that $c=0$. Similarly we can show that $a\gamma c=0$ and $\gamma\neq 0$, $c\neq 0$ imply that $a=0$. Lastly, let $a\gamma c=0$, $a\neq 0$, $c\neq 0$. If possible, let $\gamma\neq 0$. Then, as above, considering $a\neq 0$, $\gamma\neq 0$ we can show that $c=0$ which contradicts our assumption that $c\neq 0$. Hence M does not contain any divisor of zero. (ii) From the given condition we have $a\gamma_1 b\gamma_2 a=a$ which implies that $b\gamma_2 a\gamma_1 b\gamma_2 a=b\gamma_2 a$ i.e. $(b\gamma_2 a\gamma_1 b-b)\gamma_2 a=0$. Since M does not contain any divisor of zero $(b\gamma_2 a\gamma_1 b-$ $-b)\gamma_2 a=0$ together with $\gamma_2\neq 0$ and $a\neq 0$ implies that $b\gamma_2 a\gamma_1 b=b$. (iii) Let $a(\neq 0)\in M$ and $\gamma(\neq 0)\in\Gamma$. Now, by the given condition, there exists a unique element $b\in M$ such that $a\gamma b\gamma a=a$. We note that $b\neq 0$, for if $b=0$ then $a=0$ which is a contradiction. We claim that $a\gamma b\gamma c=c$ for all $c\in M$. If $c=0$ then we have nothing to show. So we assume that $c\neq 0$. Let $a\gamma b\gamma c=x$. Then $b\gamma a\gamma b\gamma c=$ $=b\gamma x$ which implies that $b\gamma c=b\gamma x$ (using (ii)). Again $b\gamma c=b\gamma x$ implies that $b\gamma(c-x)=0$. Then $c=x$, since M does not contain any divisor of zero. Hence $a\gamma b\gamma c=c$ for all $c\in M$. Similarly we can show that $b\gamma a\gamma c=c\gamma a\gamma b=c\gamma b\gamma a=c$ for all $c\in M$. Hence M is a Γ-division ring.

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WEAKER CHARACTERIZATIONS OF THE LIMIT OF A FUNCTION

BY

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We have found in the literature the following weaker characterizations of the limit of a function at ∞.

C 1. (D. J. Newman [1], Problem 107) : Let $f:[0,+\infty)\to\mathbf{R}$ be a continuous function. If

$$\lim_{n\to\infty} f(na)=0 \quad \forall a>0 \tag{1}$$

then actually $\lim_{x\to\infty} f(x)=0$. (Croft Lemma).

C 2. (Sadovnicii and Podkolzin [2], Problem 107) : Let $f:[0,+\infty)\to\mathbf{R}$ be an uniformly continuous function such that

$$\lim_{n\to\infty} f(n+a)=0, \quad \forall a>0. \tag{2}$$

Then actually $\lim_{x\to\infty} f(x)=0$.

The common feature of the two characterizations above is that from the convergence to zero on a family of unbounded sequences one infers the convergence to zero on all sequences tending to ∞. The aim of our work is to find other families of sequences exhibiting the same properties. With this respect the following definition arises naturally.

Definition 1. *Let X be a family of real functions defined on* $[0,+\infty)$.

a) *A family $\mathcal{F}$ of sequences that tend to ∞ will be called a X-basis of convergence (or a X—B. C. for simplicity) if from*

$$f\in X \text{ and } \lim_{n\to\infty} f(x_n)=l\in\mathbf{R}, \qquad \forall (x_n)\in\mathcal{F} \tag{3}$$

it follows that $\lim_{x\to\infty} f(x)=l$.

b) *The family $\mathcal{F}$ is called a strong X-basis of convergence (or S. X-B. C. for simplicity) if from*

$$f\in X \text{ such that the } \lim_{n\to\infty} f(x_n) \text{ exists and is finite for all } (x_n)\in\mathcal{F} \tag{3'}$$

it follows that $\lim_{x\to\infty} f(x)$ exists and is finite.

