Theorem 2'. *Let $f_n : R_+ \to R_+$ be a sequence of continuous mapping. We assume the family*

$$F = \{(f_n(a))_{n \geq 1} / a \geq 0\}$$

is a X-B. C. If the sequence $(f_n)_n$ satisfies the connecting condition

$$(\mathrm{Co}) \qquad \bigcap_{k,\,n} D_k(a) \neq \emptyset$$

then the conclusion of Theorem 2 remains true.

Remark 5. A slight modification of the proofs of Theorems 1, 2 yields the following extension of Crefts Lemma:

Let $f : (0, +\infty) \to (0, +\infty)$ be a continuous function such that for every $a \in Q$ the limit $\lim_{n\to\infty} f(na)$ exists and is finite. Then the limit $\lim_{x\to\infty} f(x)$ exists and is finite.

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A MONOTONE VERSION OF THE VARIATIONAL EKELAND'S PRINCIPLE

BY

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§ 0. Introduction. Let (X, d) be a semi-metric space and $F: X \to R$, a bounded from below function. Denote

$$X(F, \varepsilon) = \{x \in X\,;\ F(x) < \inf F(X) + \varepsilon\},\quad \varepsilon > 0.$$

An important problem concerning these elements is that of determining, in the neighborhood of each $u \in X(F, \varepsilon)$, a point $v \in X(F, \varepsilon)$ with some "good" variational properties. In this direction, the following 1974 Ekeland's result [11] is basic to considerations below.

Theorem 1. *Suppose X is d-complete and F is d-lower semicontinuous (in the usual sense). Then, for each $\varepsilon > 0$, $\lambda > 0$ and $u \in X(F, \varepsilon)$ there exists $v = v(\varepsilon, \lambda, u)$ in $X(F, \varepsilon)$, such that*

a) *$d(u, v) < \lambda$ and $F(u) \leq F(v)$*

b) *$F(w) - F(v) + (\varepsilon/\lambda) d(v, w) > 0$, for each $w \in X$, $d(v, w) \neq 0$.*

This principle found a number of interesting applications to a large variety of domains including fixed point theory, topological dynamics, abstract control problems, differential geometry and others; for a survey of these, we refer to the 1979 Ekeland's paper [12]. Consequently, a brief analysis of its guiding lines would be not without profit. We start with the remark that, from a structural viewpoint, the basic interpretation of Theorem 1 is the maximal one. More specifically, assume d is a metric over X; then, with respect to the ordering

$$(D_1) \qquad x \leq y \text{ if and only if } (\varepsilon/\lambda) d(x, y) \leq F(x) - F(y)$$

conclusion of the statement above may be rephrased as:

$$\left\{\begin{array}{l}\text{For each } u \in X \text{ there may be determined a } \leq\text{-maximal}\\ \text{point } v \in X \text{ with } u \leq v\end{array}\right.$$

This, in particular, means that if $u \in X(F, \varepsilon)$ then $v \in X(F, \varepsilon)$ too and $d(u, v) < \lambda$. (Note that the last information about v is essential in many cases). In other words, Ekeland's theorem is a denumerable (metrical) version of the Zorn maximality principle. This reduction line goes back to Bishop and Phelps [3]. For the specific framework we already sketched, the Brondst e d's contribution [7] must be retained: see also V á l y i [23]. In particular, by the B o u r b a k i pattern [5] one may express Theorem 1 in a fixed point manner; this fact, explicitly underlined in Ekeland's paper, shows his result is an equivalent formulation of the C a r i s t i fixed point principle [9].

