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Received 12.VII.1989

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Tomul XXXVI, s. I a. Matematică, 1990, f. 4.

ON Γ-DIVISION RING

BY

T. K. DUTTA and N. C. ADHIKARI

1. Introduction. N. Nobusawa [13] in the year 1964 defined Γ-ring as follows: Let M and Γ be two additive abelian groups. If there exist mappings from $M\times\Gamma\times M$ to M (images to be denoted by $a\alpha b$, $a, b\in M$, $\alpha\in\Gamma$) and from $\Gamma\times M\times\Gamma\to\Gamma$ (images to be denoted by $\alpha a\beta$, $\alpha, \beta\in\Gamma$, $a\in M$), satisfying, for all $a, b, c\in M$ and $\alpha, \beta, \gamma\in\Gamma$:

$$N_1: a\alpha(b\beta c)=a(\alpha b\beta)c=(a\alpha b)\beta c;$$

$$N_2: (a+b)\alpha c=a\alpha c+b\alpha c,\quad a(\alpha+\beta)b=a\alpha b+a\beta b,\quad a\alpha(b+c)=a\alpha b+a\alpha c;$$

N_3: $a\gamma b=0$ for all $a, b\in M$ implies that $\gamma=0$

then M is called a Γ-ring.

Later in the year 1966 Barnes [1] slightly weakened the defining conditions for Nobusawa's Γ-ring and defined Γ-ring as follows:

Let M and Γ be two additive abelian groups. If there exists a mapping from $M\times\Gamma\times M$ to M (images to be denoted by $a\alpha b$, $a, b\in M$, $\alpha\in\Gamma$), satisfying, for all $a, b, c\in M$ and $\alpha, \beta\in\Gamma$

$$B_1: a\alpha(b\beta c)=(a\alpha b)\beta c;$$

$$B_2: \text{same as } N_2,$$

then M is called a Γ-ring.

Usually, the former one is called Nobusawa's Γ-ring and the latter one is called Barnes' Γ-ring. The class of Γ-rings contains not only all rings but also Hestenes' ternary ring [7]. Coppage [3], Luh [11], [12], Kyuno [8], [9], [10] studied the structure of Γ-ring and obtained various generalizations of the corresponding parts in ring theory. The first author [4], [5], [6] also studied some properties of Barnes' Γ-ring and introduced the notion of Γ-field.

In this paper we have studied further some properties of Γ-field. Also we have introduced the notion of Γ-division ring (using Barnes' Γ-ring) and studied some properties of it. We have obtained some necessary and sufficient conditions for a Γ-ring to be a Γ-division ring or to be a Γ-field. Also we have established a relation between the Γ-field and the operator ring of a Γ-ring.

2. Preliminaries. A Γ-ring M is said to be commutative if $a\gamma b=b\gamma a$ for all $a, b\in M$ and for all $\gamma\in\Gamma$.

A Γ-ring M is said to be a Γ-ring with divisor of zero if there exist elements $a(\neq 0)$, $b(\neq 0)$ in M and an element $\gamma(\neq 0)$ in Γ such that $a\gamma b=0$.

An additive subgroup I of a Γ-ring M is called a right (left) ideal of M if $x\alpha y \in I$ $(y\alpha x \in I)$ for all x in I, α in Γ and y in M.

An additive subgroup I of a Γ-ring M which is a left ideal as well as a right ideal is called an ideal of M.

A Γ-ring M is said to be simple if $M\Gamma M \neq 0$ and the only ideals of M are 0 and M where $M\Gamma M$ is the set of all finite sums $\sum_i m_i\alpha_i n_i$ where $m_i, n_i \in M$ and $\alpha_i \in \Gamma$.

Let M_1 be a Γ_1-ring and M_2 be a Γ_2-ring. A pair of mappings $\theta = (\theta_1, \theta_2)$ where θ_1 is a mapping from M_1 to M_2 and θ_2 is a mapping from Γ_1 to Γ_2 is called a homomorphism if

1) θ_1 is a group homomorphism from M_1 to M_2;

2) θ_2 is a group homomorphism from Γ_1 to Γ_2;

3) for $x, y \in M$ and $\alpha \in \Gamma$ $(x\alpha y)\theta = (x\theta_1)(\alpha\theta_2)(y\theta_1)$.

If both θ_1 and θ_2 are monomorphisms (epimorphisms, isomorphisms) then θ is called a monomorphism (epimorphism, isomorphism).