With this definition C 1 is equivalent to saying that the family

$\mathcal{F}=\{(na)_{n\geq 1}/a>0\}$ is a X-B. C. for $X=C([0,+\infty))$.

In this paper we shall indicate a general method of finding X-B. C.'s for $X=C([0, +\infty))$ and $X=UC([0, +\infty))$ — the class of uniformly continuous functions.

This paper is divided into two sections according to the two situations mentioned above.

We shall look for X-B. C.'s of the form

$$\mathcal{F}=\{(f_n(a))_{n\geq 1}/a\in J\}$$

where $f_n: J\to[0, +\infty)$ is a sequence of functions defined on closed interval J such that

(4) $$\lim_{n\to\infty} f_n(a)=+\infty, \ \forall a\in J.$$

1. The case. $X=C([0, +\infty))$. We introduce the following useful notion:

Definition 2. *Let $J\subset R$ be a closed interval and let $f_n: J\to R_+$ be a sequence of functions. We say that the sequence (f_n) satisfies the asymptotic covering property (or A.C.P.) iff*

(5) *for every interval $I\subset J$ the set $I^*=\bigcup_{n\geq 1} f_n(I)$ is a neighborhood of $+\infty$.*

We state now the main result of this section:

Theorem 1. *Let $X=C([0, +\infty))$ and let $f_n: J\to[0, +\infty)$ be a sequence of continuous functions satisfying (4) and A.C.P. Then the family*

$$\mathcal{F}=\{(f_n(a))_{n\geq 1}/a\in J\} \text{ is a } X-B.C.$$

Proof. Let $f\in X$ such that

(7) $$\lim_{n\to\infty} f(f_n(a))=0, \ \forall a\in J.$$

Let $\varepsilon>0$ be arbitrary but fixed and set

$$F_{n,\varepsilon}=\{a\in J/|f(f_k(a))|\leq\varepsilon, \ \forall k\geq n\}$$

Since f and f_n are continuous we infer that $F_{n,\varepsilon}$ are closed sets. By (7) we deduce

$$\bigcup_{n\geq 1} F_{n,\varepsilon}=J, \ \forall\varepsilon>0.$$

Therefore Baire's Theorem implies that there exist $n=n(\varepsilon)$ and $\alpha<\beta$ such that $I=[\alpha, \beta]\subset F_{n,\varepsilon}$.

Let $V_\varepsilon=\bigcup_{n\geq n(\varepsilon)} f_n(I)$. By A. C. P. V_ε is a neighborhood of ∞. For $x\in V_\varepsilon$ $\exists$ $k\geq n(\varepsilon)$ and $a\in I: x=f_k(a)$ and therefore $|f(x)|\leq\varepsilon$.

This yields $\lim_{x\to\infty} f(x)=0$.

Q. E. D.

Examples. In order to draw significant consequences from our Theorem 1 we have to look upon A. C. P. more attentively. We consider the special case when f_n are increasing functions on J. The A. C. P. is equivalent to the following condition

(8) $\forall I\subset J$ $f_k(I)\cap f_{k-1}(I)\neq\varnothing$ for k large enough, or taking account of the monotonicity of f_k's.

(9) $$\forall[a, b]\subset J \ f_{k-1}(b)>f_k(a) \quad \text{for } k \text{ large enough.}$$

We shall call (9) the *chaining condition* (C. C. for brevity). Hence we can state the following.

Corollary 1. *Let $f_n: J\to\mathbf{R}$ be a sequence of continuous increasing functions satisfying (4) and C. C. Then*

$$F=\{(f_n(a))_{n\geq 1}/a\in J\} \text{ is a } X-B. \ C.$$

Corollary 2. *The families $\mathcal{F}_1=\{(na)_{n\geq 1}/a>1\}$ and $\mathcal{F}_2=\{(\log na)_{n\geq 1}/a>1\}$ are X-B. C.*

Indeed, the sequences f_n, $g_n: [1, +\infty)\to\mathbf{R}$ defined $f_n(a)=na$, $g_n(a)=\log na$ satisfy the conditions of Corollary 1.