The second interpretation of Theorem 1 is the constructive one. To describe it, let us again assume d is a metric over X. We define a relation S over X by means of the ordering introduced by (D_1), namely

$$S(x)=\{y \in X;\ x \leqslant y\},\quad x \in X.$$

Now, inductively, construct a sequence (v_n) as follows. Put $v_0=u$. Supposing $(v_i;\ 0 \leq i \leq n)$ have been defined, we introduce the point v_{n+1} by the conventions

$$(\mathrm{D}_2)\qquad v_{n+1} \in S(v_n),\qquad F(v_{n+1}) \leqslant (1/2)(F(v_n)+\inf F(S(v_n))).$$

To see this is meaningful, note that, in case $F(v_n)=\inf F(S(v_n))$, (D_2) is holding with the choice $v_{n+1}=v_n$; otherwise, use the definition of the infimum to get v_{n+1}. Hence $(v_n;\ n \in \mathrm{N})$ may be indeed constructed by this process. Now, the sequence in question is shown to be Cauchy (see E k e l a n d [12] for details); hence, by completeness, it must converge to some $v \in X$, which is our desired point. This line of argument has been refined by B o r w e i n and P r e i s s [4] in the following manner. For $p \geqslant 1$, let Γ_p denote the class of all functions

$$G(x)=\sum_n \mu_n(d(x,\ v_n))^p,\quad x \in X,$$

where $(\mu_n;\ n \in N)$ is a sequence of positive numbers with $\sum_n \mu_n=1$, and $(v_n;\ n \in \mathrm{N})$ is a convergent (in X) sequence. That G is continuous, follows immediately by the classical Weierstrass criterion; also, when X is a linear space, and its metric, d, is generated by a norm, G is convex as it can be directly seen.

Theorem 1*. *Let X and F be as in Theorem 1. Then, given the constants $\varepsilon, \lambda>0$ and $p \geqslant 1$ as well as the point u in $X(F, \varepsilon)$, there may be determined a $G \in \Gamma_p$ and a $v \in X(F, \varepsilon)$ so that conclusion a) in that result plus*

c) $\quad F(w)-F(v)+(\varepsilon/\lambda^p)(G(w)-G(v)) \geqslant 0,$ *for each* $w \in X$

are valid.

When $p=1$, this statement contains Ekeland's; indeed, it suffices to note that $G(w)-G(v) \leqslant d(v, w)$, $v, w \in X$, for any G in Γ_1.
It is therefore natural to ask whether the converse is also holding. The answer is not very simple. For a partial one, remark that, by the recurrence formula (D_2), the variational Ekeland's result is reductible to the Principle of Dependent Choices [24]:
(DC) Given a nonempty set X and a relation $\mathfrak{R}$ over X with $\mathrm{dom}(\mathfrak{R})=X$, there may be determined, for each starting point $u \in X$, a sequence (v_n) in X with $v_0=u$ and $(v_n, v_{n+1}) \in \mathfrak{R}$ for all $n \in \mathbf{N}$.
As a matter of fact, these two results are equivalent; see B r u n n e r [8].
Now, the iterative process over $Y=\mathbf{R}^X \times X$ required by Theorem 1* is of the form

$$F_{n+1}(x)=F_n(x)+\beta_n(d(x,\ v_n))^p,\quad x \in X$$

$$F_{n+1}(v_{n+1}) \leqslant \gamma F_n(v_n)+(1-\gamma)\inf F_{n+1}(X)$$

where $(\beta_n;\ n \in \mathbf{N})$ and γ are suitably chosen. But, in this case, it is hard to associate it a relation $\mathfrak{R}$ over Y, because at each step, n, the function F_{n+1} depends on β_n and the point v_{n+1} is also depending on F_{n+1}. Therefore at least at a first glance — Theorem 1* is not reductible to Theorem 1.

Finally, from a practical viewpoint, the most useful interpretation of these results is the variational one. This is related to b) or c) above and consists in writing the differential consequences of these relations in a normed context (modulo X). (And, naturally, it needs some appropriate hypotheses about F). For example (cf. E k e l a n d [11]) if the function in question is Gâteaux/Fréchet differentiable, then b) gives

$$\|F'(v)\| \leqslant (\varepsilon/\lambda)\ (\text{where } \|\cdot\| \text{ is the norm in } X^*).$$

Accordingly, when the norm of X is $(\#)$—smooth, where $(\neq)$ is a certain bornology over X (see B o r w e i n and P r e i s s [4] for details), a differential consequence of c) is (for $p>1$).

