial y^h} - C_{jh|k}^i + C_{jr}^i P_{kh}^r, \end{aligned} \tag{1.2}$$

where

$$R_{jk}^i = \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j}, \quad P_{jk}^i = \frac{\partial N_j^i}{\partial y^k} - F_{kj}^i. \tag{1.3}$$

We define also the Finsler tensor

$$K_{jkh}^i \overset{\text{def}}{=} R_{jkh}^i - C_{jr}^i R_{kh}^r. \tag{1.4}$$

R_{kh}^r is the curvature tensor of the non-linear connection N_j^i.

If the non-linear connection N is integrable, then $K_{jkh}^i = R_{jkh}^i$ with respect to a special coordinate system.

Let a nondegenerate tensor field Φ^i_j of type (1, 1) be defined on M. Since the tensor Φ^i_j is nondegenerate, there exists a (1, 1) — type tensor field J^i_j reciprocal to Φ^i_j, that is, $\Phi^i_j J^j_k = \delta^i_k$, where δ is the Kronecker symbol.

Following [8], [9] we consider the notion of Finsler quasi-connection on M. If we consider a Finsler quasi connection $F\mathring{\Gamma}=(\mathring{N}, \mathring{F}, \mathring{C})$ over M, every other Finsler quasi-connections $F\Gamma=(N, F, C)$ on M will be given by the formula

$$(1.5)\qquad N^i_j=\mathring{N}^i_j,\quad F^i_{jk}=\mathring{F}^i_{jk}+X^i_{jk},\quad C^i_{jk}=\mathring{C}^i_{jk}+Y^i_{jk},$$

where X^i_{jk} and Y^i_{jk} are arbitrary Finsler tensor fields of tpe (1, 2).

2. The operators Ω and Ω^* on M. Following [5], let us introduce the tensors

$$(2.1)\qquad \Omega^{ih}_{km}=\frac{1}{n}\Phi^i_m J^h_k,\qquad \Omega^{*ih}_{km}=\delta^i_k\delta^h_m-\Omega^{ih}_{km}$$

and consider the operations

$$(2.2)\qquad \begin{aligned} (\Omega V)^i_{mh}&=\Omega^{ij}_{kh}V^k_{mj}, & (\Omega^* V)^i_{mh}&=\Omega^{*ij}_{kh}V^k_{mj},\\ (\Omega H)^{is}_{mh}&=\Omega^{ij}_{qh}H^{qs}_{mj}, & (\Omega^* H)^{is}_{mh}&=\Omega^{*ij}_{qh}H^{qs}_{mj}, \end{aligned}$$

with V and H being Finsler tensor fields of type (1, 2) and (2, 2), respectively. It can readily be seen that the following relations hold

$$(2.3)\qquad \Omega+\Omega^*=I\otimes I,\ \Omega\Omega=\Omega,\ \Omega^*\Omega^*=\Omega^*,\ \Omega\Omega^*=\Omega^*\Omega=0$$

where I is the identity tensor field. Hence Ω and Ω^* are operators of Obata type [4].

3. Family of Finsler Quasi-connections on M. We now consider the Finsler tensor fields U^i_{jk} and V^i_{jk} defined by

$$(3.1)\qquad U^i_{jk}=n\Phi^i_m\Phi^m_{j\overset{\circ}{|}k}-\Phi^i_k\Phi^m_{j\overset{\circ}{|}m},\quad V^i_{jk}=n\Phi^i_m\Phi^m_{j}\overset{\circ}{|}_k-\Phi^i_k\Phi^m_{j}\overset{\circ}{|}_m,$$

where $\overset{\circ}{|}$ and $\overset{\circ}{|}$ are the h- and v-covariant differentiations with respect to $F\mathring{\Gamma}=(\mathring{N}, \mathring{F}, \mathring{C})$ [8].