Now let (x_n) be an increasing sequence of nonnegative numbers such that

(10) $$x_n\nearrow\infty \text{ as } n\to\infty,$$

(11) $$x_{n+1}-x_n\to 0 \text{ as } n\to\infty.$$

An example of such a sequence is given by

$$x_n=1+\frac{1}{2}+\ldots+\frac{1}{n}.$$

Let $h_n: [0, +\infty)\to\mathbf{R}$, $h_n(a)=x_n+a$.

Clearly it follows by (11) that (h_n) satisfies the C. C. So we can state

Corollary 3. *The family $\mathcal{F}^*=\{(x_n+a)_{n\geq 1}/a\geq 0\}$ with (x_n) satisfying (10), (11) is a X-B. C. In particular, if $f\in C([0, +\infty))$ satisfies*

$$\lim_{n\to\infty} f\left(a+\frac{1}{2}+\ldots+\frac{1}{n}\right)=0, \ \forall a\geq 0$$

it follows that $\lim_{x\to\infty} f(x)=0$.

We conclude this section by pointing out a sufficient condition so that C. C. holds.

Let $F: [0, +\infty)\times J\to\mathbf{R}$ be a C^1 function. We are interested in differential conditions on F such that the sequence $f_n(a)=F(n, a)$ satisfy C. C. We examine the difference

$$D_k=F(k-1, \beta)-F(k, \alpha), \quad [\alpha, \beta]\subset J, \ k\in\mathbf{N}.$$

By the mean value theorem for $F=F(t, a)$

$$D_k=-\frac{\partial F}{\partial t}+\frac{\partial F}{\partial a}(\beta-\alpha),$$

where the partial derivatives are calculated at a point P_k on the segment with end-points $(k-1, \beta)$ and (k, α). P_k is a point in the (t, a) plane. As $k\to\infty$, the t-coordinate of P_k tends to ∞. We notice that D_k can be rewritten as

$$D_k=\frac{\partial F}{\partial a}\left[(\beta-\alpha)-\frac{\partial F}{\partial t}\Big/\frac{\partial F}{\partial a}\right].$$

So if we impose

(12) $$\frac{\partial F}{\partial a}>0 \text{ on } \mathbf{R}_+\times J$$

and

(13) $$\lim_{t\to\infty}\frac{\partial F}{\partial t}\Big/\frac{\partial F}{\partial a}=0 \text{ uniformly on the compacts of } J$$

then it is easily seen that $D_k>0$ for k large enough.

Thus we have established

Corollary 4. *Let $F:\mathbf{R}_+\times J\to\mathbf{R}$ be a C^1 function satisfying (12), (13) and*

(14) $$\lim_{t\to\infty}F(t,a)=\infty,\quad \forall a\in J.$$

then the sequence $f_n:J\to\mathbf{R}$ given by $f_n(a)=F(n,a)$ satisfy the A. C. P.

Examples of functions F satisfying the conditions of Corollary 4 are

$$F(t,a)=ta,\ \log ta,\ e^{ta},\ t>0,\ a>1.$$

It is a straightforward remark that the family of the functions with properties as in the above corollary is a convex cone.

Thus we can obtain further examples.

If we look for $F(t,a)$ of the form $F(t,a)=G(t)H(a)$, $G,H>0$ (12), (13) and (14) become

(12′) $$H'(a)>0,\quad \forall a\in J.$$

(13′) $$\lim_{t\to\infty}\frac{G'(t)}{G(t)}=0,$$

(14′) $$\lim_{t\to\infty}G(t)=\infty.$$

If we look for $F(t,a)$ of the form $F(t,a)=G(t)+H(a)$, $G,H>0$:

(12″) $$H'(a)>0,$$

(13″) $$\lim_{t\to\infty}G'(t)=0,$$

(14″) $$\lim_{t\to\infty}G(t)=\infty.$$

The examples of Corollary 2 are modelled by (12′)–(14′) and the example of Corolary 3 is modelled by (12″)–(14″). We can see that the family of F as in Corollary 4 is quite large.

Thus Theorem 1 offers a large class of X-B. C. for $X=C([0,+\infty))$. Our next task is to show that many of the basis we have constructed are in fact strong bases.