Let M be a Γ-ring and R be the subring of the endomorphism ring of M (operators being taken on the right) consisting of all sums of the form $\sum_i [\gamma_i, x_i]$ $(\gamma_i \in \Gamma, x_i \in M)$, where $[\gamma, x]$ is defined by $m[\gamma, x] = m\gamma x$ for all $m \in M$. R is called the right operator ring of M. This definition of operator ring is equivalent to the original definition of operator ring defined by Luh [12]. Let M be a Γ-ring. M is said to be left (right) primitive if (i) the left (right) operator ring $L(R)$ of M is a left (right) primitive ring and (ii) $x\Gamma M = 0$ $(M\Gamma x = 0)$ implies $x = 0$.

A commutative Γ-ring M is called a Γ-field if for every nonzero element a of M and for every pair of nonzero elements γ_1, γ_2 of Γ there exists an element a' in M such that $a\gamma_1 a'\gamma_2 b = b$ for all $b \in M$.

A commutative Γ-ring M is primitive iff it is a Γ-field (Theorem 3.11[6]).

Let M and Γ be two nonempty sets. M is called a Γ-semigroup if there exists a mapping from $M \times \Gamma \times M \to M$ (images to be denoted by $x\alpha y$, $x, y \in M$ and $\alpha \in \Gamma$) satisfying $(x\alpha y)\beta z = x\alpha(y\beta z)$.

Let M be a Γ-semigroup (Γ-ring) and α be a fixed element of Γ. We define a binary multiplication '$\circ$' in M as follows: $a \circ b = a\alpha b$, for all $a, b \in M$. Then M is a semigroup (ring). We denote this semigroup (ring) by M_α.

A nonempty subset A of a Γ-semigroup S is called a left (right) ideal of S if $s\gamma a$ $(a\gamma s) \in A$ where $s \in S$, $\gamma \in \Gamma$, $a \in A$.

A nonempty subset A of S which is a left ideal and a right ideal is called an ideal of S.

Let S be a Γ-semigroup. If S_α is a group for some $\alpha \in \Gamma$ then S_α is a group for all $\alpha \in \Gamma$ (Theorem [15]). A Γ-semigroup S is called a Γ-group if S_α is a group for some $\alpha \in \Gamma$.

3. Γ-field. In this section we study some properties of Γ-fields.

3.1. Theorem. *A commutative Γ-ring M will be a Γ-field if and only if $M' = M \setminus \{0\}$ is a Γ'-group where $\Gamma' = \Gamma \setminus \{0\}$.*

Proof. Let M be a Γ-field. We shall first show that M' is a Γ'-semigroup. Let $a, b \in M'$ and $\gamma \in \Gamma'$. Then $a \neq 0$, $b \neq 0$ $\gamma \neq 0$. Since a Γ-field M does not contain any divisor of zero, $a\gamma b \neq 0$. So $a\gamma b \in M'$. Also $a\alpha(b\beta c) = (a\alpha b)\beta c$, for all $a, b, c \in M'$ and for all $\alpha, \beta \in \Gamma'$. Hence M' is a Γ'-semigroup. Let I be an ideal of the Γ'-semigroup M'. Let $a \in I$. Since M is a Γ-field, for the element $a(\neq 0)$ of M and for the elements $\gamma_1(\neq 0)$ and $\gamma_2(\neq 0)$ of Γ there exists an element a' in M such that $a\gamma_1 a'\gamma_2 b = b$, for all $b \in M$. Obviously $a' \neq 0$ and so $a' \in M'$. Now $a \in I$ implies that $b = a\gamma_1 a'\gamma_2 b \in I$ for all $b \in M'$. Hence $M' \subseteq I$. Also $I \subseteq M'$. So $M' = I$. Thus M' contains no proper ideal. Consequently M' is a Γ'-group (Theorem 2.1, [15]). Conversely let M' be a Γ'-group. Let $a \in M'$ and $\gamma_1, \gamma_2 \in \Gamma'$. Since M' is a Γ'-group, M'_{γ_2} is a group. Let e be the identity element of M'_{γ_2}. Then $e\gamma_2 b = e\circ b = b$ for all $b \in M'$. Again M_{γ_1} is also a group, so for the elements a and e of M' there exists an element a' in M such that $a\gamma_1 a' = a\circ a' = e$. Thus $b = e\gamma_2 b = a\gamma_1 a'\gamma_2 b$ for all $b \in M'$. If $b = 0$ then obviously $b = a\gamma_1 a'\gamma_2 b$. So $a\gamma_1 a'\gamma_2 b = b$ for all $b \in M$. In other words M is a Γ-field.