It can be verified directly that

$$(3.2)\qquad (\Omega U)^i_{jk}=0,\qquad (\Omega V)^i_{jk}=0.$$

Now, we are interested in finding of the Finsler quasi-connections $F\Gamma=(N, F, C)$ given by (1.5) when the Finsler tensor fields X and Y satisfying the equations

$$(3.3)\qquad (\Omega^* X)^i_{jk}=U^i_{jk},\quad (\Omega^* Y)^i_{jk}=V^i_{jk}.$$

The equation (3.3) admits a solution if and only if $(\Omega U)^i_{jk}=0$ and $(\Omega V)^i_{jk}=0$ [5], and then the general solution of this equation is given by

$$(3.4)\qquad X^i_{jk}=U^i_{jk}+W^i_{jk},\quad Y^i_{jk}=V^i_{jk}+Z^i_{jk},$$

where W and Z are arbitrary Finsler tensor fields of type (1, 2) possessing the properties

$$(3.5)\qquad (\Omega^* W)^i_{jk}=0,\quad (\Omega^* Z)^i_{jk}=0.$$

Hence the general family of Finsler quasi-connections $F\Gamma=(N, F, C)$ is given by

$$(3.6)\qquad \begin{aligned} N^i_j=\mathring{N}^i_j,\quad F^i_{jk}&=\mathring{F}^i_{jk}+n\Phi^i_m\Phi^m_{j\overset{\circ}{|}k}-\Phi^i_k\Phi^m_{j\overset{\circ}{|}m}+W^i_{jk},\\ C^i_{jk}&=\mathring{C}^i_{jk}+n\Phi^i_m\Phi^m_{j}\overset{\circ}{|}_k-\Phi^i_k\Phi^m_{j}\overset{\circ}{|}_m+Z^i_{jk}. \end{aligned}$$

Let $F\Gamma=(N, F, C)$ and $F\bar{\Gamma}=(\bar{N}, \bar{F}, \bar{C})$ be two Finsler quasi-connections of the family (3. 6). For $F\bar{\Gamma}=(\bar{N}, \bar{F}, \bar{C})$ we have

$$(3.7)\qquad \begin{aligned} \bar{N}^i_j=\mathring{N}^i_j,\quad \bar{F}^i_{jk}&=\mathring{F}^i_{jk}+n\Phi^i_m\Phi^m_{j\overset{\circ}{|}k}-\Phi^i_k\Phi^m_{j\overset{\circ}{|}m}+\bar{W}^i_{jk},\\ \bar{C}^i_{jk}&=\mathring{C}^i_{jk}+n\Phi^i_m\Phi^m_{j}\overset{\circ}{|}_k-\Phi^i_k\Phi^m_{j}\overset{\circ}{|}_m+\bar{Z}^i_{jk}. \end{aligned}$$

From (3.6) and (3.7) it follows

$$(3.8)\qquad \bar{N}^i_j=N^i_j,\quad \bar{F}^i_{jk}=F^i_{jk}+A^i_{jk},\quad \bar{C}^i_{jk}=C^i_{jk}+B^i_{jk},$$

where $A^i_{jk}=\bar{W}^i_{jk}-W^i_{jk}$ and $B^i_{jk}=\bar{Z}^i_{jk}-Z^i_{jk}$ are arbitrary Finsler tensor fields which satisfy conditions

$$(3.9)\qquad (\Omega^* A)^i_{jk}=0,\qquad (\Omega^* B)^i_{jk}=0.$$

Now, if we consider the transformation $F\Gamma=(N, F, C)\rightarrow F\bar{\Gamma}=(\bar{N}, \bar{F}, \bar{C})$ defined by (3.8) with A and B satisfying (3.9), the set of these transformations is a group which is isomorphic to the additive group of Finsler tensors A and B.

Next as an interesting particular case, we take

$$(3.10)\qquad A^i_{jk}=\Phi^i_k U_j,\qquad B^i_{jk}=\Phi^i_k V_j,$$

with U_j and V_j are arbitrary Finsler covector fields.

It easily follows that A and B defined by (3.10) obey (3.9), whence

$$(3.11)\qquad \bar{N}^i_j=N^i_j,\quad \bar{F}^i_{jk}=F^i_{jk}+\Phi^i_k U_j,\quad \bar{C}^i_{jk}=C^i_{jk}+\Phi^i_k V_j.$$

The set of these transformations defines the group G.