This is done in the following general result.

Theorem 2. *Let $f_n:\mathbf{R}_+\to\mathbf{R}_+$ be a sequence of homeomorfisms such that the family $\mathcal{F}=\{(f_n(a))_{n\geq 1}/\ a\geq 0\}$ is a X-B. C.*

If the sequence $(f_n)_{n\geq 1}$ satisfies the density condition

(D) $$\forall k\in\mathbf{N},\ \forall a\geq 0,\ \textit{the set } D_k(a)=\{f_m^{-1}\circ f_n(a)\}_{m,n\geq k}$$

is dense in $\mathbf{R}_+$, then $\mathcal{F}$ is in fact a S. X-B. C.

Proof. Let $f\in C([0,+\infty))$ such that for every $a>0$ the limit $\lim\limits_{n\to\infty}f(f_n(a))$ exists and is finite. We denote this limit by $l(a)$.

We are going to prove that $l(a)=l(b)=l$, $\forall a,b\in\mathbf{R}_+$. From the fact that $\mathcal{F}$ is a X-B. C. it will follow that $\lim\limits_{x\to\infty}f(x)=l$, hence our theorem.

We reason by reductio ad absurdum. We assume that there exists $a>0$ such that $l(a)\neq l(0)$ i. e.

$$\lim_{n\to\infty}f(f_n(a))\neq\lim_{n\to\infty}f(f_n(0)).$$

We can set $l(0)=0$, $l(a)=1$ without any loss of generality.

Since f is continuous one can find some small enough open neighborhoods I_n of $f_n(0)$ and J_n of $f_n(a)$, $\forall n\geq 1$ such that

$$|f(f_n(0))-f(x)|\leq\frac{1}{n}\ \forall x\in I_n,\quad |f(f_n(a))-f(x)|\leq\frac{1}{n}\ \ \forall x\in J_n.$$

We shall prove the existence of a $b>0$ such that infinitely many terms of $f_n(b)$ lie in neighborhoods of the type I_n and infinitely many in neighborhoods of the type J_n. It will follow that a subsequence of $f(f_n(b))$ tends to 0 while another subsequence tends to 1 which contradicts the existence of $l(b)$.

Let us analyze the conditions

$$f_m(b)\in\bar{I}_n,\quad f_p(b)\in\bar{J}_r.$$

They are equivalent with

$$b\in\overline{f_m^{-1}(I_n)},\qquad b\in\overline{f_p^{-1}(J_r)}.$$

If

$$S_{m,n,p,r}=f_m^{-1}(I_n)\cap f_p^{-1}(J_r)$$

actually one has to find infinitely many (m,n,p,r), $m,n,p,r\to\infty$, such that

$$S=\bigcap\bar{S}_{m,n,p,r}\neq\varnothing.$$

Then any $b\in S$ has the desired properties. These quadruples will be described inductively.

Step 0. Set $p_0=1$, $r_0=2$ and $K_0=f_1^{-1}\circ(J_2)=f_{p_0}^{-1}(J_{r_0})$.

K_0 is an open interval since f_n are homeomorfisms. Since the set $\{f_m^{-1}\circ f_n(0)\}_{m,n\geq 1}$ is dense one can choose m_0, n_0 so that $f_{m_0}^{-1}\circ f_{n_0}(0)\in K_0$ and therefore

$$S_{m_0,n_0,p_0,r_0}=f_{m_0}^{-1}(I_{n_0})\cap f_{p_0}^{-1}(J_{r_0})\neq\varnothing.$$

Clearly, $S_{m_0, n_0, p_0, r_0} = S_0$ is an interval (with nonvoid interior).

Step 1. We choose $k > \max(m_0, n_0, p_0, r_0)$. Since the set $D_k(a)$ is dense in $\mathbf{R}_+$ we can find p_1, r_1 such that $f_{p_1}^{-1} \circ f_{r_1}(a) \in S_0$, and therefore $K_1 = f_{p_1}^{-1}(J_{r_1}) \cap S_0 \neq \emptyset$.