3.2. Theorem. *A commutative Γ-ring M without any divisor of zero will be a Γ-field if and only if, for every nonzero element a of M and for some $\gamma(\neq 0)$ of Γ there exists an element a' in M such that $a\gamma a'\gamma b = b$ for all $b \in M$.*

Proof. Since the necessary part follows from the definition of the Γ-field, we shall prove only the sufficiency part. Let us assume that the given condition holds in a commutative Γ-ring M without any divisor of zero. Since M contains no divisor of zero, M' is a Γ'-semigroup where $M' = M \setminus \{0\}$ and $\Gamma' = \Gamma \setminus \{0\}$. Now M'_γ is a semigroup which contains no proper ideal since $b = a\gamma a'\gamma b$ for all $b \in M$. So M'_γ is a group. Hence M' is a Γ'-group. So M is a Γ-field.

3.3. Theorem *A Γ-field has no proper homomorphic image.*

Proof. Let M be a Γ_1-field and N be a Γ_2-field. Let $\theta = (\theta_1, \theta_2)$ be an epimorphism from the Γ_1-field M onto the Γ_2-field N. Let $K = \{m \in M / \theta_1(m) = 0\}$. Let $m_1, m_2 \in K$. Then $\theta_1(m_1 - m_2) = \theta_1(m_1) - \theta_1(m_2) = 0$, so $m_1 - m_2 \in K$. Let $\gamma \in \Gamma_1$ and $x \in M$. Then $\theta(m_1\gamma x) = \theta_1(m_1)\theta_2(\gamma)\theta_1(x) = 0$. So $m_1\gamma x \in K$. Hence K is an ideal of M. Since M is a Γ-field, M contains no proper ideal (Prop. 3.5 [6]), so either $K = 0$ or $K = M$. If $K = M$ then θ_1 is a zero morphism and hence θ is a zero morphism. On the other hand, if $K = \{0\}$ then θ_1 is a monomorphism and hence an isomorphism since θ_1 is an epimorphism. Now, let $\gamma_1, \gamma_2 \in \Gamma$ be such that $\theta_2(\gamma_1) = \theta_2(\gamma_2)$. Let $a(\neq 0)$, $b(\neq 0) \in M$. Then $\theta_1(a) \neq 0$, $\theta_1(b) \neq 0$. Now $\theta_2(\gamma_1) = \theta_2(\gamma_2)$ implies that $\theta_1(a)\theta_2(\gamma_1)\theta_1(b) = \theta_1(a)\theta_2(\gamma_2)\theta_1(b)$ i.e. $\theta(a\gamma_1 b) = \theta(a\gamma_2 b)$ which implies that $a\gamma_1 b - a\gamma_2 b \in K$. Since $K = \{0\}$, $a\gamma_1 b = a\gamma_2 b$. So $a(\gamma_1 - \gamma_2)b = 0$. Since M does not contain any divisor of zero, $a(\gamma_1 - \gamma_2)b = 0$ implies that $\gamma_1 = \gamma_2$. Hence θ_2 is also a monomorphism and hence an isomorphism, since θ_2 is already an epimorphism. Thus $\theta = (\theta_1, \theta_2)$ is an isomorphism. Hence the theorem.

3.4. Lemma. *If a Γ-ring M is commutative then its right (left) operator ring $R(L)$ is also commutative.*

Proof. Let M be a commutative Γ-ring and R be its right operator ring. Then R is the subring of the endomorphism ring of M (operators being taken to act on the right), consisting of all sums of the form $\sum_i [\alpha_i, x_i]$ $(\alpha_i \in \Gamma, x_i \in M)$ where $[\alpha, x]$ is defined by $m[\alpha, x] = m\alpha x$, for all $m \in M$. Now, let $\sum_i [\alpha_i, x_i]$, $\sum_j [\beta_j, y_j] \in R$.