4. The invariants of the group G. If we denote by $|$ and $|$; $\bar{|}$ and $\bar{|}$ the operator of h- and v- covariant differentiation with respect to $F\Gamma=(N, F, C)$, $F\bar{\Gamma}=(\bar{N}, \bar{F}, \bar{C})$ correspondingly, then, after straight forward calculations we find the following invariants

$$(4.1)\qquad \begin{aligned} \Phi^i_{j\bar{|}k}&=\Phi^i_{j|k}, & \Phi^i_j\bar{|}_k&=\Phi^i_j|_k,\\ J^i_{j\bar{|}k}&=J^i_{j|k}, & J^i_j\bar{|}_k&=J^i_j|_k,\\ \Omega^{ih}_{mj\bar{|}k}&=\Omega^{ih}_{mj|k}, & \Omega^{ih}_{mj}\bar{|}_k&=\Omega^{ih}_{mj}|_k,\\ \Omega^{*ih}_{mj\bar{|}k}&=\Omega^{*ih}_{mj|k}, & \Omega^{*ih}_{mj}\bar{|}_k&=\Omega^{*ih}_{mj}|_k. \end{aligned}$$

Let T^i_{jk}, S^i_{jk}, P^i_{jk} ; $\bar{T}^i_{jk}$, $\bar{S}^i_{jk}$, $\bar{P}^i_{jk}$ be the torsion tensors and K^k_{hij}, S^k_{hij} ; $\bar{K}^k_{hij}$, $\bar{S}^k_{hij}$ be the curvature tensors of $F\Gamma$ and $F\bar{\Gamma}$, respectively.

Then we have

$$(4.2)\qquad \begin{aligned} \bar{T}^i_{jk}&=T^i_{jk}+\Phi^i_k U_j-\Phi^i_j U_k,\\ \bar{S}^i_{jk}&=S^i_{jk}+\Phi^i_k V_j-\Phi^i_j V_k,\quad \bar{P}^i_{jk}=P^i_{jk}-\Phi^i_j p_k\ ; \end{aligned}$$

$$\bar{K}^k_{hij}=K^k_{hij}+(\Phi^k_{h|i})U_j-(\Phi^k_{h|j})U_i+\Phi^k_h(U_{j|i}-U_{i|j}+T^m_{ij}U_m), \tag{4.3}$$
$$\bar{S}^k_{hij}=S^k_{hij}+(\Phi^k_h|_i)V_j-(\Phi^k_h|_j)V_i+\Phi^k_h(V_j|_i-V_i|_j+S^m_{ij}V_m).$$

From (2.2) and (4.2) we obtain

$$(\Omega\bar{T})^i_{kj}=(\Omega T)^i_{kj}+\frac{n-1}{n}\Phi^i_jU_k,$$
$$(\Omega\bar{S})^i_{kj}=(\Omega S)^i_{kj}+\frac{n-1}{n}\Phi^i_jV_k, \tag{4.4}$$
$$(\Omega\bar{P})^i_{kj}=(\Omega P)^i_{kj}-\frac{1}{n}\Phi^i_jp_k.$$

It follows

$$\Phi^i_jU_k=\frac{n}{n-1}[(\Omega\bar{T})^i_{kj}-(\Omega T)^i_{kj}],$$
$$\Phi^i_jV_k=\frac{n}{n-1}[(\Omega\bar{S})^i_{kj}-(\Omega S)^i_{kj}], \tag{4.5}$$
$$\Phi^i_jp_k=-n[(\Omega\bar{P})^i_{kj}-(\Omega P)^i_{kj}].$$