Since $D_k(0)$ is dense we can find m_1, n_1 such that $f_{m_1}^{-1} \circ f_{n_1}(0) \in K_1$, i. e. $f_{m_1}^{-1}(J_{n_1}) \cap K_1 \neq \emptyset$.

Therefore

$$S_{m_1, n_1, p_1, r_1} \supset K_1 \cap f_{m_1}^{-1}(J_{n_1}) \neq \emptyset$$

and

$$S_1 = S_0 \cap S_{m_1, n_1, p_1, r_1} \neq \emptyset \text{ is an open interval.}$$

Step J+1. We assume that we have found $S_{m_1, n_1, p_1, r_1}, \ldots, S_{m_j, n_j, p_j, r_j}$ such that

$$S_j = \bigcap_{i=1}^{j} S_{m_i, n_i, p_i, r_i} \neq \emptyset \text{ is an open interval.}$$

Let $k > \max\{m_i, n_i, p_i, r_i / i = 1, \ldots, j\}$. Since the set $D_k(a)$ is dense in $\mathbf{R}_+$ we can find p_{j+1}, r_{j+1} such that

$$f_{p_{j+1}}^{-1} \circ f_{r_{j+1}}(a) \in S_j, \text{ and therefore } K_{j+1} = f_{p_{j+1}}^{-1}(J_{r_{j+1}}) \cap S_j \neq \emptyset.$$

Since $D_k(0)$ is dense we can find m_{j+1}, n_{j+1} such that

$$f_{m_{j+1}}^{-1} \circ f_{n_{j+1}}(0) \in K_{j+1}, \text{ i. e. } f_{m_{j+1}}^{-1}(J_{n_{j+1}}) \cap K_{j+1} \neq \emptyset.$$

Clearly, $S_{m_{j+1}, n_{j+1}, p_{j+1}, r_{j+1}} \neq \emptyset$ and $S_{j+1} = S_j \cap S_{m_{j+1}, n_{j+1}, p_{j+1}, r_{j+1}} \neq \emptyset$.

Thus $(\bar{S}_j)$ is a decreasing sequence of nonempty, closed and bounded intervals. By Cantor's Theorem, we infer $\bigcap \bar{S}_j \neq \emptyset$, which is in fact what we wanted to prove.

Q. E. D.

The condition that f_n should be homeomorfism is not needed in proof. (It was assumed only to assure the sets $f_m^{-1} \circ f_n(a)$ are singletons).

If we do not impose f_n to be one-to-one, the density conditions can be restated as

(D) $\forall k \in \mathbf{N}$, $\forall a \geq 0$, the set $D_k(a) = \bigcup_{m, n \geq k} f_m^{-1} \circ f_n(a)$ is dense in $\mathbf{R}_+$.

and the statement of Theorem 2 remains true with exactly the same proof.

We now present some applications of Theorem 2.

We first prove an auxiliary result.

Lemma 1. *Let $f_n, g_n : \mathbf{R} \to \mathbf{R}$ be defined by*

$$f_n(a) = na, \quad g_n(a) = a + x_n,$$

where $(x_n) \subset [0, +\infty)$ satisfies (10) and (11).

Then (f_n), (g_n) satisfy the density condition (D).

Proof of lemma 1. The statement about (f_n) follows from the density of the set $\mathbf{Q}$ since

$$f_m^{-1} \circ f_n(a) = (n/m)a.$$

The statement about (g_n) is a classical result of analysis. Indeed $g_m^{-1} \circ g_n(a) = x_n - x_m + a$ and setting $d_n = x_n - x_{n-1}$ it is easily seen that

$$x_n - x_m = \sum_{i=m+1}^{n} d_i \quad (n > m).$$

Condition (D) follows from the fact that the series $\sum_{n \geq k} d_n$ is divergent and has his terms positive and tending to 0. (For more details we refer the reader to Polya—Szegö [3], Problems 100—102).