$$0 \in \partial^{\#} F(v)+p(\varepsilon/\lambda)B^*\ (\text{where } B^* = \text{the unit ball in } X^*).$$

A common assumption encountered in these results is that concerning the *lsc* property of the function F. In the amorph (modulo X) case, there is no substitute for this property. But, when the ambient (semi-metric) space is endowed with a quasi-ordering $\leqslant$, the natural substitutes for the *lsc* property are, as underlined in T u r i n i c i [20], the $\leqslant$ —decreasing property of the function F or the $\leqslant$ —*lsc* property of the same. It is the main aim of the present exposition to emphasize that, roughly speaking, the former (latter) of these can replace *lsc* in the statement of Theorem 1 (Theorem 1*) to retain its conclusion (modulo $\leqslant$); this will be the content of Section 2. The preliminary material is collected in Section 1; it is concerned with (abstract) maximality principles of denumerable type. The variational aspects of the main results are being delineated in Section 3; it is worth noting that the natural framework of these is represented by the normed lattices. Also, an attention is paid to the existence of an exact solution to the non-constrained minimization problem associated to F. (This has been solved by means of a P a l a i s-S m a l e technique [16]). The constrained case will be considered elsewhere.

§ 1. Maximality principles of denumerable type. Let X be a nonempty set and $\leqslant$, a *qasi-ordering* (i.e., a reflexive and transitive relation) over X. Under a terminology comparable with N a c h b i n's [14, ch. II, § 2], we say that a family $\mathfrak{U}$ of subsets of X^2 is a *pseudo-uniformity* over X when each $U \in \mathfrak{U}$ includes the diagonal $\Delta(X)=\{(x, x);\ x \in X\}$. Supposing we fixed such an object, let us term the point $z \in X$, $\mathfrak{U}$ — *maximal* when $z \leqslant u \leqslant v$ implies $(u, v) \in \bigcap \mathfrak{U}$. We are interested in the following te establish sufficient conditions under which $\leqslant$ be a *normal* quasi-ordering, in the sense

(NO) For each $x \in X$ there exists a $\mathfrak{U}$ — maximal element $z \in X$ with $x \leqslant z$.

To this end, a basic assumption must be made about the ambient quasi-ordering; namely

(C_1) each ascending sequence in X is bounded above.

Concerning the other specific hypotheses to be added, let us call the (ascending) sequence (x_n) in X $\mathfrak{U}$ — *Cauchy* when for each $U\in\mathfrak{U}$ there exists $n(U)\in N$ such that

$$(x_p,\ x_q)\in U,\ \text{whenever}\ n(U)\leqslant p\leqslant q.$$

This property is, for an individual sequence, stronger than that of being $\mathfrak{U}$ — *asymptotic*, expressed as: for each $U\in\mathfrak{U}$ there exists $n(U)\in N$, such that

$$(x_p,\ x_{p+1})\in U,\ \text{for all}\ p\geqslant n(U).$$

However, when all (ascending) sequences are involved, the reciprocal of this implication also holds, making the global conditions

(C_2) each ascending sequence is $\mathfrak{U}$ — Cauchy

(C_2') each ascending sequence is $\mathfrak{U}$ — asymptotic

be mutually equivalent. (The proof of this assertion being almost evident, we omit the details). Now, for a number of reasons we shall explain further, let us introduce for the pseudo-uniformity $\mathfrak{U}$, the property of being $\leqslant$ —*triangular*, by the convention/condition

(C_3) for each $U\in\mathfrak{U}$ there exists $V\in\mathfrak{U}$ such that $x\leqslant y\leqslant z$, $(x,\ y)$, $(x,\ z)\in V\Rightarrow(y,\ z)\in U$.