Substituting (4.5) in (4.2) we have

$$\bar{T}^i_{jk}=T^i_{jk}+\frac{n}{n-1}[(\Omega\bar{T})^i_{jk}-(\Omega T)^i_{jk}-(\Omega\bar{T})^i_{kj}+(\Omega T)^i_{kj}],$$

$$\bar{S}^i_{jk}=S^i_{jk}+\frac{n}{n-1}[(\Omega\bar{S})^i_{jk}-(\Omega S)^i_{jk}-(\Omega\bar{S})^i_{kj}+(\Omega S)^i_{kj}],$$

$$\bar{P}^i_{jk}=P^i_{jk}+n[(\Omega\bar{P})^i_{kj}-(\Omega P)^i_{kj}].$$

Thus we have arrived at the invariant tensor fields of the group G given by

$$\underset{1}{I}{}^i_{jk}=T^i_{jk}+\frac{n}{n-1}[(\Omega T)^i_{kj}-(\Omega T)^i_{jk}],$$
$$\underset{2}{I}{}^i_{jk}=S^i_{jk}+\frac{n}{n-1}[(\Omega S)^i_{kj}-(\Omega S)^i_{jk}], \tag{4.6}$$
$$\underset{3}{I}{}^i_{jk}=P^i_{jk}-n(\Omega P)^i_{kj}.$$

From (4.1), (4.2), (4.3) and (4.5) we can obtain, by some easy computations, the formula

$$\bar{K}^k_{hij}=K^k_{hij}+\frac{n}{n-1}[(\Omega\bar{T})^k_{jh\bar{|}i}-(\Omega\bar{T})^k_{ih\bar{|}j}+\bar{T}^m_{ij}(\Omega\bar{T})^k_{mh}]$$
$$-\frac{n}{n-1}[(\Omega T)^k_{jh|i}-(\Omega T)^k_{ih|j}+T^m_{ij}(\Omega T)^k_{mh}], \tag{4.7}$$
$$\bar{S}^k_{hij}=S^k_{hij}+\frac{n}{n-1}[(\Omega\bar{S})^k_{jh}\bar{|}_i-(\Omega\bar{S})^k_{ih}\bar{|}_j+\bar{S}^m_{ij}(\Omega\bar{S})^k_{mh}]$$
$$-\frac{n}{n-1}[(\Omega S)^k_{jh}|_i-(\Omega S)^k_{ih}|_j+S^m_{ij}(\Omega S)^k_{mh}].$$

Thus we find other invariant tensor fields of the group G as follows

$$\underset{4}{I}{}^k_{hij}=K^k_{hij}-\frac{n}{n-1}[(\Omega T)^k_{jh|i}-(\Omega T)^k_{ih|j}+T^m_{ij}(\Omega T)^k_{mh}]$$
$$\underset{5}{I}{}^k_{hij}=S^k_{hij}-\frac{n}{n-1}[(\Omega S)^k_{jh}|_i-(\Omega S)^k_{ih}|_j+S^m_{ij}(\Omega S)^k_{mh}] \tag{4.8}$$

Now, we shall give a geometric interpretation for these invariants. Let $F\overset{*}{\Gamma}=(\overset{*}{N},\overset{*}{F},\overset{*}{C})$ be a Finsler quasi-connection of the group G given by

$$\overset{*}{N}{}^i_j=N^i_j,\quad \overset{*}{F}{}^i_{jk}=F^i_{jk}+\Phi^i_k\overset{*}{U}_j,\quad \overset{*}{C}{}^i_{jk}=C^i_{jk}+\Phi^i_k\overset{*}{V}_j. \tag{4.9}$$

From (4.2) and (4.9) we have

$$\overset{*}{T}{}^i_{jk}=T^i_{jk}+\Phi^i_k\overset{*}{U}_j-\Phi^i_j\overset{*}{U}_k,$$
$$\overset{*}{S}{}^i_{jk}=S^i_{jk}+\Phi^i_k\overset{*}{V}_j-\Phi^i_j\overset{*}{V}_k,\qquad \overset{*}{P}{}^i_{jk}=P^i_{jk}-\Phi^i_j\overset{*}{p}_k. \tag{4.10}$$