Q. E. D.

Combining theorem 2, Lemma 1 and the previous results we deduce the following result:

Corollary 5. *The families $\mathcal{F}_1 = \{(na)_{n \geq 1} / a \geq 0\}$ and $\mathcal{F}^* = \{(a + x_n)_{n \geq 1} / a \geq 0\}$ with (x_n) satisfying (10), (11) are strong X-basis of convergence.*

2. The case $X = UC([0, +\infty))$

In the sequel the crucial part will play the following notion:

Definition 3. *Let $J \subset \mathbf{R}$ be a closed interval and let $f_n : J \to \mathbf{R}_+$ be a sequence of functions. We say that $(f_n)_{n \geq 1}$ satisfy the weak asymptotic covering property (or W.A.C.P.) iff*

(15) *there exists a bounded interval $I \subset J$ such that the set $\bigcup_{n \geq 1} f_n(I)$ is a neighborhood of ∞.*

The following is the main result of this section:

Theorem 3. *Let $X = UC([0, \infty))$ and let $f_n : J \to \mathbf{R}$ be a sequence of functions satisfying (4), the W.A.C.P. and a uniform Lipschitz condition*

(16) $\exists L > 0$ *such that* $|f_n(x) - f_n(y)| \leq L|x - y|$ $\forall x, y \in J$, $n \geq 1$

Then the family of sequences $\mathcal{F} = \{(f_n(a))_{n \geq 1} / a \in J\}$ is a X-B.C.

Proof. Let $f \in X$ be such that

(17) $$\lim_{n \to \infty} f(f_n(a)) = 0 \qquad \forall a \in J.$$

Let $\varepsilon > 0$ be arbitrary but fixed and set

$$F_{n, \varepsilon} = \{a \in J / |f(f_k(a))| < \varepsilon, \ \forall k \geq n\}, \ n \in \mathbf{N}.$$

By (15) we know that there exists $I \subset J$ such that $I^* = \bigcup_{n \geq 1} f_n(I)$ is a neighborhood of ∞. We claim that

(18) given $\varepsilon > 0$ there exists $n = n(\varepsilon)$ such that $I \subset F_{n, \varepsilon}$.

We assume the contrary i. e. $\forall n \in \mathbf{N}$ there exists $a_n \in I$ such that $a_n \in F_{n, \varepsilon}$. Since I is bounded we may extract a convergent subsequence of (a_n) still denoted with (a_n). By (17) we infer

(19) $\forall n \in \mathbf{N}$, there exists a minimal $k_n \in \mathbf{N} : a_n \in F_{k_n, \varepsilon/2}$.

By the uniform continuity of f we get

(20) $\exists \delta > 0 : \forall x, y$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| \leq \varepsilon/2$.

Since (a_n) is convergent there exists $n_0 \in \mathbf{N}$ such that

(21) $$\forall n_1, n_2 \geq n_0 : |a_{n_1} - a_{n_2}| \leq \frac{\delta}{L}.$$

By (16), (20), (21) we infer:

(22) $$\forall n_1, n_2 \geq n_0,\ k \geq 1,\ |f(f_k(a_{n_1})) - f(f_k(a_{n_2}))| \leq \varepsilon/2.$$

Let $k_0 = k_{n_0}$. From (19) we get that

(19′) $$|f(f_k(a_n))| \leq \varepsilon/2 \qquad \forall k \geq k_0$$

Let $n \geq n_0$, $k \geq k_0$

$$|f(f_k(a_n))| \leq |f(f_k(a_n)) - f(f_k(a_{n_0}))| + |f(f_k(a_{n_0}))| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

where we have taken account of (22) and (19′) respectivelly.

Therefore $\forall n > n_0$, $a_n \in F_{k_0}$, which is in contradiction with our choice of a_n's.

Now the proof follows the same lines as in Theorem 1.

Let $V_\varepsilon = \bigcup_{k \geq n} f_k(I)$ where $n(\varepsilon)$ is given by (18). By W.A.C.P. V_ε is a neighborhood of ∞. For $x \in V_\varepsilon$, $\exists k \geq n_\varepsilon : x = f_k(a)$ and therefore by (18) $|f(x)| \leq \varepsilon$. This yields

$$\lim_{x \to \infty} f(x) = 0.$$

Q. E. D.

The characterization C. 2 we have presented in introduction follows easily from Theorem 3 taking $f_n(a) = a + n$.