Then

$$m(\sum_i[\alpha_i, x_i] \sum_j [\beta_j, y_j]) = (m\sum_i[\alpha_i, x_i]) \sum_j [\beta_j, y_j] = (\sum_i m\,\alpha_i x_i) \sum_j[\beta_j, y_j] =$$

$$= \sum_i (m\, \alpha_i x_i \sum_j [\beta_j, y_j]) = \sum_i (\sum_j (m\,\alpha_i x_i)\beta_j y_j) = \sum_i (\sum_j m\,\alpha_i(x_i\beta_j y_j)) =$$

$$= \sum_i(\sum_j(x_i\beta_j y_j)\alpha_i m) = \sum_i(\sum_j(x_i\beta_j(y_j\alpha_i m)) = \sum_i(\sum_j(x_i\beta_j(m\alpha_i y_j)) = \sum_i(\sum_j(m\beta_j(x_i\alpha_i y_j)) =$$

$$= \sum_i (\sum_j(m\,\beta_j y_j)\alpha_i x_i) = \sum_j (\sum_i(m\,\beta_j y_j)\alpha_i x_i) = \sum_j(m\,\beta_j y_j \sum_i[\alpha_i, x_i]) =$$

$$= m(\sum_j[\beta_j, y_j] \sum_i [\alpha_i, x_i]).$$

This is true for all $m \in M$. Hence

$$\sum_i[\alpha_i, x_i] \sum_j [\beta_j, y_j] = \sum_j[\beta_j, y_j] \sum_i [\alpha_i, x_i].$$

Hence R is commutative.

3.5. Theorem. *A commutative Γ-ring M will be a Γ-field if and only if its right operator ring R is a field.*

Proof. Let the commutative Γ-ring M be a Γ-field. Since M is a Γ-field, it is primitive in the sense of Ravisankar and Shukla (Proposition 3.11 [14]). Hence it is primitive in the sense of Luh. (Theorem 1.6 [14]) i.e. its right operator ring R is primitive. Also, since M is commutative, R is commutative (Lemma 3.4). Hence R is a commutative primitive ring; in other words R is a field. Conversely, we assume that the right operator ring R of the commutative Γ-ring M is a field. Then R is a primitive ring. Let $M\Gamma x = 0$. Let $[\gamma, m]$ $(\neq 0) \in R$. Now $x[\gamma, m] = x\gamma m = m\gamma x$ (since M is commutative) $= 0$. Since $[\gamma, m]^{-1} \in R$. Now $x[\gamma, m] = 0$ implies that $x[\gamma, m]\,[\gamma, m]^{-1} = 0$ i.e. $x = 0$. Thus $M\Gamma x = 0$ implies that $x = 0$. So M is primitive in the sense of Luh and hence it is primitive in the sense of Ravisankar and Shukla. Thus M is a commutative primitive Γ-ring. So M is a Γ-field.

4. Γ-division ring. In this section we define a Γ-division ring and study some properties of it.

4.1. Definition. *Let M be a Γ-ring. M is called a Γ-division ring if $M' = M \setminus \{0\}$ is a Γ′-group where $\Gamma' = \Gamma \setminus \{0\}$.*

4.2. Corollary. *Every commutative Γ-division ring is a Γ-field.*

Proof. The corollary follows from the theorem 3.1 and the definition of the Γ-division ring.

4.3. Theorem. *Let M be a Γ-ring. Then M is a Γ-division ring if and only if for every a $(\neq 0)$ of M and for every $\gamma(\neq 0)$ of Γ there exists an element a' in M such that*

$$a\gamma a'\gamma b = a'\gamma a\gamma b = b\gamma a\gamma a' = b\gamma a'\gamma a = b \quad \textit{for all} \quad b \in M.$$

Proof. Let M be a Γ-division ring. Then M' is a Γ′-group. Hence from the definiton of Γ-group it follows that M'_γ is a group for every $\gamma \in \Gamma'$. Let $a \in M' = M'_\gamma$. Since M'_γ is a group there exists an element a' in M' such that $a\circ a' = a' \circ a = e_\gamma$, where e_γ is the identity element of M'_γ. Now, for any $b \in M' = M'_\gamma$, $e_\gamma \circ b = b \circ e_\gamma = b$. Hence $b = a \circ a' \circ b = a' \circ a \circ b = b \circ a \circ a' = b \circ a' \circ a$ i.e., $b = a\gamma a'\gamma b = a'\gamma a\gamma b = b\gamma a\gamma a' = b\gamma a'\gamma a$. Now, if $b = 0$ then obviously $b = a\gamma a'\gamma b = a'\gamma a\gamma b = b\gamma a\gamma a' = b\gamma a'\gamma a$. Hence the necessary part.