We should be interested in obtaining a Finsler quasi-connection $F\overset{*}{\Gamma}=(\overset{*}{N},\overset{*}{F},\overset{*}{C})$ with the Finsler torsion tensor fields $\overset{*}{T}{}^i_{jk}$, $\overset{*}{S}{}^i_{jk}$, $\overset{*}{P}{}^i_{jk}$ such that

$$\overset{*}{T}{}^i_{jk}=\underset{1}{I}{}^i_{jk},\qquad \overset{*}{S}{}^i_{jk}=\underset{2}{I}{}^i_{jk},\qquad \overset{*}{P}{}^i_{jk}=\underset{3}{I}{}^i_{jk}. \tag{4.11}$$

From (4.6), (4.10) and (4.11) we have

$$\Phi^i_k\overset{*}{U}_j-\Phi^i_j\overset{*}{U}_k=\frac{n}{n-1}[(\Omega T)^i_{kj}-(\Omega T)^i_{jk}],$$
$$\Phi^i_k\overset{*}{V}_j-\Phi^i_j\overset{*}{V}_k=\frac{n}{n-1}[(\Omega S)^i_{kj}-(\Omega S)^i_{jk}], \tag{4.12}$$
$$\Phi^i_j\overset{*}{p}_k=n(\Omega P)^i_{kj}.$$

Now, multiplying (4.12) by J^k_m, we obtain

$$\delta^i_m\overset{*}{U}_j-\Phi^i_jJ^k_m\overset{*}{U}_k=\frac{n}{n-1}[(\Omega T)^i_{kj}-(\Omega T)^i_{jk}]J^k_m,$$
$$\delta^i_m\overset{*}{V}_j-\Phi^i_jJ^k_m\overset{*}{V}_k=\frac{n}{n-1}[(\Omega S)^i_{kj}-(\Omega S)^i_{jk}]J^k_m, \tag{4.13}$$
$$\Phi^i_jJ^k_m\overset{*}{p}_k=n(\Omega P)^i_{kj}J^k_m.$$

By contracting i and m in (4.13) we find

$$\overset{*}{U}_j=\frac{n}{(n-1)^2}[(\Omega T)^m_{kj}-(\Omega T)^m_{jk}]J^k_m,$$
$$\overset{*}{V}_j=\frac{n}{(n-1)^2}[(\Omega S)^m_{kj}-(\Omega S)^m_{jk}]J^k_m, \tag{4.14}$$
$$\overset{*}{p}_j=n(\Omega P)^m_{kj}J^k_m.$$

Using (2.1) and (2.2) equation (4.14) can be written as

$$(4.15)\qquad \overset{*}{U}_j = -\frac{1}{n-1} T^m_{jk} J^k_m, \qquad \overset{*}{V}_j = -\frac{1}{n-1} S^m_{jk} J^k_m, \qquad \overset{*}{P}_j = P^m_{jk} J^k_m.$$

Substituting (4.15) in (4.9) we obtain the Finsler quasi-connection $F\overset{*}{\Gamma} = (\overset{*}{N}, \overset{*}{F}, \overset{*}{C})$ given by

$$(4.16)\qquad \begin{aligned} &\overset{*}{N}^i_j = N^i_j, \qquad \overset{*}{F}^i_{jk} = F^i_{jk} - \frac{1}{n-1} \Phi^i_k T^m_{jh} J^h_m, \\ &\overset{*}{C}^i_{jk} = C^i_{jk} - \frac{1}{n-1} \Phi^i_k S^m_{jh} J^h_m. \end{aligned}$$

Let $\overset{*}{K}^k_{hij}$ and $\overset{*}{S}^k_{hij}$ be the Finsler curvature tensor fields of the Finsler quasi-connection $F\overset{*}{\Gamma} = (\overset{*}{N}, \overset{*}{F}, \overset{*}{C})$. From equations (4.3), (4.8), (4.15) and (4.16) we can conclude, by some easy computations, that the Finsler curvature tensor fields of the Finsler quasi-connection $F\overset{*}{\Gamma} = (\overset{*}{N}, \overset{*}{F}, \overset{*}{C})$ is given by

$$(4.17)\qquad \overset{*}{K}^k_{hij} = \underset{4}{I}^k_{hij}, \qquad \overset{*}{S}^k_{hij} = \underset{5}{I}^k_{hij}$$