Let us clarify the W.A.C.P. We further specialize considering (f_k) increasing functions. A straightforward analogy with the previous case yields the *weak chaining condition* (W. C. C.).

(23) $$\exists [\alpha, \beta] \subset J : f_{k+1}(\beta) > f_k(\alpha) \text{ for } k \text{ large enough.}$$

Accordingly we can state a corollary similar to Corollary 1.

We now look for an analogue of Corollary 4.

Let $F : \mathbf{R}_+ \times J \to \mathbf{R}_+$ be a C^1 mapping satisfying (14) i. e.

$$\lim_{t \to \infty} F(t, a) = +\infty \quad \forall a \in J.$$

We assume $\partial F / \partial a > 0$. When does the sequence $f_n(a)$ satisfy the W. A. C. P.?

We see that (16) is fulfilled when

(16′) $$\exists L > 0 : 0 \leq \frac{\partial F}{\partial a} < L \qquad \forall (t, a) \in \mathbf{R}_+ \times J.$$

A sufficient condition so that W.C.C. holds is

(24) there exist $[\alpha, \beta] \subset J$ and $\varphi : [\alpha, \beta)] \to \mathbf{R}_+$ such that

a) $\Phi(a) < \beta - \alpha \quad \forall a \in [\alpha, \beta]$. b) $\Phi(a) \geq \overline{\lim_{t \to \infty}} \dfrac{\partial F}{\partial t} \Big/ \dfrac{\partial F}{\partial a}$.

Hence we have the following result:

Corollary 5. *Let $F : \mathbf{R}_+ \times J \to \mathbf{R}_+$ satisfying (14), (16′) and (24). Then the family of sequences $\{(f_n(a))\}_{a \in J}$ is a X-B. C. (here $f_n(a) = F(n, a)$, $\forall n$, $\forall a \in J$).*

If we look for F of the form $F(t, a) = G(t) + H(a)$, (14), (16′) and (24) become

(25) $$0 < H'(a) \leq L,$$

(26) $$\lim_{t \to \infty} G(t) = +\infty,$$

(27) $$\overline{\lim_{t \to \infty}}\, G'(t) \leq H'(a)(\beta - \alpha) \quad \forall a \in [\alpha, \beta].$$

Remark 2. The family $\{a + n\}_{a \geq 0}$ is not a strong basis of convergenge for $f \in C([0, +\infty))$ as it is easily seen considering the example $f(x) = \sin 2\pi x$, $x \geq 0$.

Since the sequence $(\sin n)_{n \geq 1}$ is dense in $[-1, 1]$ it follows that the family

$$\mathcal{F} = \{a + n\}_{a \geq 0} \cup (n + \sin n) \text{ is a strong basis.}$$

It would be an interesting thing to clarify the conditions that a basis should satisfy in order to be a strong basis.

Remark 3. The family of sequences $\{(a + n)_{n \leq N}\}_{a \geq 0}$ is not a X-B. C. for $X = C([0, \infty))$ as it easily seen considering the following example: $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ is the continuous function defined by

$$\psi(x) = \begin{cases} 0, & x \in \left[n - 1,\ n - \dfrac{2}{n}\right], \\ 1, & x = n - \dfrac{1}{n}, \\ \text{linear on each of the intervals} & \left(n - \dfrac{2}{n},\ n - \dfrac{1}{n}\right) \text{ and } \left(n - \dfrac{1}{n},\ n\right). \end{cases}$$

The graph of ψ on $[n - 1, n]$ is illustrated in figure 1.