Conversely we assume that the given condition holds in a Γ-ring M. First we shall show that M does not contain any divisor of zero. Let $a\gamma b = 0$ and $a \neq 0$ and $\gamma \neq 0$ where $a, b \in M$ and $\gamma \in \Gamma$. Since $a \neq 0$ and $\gamma \neq 0$ there exists an element a' in M such that $a'\gamma a\gamma c = c$ for all $c \in M$. Hence $b = a'\gamma a\gamma b = 0$. Similarly we can show that if $a\gamma b = 0$ and $\gamma \neq 0$ $b \neq 0$ then $a = 0$. Lastly we assume that $a\gamma b = 0$ and $a \neq 0$ and $b \neq 0$. If possible, let $\gamma \neq 0$. Then for the element $a(\neq 0)$ of M and $\gamma(\neq 0)$ of Γ there exists an element a' in M such that $a'\gamma a\gamma c = c$ for all $c \in M$. Hence $b = a'\gamma a\gamma b = 0$ which contradicts out assumption that $b \neq 0$. Hence $a\gamma b = 0$ and $a \neq 0$, $b \neq 0$ imply that $\gamma = 0$. Thus M does not contain any divisor of zero. This implies that $M' = M \setminus \{0\}$ is a Γ′-semigroup where $\Gamma' = \Gamma \setminus \{0\}$. Also using the given condition we can show that M' does not contain any proper left or right ideal. Hence M' is a Γ′-group (Theorem 2.1 [15]). In other words M is a Γ-division ring.

4.4. Corollary. *A Γ-division ring does not contain any divisor of zero.*

Proof. The corollary follows from the sufficiency part of the Theorem 4.3.

4.5. Theorem. *Let M and N be two simple R-modules. Then $\mathrm{Hom}_R(M, N)$ is a Γ-division ring where* $\Gamma = \mathrm{Hom}_R(N, M)$.

Proof. It can be easily verified that $\mathrm{Hom}_R(M, N)$ is a Γ-ring where $\Gamma = \mathrm{Hom}_R(N, M)$ where $f + g(f, g \in \mathrm{Hom}_R(M, N)$ is defined by $(f+g)(m) = f(m) + g(m)$, $m \in M$ and $f\alpha g$ $(f, g \in \mathrm{Hom}_R(M, N)$ and $\alpha \in \mathrm{Hom}_R(N, M))$ is defined by $(f\alpha g)(m) = f(\alpha(g(m)))$. Let $f(\neq 0) \in \mathrm{Hom}_R(M, N)$. Since both M and N are simple, f is an R-isomorphism. Similarly if $\gamma(\neq 0) \in \mathrm{Hom}_R(N, M)\gamma$ is also an R-isomorphism. Now it can be easily verified that the the condition stated in theorem 4.3 for a Γ-ring to be a Γ-division ring holds in $\mathrm{Hom}_R(M, N)$. Hence the theorem.

4.6. Lemma. *A Γ-ring M is commutative if and only if, for every $\alpha(\neq 0) \in \Gamma$, M_α is commutative.*

4.7. Theorem. *A finite Γ-division ring uis a Γ-field.*

Proof. Let M be a finite Γ-division ring. Let $\alpha(\neq 0) \in \Gamma$. Then M_α is a finite ring. Since M is a Γ-division ring, M does not contain any divisor of zero and hence M_α also does not contain any divisor of zero. So $M_\alpha \setminus \{0\}$ is a semigroup. Now using theorem 4.3 we can show that $M_\alpha \setminus \{0\}$ is left simple as well as right simple. Consequently $M_\alpha \setminus \{0\}$ is a group and hence M_α is a division ring which is finite. So, by Wedderburn's theorem M_α is a field. So M is commutative. This is true for all $\alpha(\neq 0) \in \Gamma$. Hence from the above lemma it follows that M is a commutative Γ-division ring. So M is a Γ-field.

4.8. Theorem. *Let M be a Γ-ring. Let for each nonzero element a of M and for each pair of nonzero elements γ_1, γ_2 of Γ there exists a unique element $b \in M$ such that $a\gamma_1 b\gamma_2 a = a$ then (i) M has no divisor of zero (ii) $b\gamma_2 a\gamma_1 b = b$ (iii) M is a Γ-division ring.*