Thus we have proved the following

Theorem (4.1) *There exists a Finsler quasi-connection $F\overset{*}{\Gamma} = (\overset{*}{N}, \overset{*}{F}, \overset{*}{C})$ of the group G with the Finsler torsion tensor fields $\overset{*}{T}^i_{jk}$, $\overset{*}{S}^i_{jk}$, $\overset{*}{P}^i_{jk}$ and the Finsler curvature tensor fields $\overset{*}{K}^k_{hij}$, $\overset{*}{S}^k_{hij}$ given by (4.11) and (4.17) respectively.*

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RECENZII BOOK REVIEWS

BAMBERG, PAUL and SHILOMO STERNBERG: *A Course in Mathematics for Students of Physics,* Cambridge University Press ,1990, vol. I, XVIII + 406 pag. vol. II, XVIII + 446 (407—852) pag.

Acest amplu curs, în două volume, rezultat al unei experiențe didactice de un deceniu pe care autorii, cunoscuți matematicieni, au desfășurat-o la Harvard University, se adresează studenților în fizică. Parcurgerea conținutului acestui curs presupune o bună cunoaștere a unor elemente de algebră, geometrie analitică plană și calcul diferențial și integral pentru funcții reale de o variabilă reală. În principal, conținutul cursului constă în teoria spațiilor vectoriale, teoria diferențială și integrală a funcțiilor vectoriale de mai multe variabile reale, bazele teoriei calculului diferențial exterior, topologia complexelor unidimensionale și multidimensionale, bazele analizei complexe și metode de evaluare asimptotică a integralelor. Acest aparat matematic este în mod organic folosit pentru a descrie riguros teoria gaussiană a opticii, teoria rețelelor electrice, electrostatica, magnetostatica, ecuațiile lui Maxwell și termodinamica clasică a lui Caratheodory și Born. În desfășurarea materialului prezentat autorii folosesc metoda *„în spirală"* prin care unele subiecte, noțiuni și metode sînt reluate gradat, de mai multe ori, la diferite nivele de abstractizare și rafinament. Acest procedeu este preferat metodei *„liniare"* în care accentul este pus mai ales pe înlănțuirea subiectelor în ordine strict logică, fără reveniri și reluări. Este astfel asigurată fundamentarea conceptelor, metodelor și teoriilor matematice expuse, înlănțuirea naturală a acestora și înțelegerea aplicațiilor fizice respective. Pentru a ne face o imagine de ansamblu asupra arhitecturii și conținutului acestui elegant curs prezentăm titlurile celor 22 de capitole, dispuse în număr egal în cele două volume. I. Transformări liniare în plan, II. Autovectori și autovalori, III. Ecuații diferențiale în plan IV. Produse scalare, V. Calcul diferențial în plan, VI. Teoreme ale calculului diferențial, VII. Forme diferențiale și integrale curbilinii, VIII. Integrale duble, IX. Optica gaussiană, X. Spații vectoriale și transformări liniare, XI. Determinanți, XII. Teoria rețelelor electrice, XIII. Metoda proiecției ortogonale, XIV. Complexe multidimensionale, XV. Complexe în R^n, XVI. Electrostatică în R^3, XVII. Curenți, fluxuri și magnetostatică, XVIII. Operatorul „stea", XIX. Ecuațiile lui Maxwell, XX. Analiză complexă, XXI. Evaluarea asimptotică a integralelor, XXII. Termodinamică. Fiecare capitol începe cu prezentarea succintă a conținutului său și se încheie cu un sumar al rezultatelor obținute, cu o listă de întrebări menite a fixa aceste rezultate și cu un număr apreciabil de probleme propuse, cele două volume conținînd peste 300 de probleme. Fiecare din cele două volume se încheie cu sugestii privind folosirea bibliografiei menționate și cu un indice de noțiuni. Forma frumoasă, riguroasă și modernă în care sînt prezentate conceptele și teoriile matematice, precum și modul în care acestea sînt aplicate în abordarea unor capitole de fizică fac ca acest curs să fie de mare interes mai ales pentru cei ce predau matematica studenților în fizică și inginerie.

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