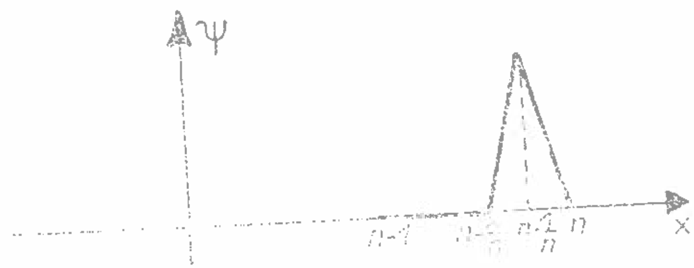

Fig. 1

That peak of ψ becomes narrawer as $n \to \infty$ so that f is not uniformly continuous. Clearly f has no limit as $x \to \infty$. However, it is easily seen that for every $a \geq 0$, $f(a + n) = 0$, $\forall n \geq n(a)$: $n(a)$ can be chosen to be the least natural number n such that $2/n \leq 1 - \{a\}$ where $\{a\} = a - [a]$.

Therefore the family of sequences $(a + n)_{n \geq 1,\, a \geq 0}$ is not a $C((0, +\infty))$ − B.C.

Remark 4. A carefull analysis of the proof of Theorem 2 shows that the density condition can be replaced by a slightly weaker one yielding the following more general result.

Theorem 2'. *Let $f_n : R_+ \to R_+$ be a sequence of continuous mapping. We assume the family*

$$F = \{(f_n(a))_{n \geq 1} / a \geq 0\}$$

is a X-B. C. If the sequence $(f_n)_n$ satisfies the connecting condition

$$(\mathrm{Co}) \qquad \bigcap_{k,\, a} D_k(a) \neq \emptyset$$

then the conclusion of Theorem 2 remains true.

Remark 5. A slight modification of the proofs of Theorems 1, 2 yields the following extension of Crefts Lemma:

Let $f : (0, +\infty) \to (0, +\infty)$ be a continuous function such that for every $a \in Q$ the limit $\lim\limits_{n \to \infty} f(na)$ exists and is finite. Then the limit $\lim\limits_{x \to \infty} f(x)$ exists and is finite.

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A MONOTONE VERSION OF THE VARIATIONAL EKELAND'S PRINCIPLE

BY

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§ 0. Introduction. Let (X, d) be a semi-metric space and $F : X \times R$, a bounded from below function. Denote

$$X(F, \varepsilon) = \{x \in X ;\ F(x) < \inf F(X) + \varepsilon\},\quad \varepsilon > 0.$$

An important problem concerning these elements is that of determining, in the neighborhood of each $u \in X(F, \varepsilon)$, a point $v \in X(F, \varepsilon)$ with some "good" variational properties. In this direction, the following 1974 Ekeland's result [11] is basic to considerations below.

Theorem 1. *Suppose X is d-complete and F is d-lower semicontinuous (in the usual sense). Then, for each $\varepsilon > 0$, $\lambda > 0$ and $u \in X(F, \varepsilon)$ there exists $v = v(\varepsilon, \lambda, u)$ in $X(F, \varepsilon)$, such that*

a) *$d(u, v) < \lambda$ and $F(u) \leq F(v)$*

b) *$F(w) - F(v) + (\varepsilon/\lambda) d(v, w) > 0$, for each $w \in X$, $d(v, w) \neq 0$.*

This principle found a number of interesting applications to a large variety of domains including fixed point theory, topological dynamics, abstract control problems, differential geometry and others; for a survey of these, we refer to the 1979 Ekeland's paper [12]. Consequently, a brief analysis of its guiding lines would be not without profit. We start with the remark that, from a structural viewpoint, the basic interpretation of Theorem 1 is the maximal one. More specifically, assume d is a metric over X; then, with respect to the ordering

$$(D_1) \qquad x \leq y \text{ if and only if } (\varepsilon/\lambda) d(x, y) \leq F(x) - F(y)$$

conclusion of the statement above may be rephrased as:

$$\begin{cases} \text{For each } u \in X \text{ there may be determined a } \leq\text{-maximal} \\ \text{point } v \in X \text{ with } u \leq v \end{cases}$$

This, in particular, means that if $u \in X(F, \varepsilon)$ then $v \in X(F, \varepsilon)$ too and $d(u, v) < \lambda$. (Note that the last information about v is essential in many cases). In other words, Ekeland's theorem is a denumerable (metrical) version of the Zorn maximality principle. This reduction line goes back to Bishop and Phelps [3]. For the specific framework we already sketched, the Bronds-