Proof. (i) Let $a\gamma c = 0$, where $a, c \in M$ and $\gamma \in \Gamma$. Let $a \neq 0$ and $\gamma \neq 0$. Then, by the given condition, there exists a unique element $b \in M$ such that $a\gamma b\gamma a = a$. Now $a\gamma c = 0$ implies that $a\gamma c\gamma a = 0$. Hence $a\gamma(b+c)\gamma a = a\gamma b\gamma a + a\gamma c\gamma a = a\gamma b\gamma a = a$. Now, by the uniqueness of the element b we have $b + c = b$ which implies that $c=0$. Similarly we can show that $a\gamma c=0$ and $\gamma\neq 0$, $c\neq 0$ imply that $a=0$. Lastly, let $a\gamma c=0$, $a\neq 0$, $c\neq 0$. If possible, let $\gamma\neq 0$. Then, as above, considering $a\neq 0$, $\gamma\neq 0$ we can show that $c=0$ which contradicts our assumption that $c\neq 0$. Hence M does not contain any divisor of zero. (ii) From the given condition we have $a\gamma_1 b\gamma_2 a=a$ which implies that $b\gamma_2 a\gamma_1 b\gamma_2 a=b\gamma_2 a$ i.e. $(b\gamma_2 a\gamma_1 b-b)\gamma_2 a=0$. Since M does not contain any divisor of zero $(b\gamma_2 a\gamma_1 b-b)\gamma_2 a=0$ together with $\gamma_2\neq 0$ and $a\neq 0$ implies that $b\gamma_2 a\gamma_1 b=b$. (iii) Let $a(\neq 0)\in M$ and $\gamma(\neq 0)\in\Gamma$. Now, by the given condition, there exists a unique element $b\in M$ such that $a\gamma b\gamma a=a$. We note that $b\neq 0$, for if $b=0$ then $a=0$ which is a contradiction. We claim that $a\gamma b\gamma c=c$ for all $c\in M$. If $c=0$ then we have nothing to show. So we assume that $c\neq 0$. Let $a\gamma b\gamma c=x$. Then $b\gamma a\gamma b\gamma c=b\gamma x$ which implies that $b\gamma c=b\gamma x$ (using (ii)). Again $b\gamma c=b\gamma x$ implies that $b\gamma(c-x)=0$. Then $c=x$, since M does not contain any divisor of zero. Hence $a\gamma b\gamma c=c$ for all $c\in M$. Similarly we can show that $b\gamma a\gamma c=c\gamma a\gamma b=c\gamma b\gamma a=c$ for all $c\in M$. Hence M is a Γ-division ring.

Acknowledgement. We are thankful to Dr. M. K. Sen, for his valuable suggestions in the preparation of this paper.

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Received 15.VIII.1987
Revised 25.11.1990

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ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA"
Tomul XXXVI, s. I a, Matematică, 1990, f. 4.

WEAKER CHARACTERIZATIONS OF THE LIMIT OF A FUNCTION

BY

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We have found in the literature the following weaker characterizations of the limit of a function at ∞.

C 1. (D. J. Newman [1], Problem 107) : Let $f:[0,+\infty)\to\mathbf{R}$ be a continuous function. If

$$\lim_{n\to\infty} f(na)=0 \quad \forall a>0 \tag{1}$$

then actually $\lim_{x\to\infty} f(x)=0$. (Croft Lemma).

C 2. (Sadovnicii and Podkolzin [2], Problem 107) : Let $f:[0,+\infty)\to\mathbf{R}$ be an uniformly continuous function such that

$$\lim_{n\to\infty} f(n+a)=0, \quad \forall a>0. \tag{2}$$

Then actually $\lim_{x\to\infty} f(x)=0$.

The common feature of the two characterizations above is that from the convergence to zero on a family of unbounded sequences one infers the convergence to zero on all sequences tending to ∞. The aim of our work is to find other families of sequences exhibiting the same properties. With this respect the following definition arises naturally.

Definition 1. *Let X be a family of real functions defined on* $[0,+\infty)$.

a) *A family* $\mathcal{F}$ *of sequences that tend to* ∞ *will be called a X-basis of convergence (or a X—B. C. for simplicity) if from*

$$f\in X \text{ and } \lim_{n\to\infty} f(x_n)=l\in\mathbf{R}, \qquad \forall (x_n)\in\mathcal{F} \tag{3}$$

it follows that $\lim_{x\to\infty} f(x)=l$.

b) *The family* $\mathcal{F}$ *is called a strong X-basis of convergence (or S. X-B. C. for simplicity) if from*

(3′) $f\in X$ *such that the* $\lim_{n\to\infty} f(x_n)$ *exists and is finite for all* $(x_n)\in\mathcal{F}$

it follows that $\lim_{x\to\infty} f(x)$ *exists and is finite.*

With this definition C 1 is equivalent to saying that the family

$$\mathcal{F}=\{(na)_{n\geq 1}/a>0\} \text{ is a } X\text{-}B.\ C. \text{ for } X=C([0,+\infty)).$$