Note that, under such an assumption, the $\mathfrak{U}$—maximality of $z\in X$ may be written as: $z\leqslant w$ implies $(z,\ w)\in\cap\,\mathfrak{U}$. Also, the property of being $\mathfrak{U}$ — Cauchy is equivalent, for an ascending sequence (x_n), with the (weaker in general) property of being $\mathfrak{U}$ — *semi-Cauchy*: for each $U\in\mathfrak{U}$ there exists $n=n(U)\in N$ such that

$$(x_n,\ x_p)\in U,\ \text{for all}\ p\geqslant n.$$

This, in turn, may be connected with a quasi-asymptotic property of a sequence described as: (x_n) is $\mathfrak{U}$ — *quasi-asymptotic* when, for each $n\in N$, $U\in\mathfrak{U}$,

$$(x_m,\ x_{m+1})\in U,\ \text{for some}\ m=m(n,\ U)\geqslant n.$$

Evidently, $\mathfrak{U}$ — semi-Cauchy is stronger than $\mathfrak{U}$ — quasi-asymptotic. The converse relation is also valid when all (ascending) sequences are involved; in other words, the global conditions

(C_4) each ascending sequence is $\mathfrak{U}$ — semi-Cauchy

(C_4') each ascending sequence is $\mathfrak{U}$ — quasi-asymptotic

are equivalent each other. (This equivalence continues to hold even in the absence of (C_3); but, in this case, (C_2) is (generally) not reductible to (C_4)).

Given two pseudo-uniformities $\mathfrak{U}$ and $\mathfrak{V}$ we shall say $\mathfrak{V}$ is *finer* than $\mathfrak{U}$ (and we denote this as $\mathfrak{V}\geqslant\mathfrak{U}$) in case for each $U\in\mathfrak{U}$ there exists $V\in\mathfrak{V}$ with $U\supseteq V$; also, we say $\mathfrak{U}$ and $\mathfrak{V}$ are *equivalent* when $\mathfrak{V}\geqslant\mathfrak{U}\geqslant\mathfrak{V}$. In such a situation, each $\mathfrak{U}$—notion (from the above introduced ones) is identical with its corresponding $\mathfrak{V}$—notion. (For example, $\mathfrak{U}$—maximality and $\mathfrak{V}$—maximality are equal, $\mathfrak{U}$—Cauchy is identical with $\mathfrak{V}$ — Cauchy, etc.). We shall term the pseudo-uniformity $\mathfrak{U}$ (over X), *pseudo-metrizable*, when it is equivalent with a denumerable one. It is now important to specify that the answer to the above posed problem is to be given in the context of such pseudo-uniformities. (The general case will be treated elsewhere). A basic result of this type is concentrated in

Theorem A. *Let the pseudo-metrizable pseudo-uniformity $\mathfrak{U}$ over X be such that, in addition to (C_1), either of the (mutually equivalent) conditions (C_2)/(C_2') is being fulfilled. Then, $\leqslant$ is a normal quasi-ordering. (Corollary: Let the pseudo-metrizable $\leqslant$—triangular pseudo-uniformity $\mathfrak{U}$ over X be such that, either of the (mutually equivalent) conditions (C_2)/(C_2') or (C_4)/(C_4') is fulfilled. Then, for each $x\in X$ there is a $z\in X$ with a) $x\leqslant z$, b) $z\leqslant w\Rightarrow(z,\ w)\in\cap\,\mathfrak{U}$).*

Before passing to the proof of this statement, let us indicate a particular case of it, whose usefulness will be clear in the sequel. Let $(Y,\ \tau)$ be a topological space and $0\in Y$, a distinguished point. By convention, any mapping $d: X^2\to Y$ with

$$d(x,\ x)=0,\ \text{for all}\ x\in X$$

will be referred to as a Y — *pseudometric* over X. Given such an object, we associate it a pseudo-uniformity $\mathfrak{U}(d)$ over X consisting of all subsets

$$U(d,\ G)=\{(x,\ y)\in X^2:\ d(x,\ y)\in G\},$$

where G describes the family $\Omega(0)$ of all open *neighberhoods* of $0\in Y$. By definition, any property involving $\mathfrak{U}(d)$ will be attached to d itself. (For example, d is $\leqslant$—triangular if and only if $\mathfrak{U}(d)$ fulfils this condition). Equivalently, any notion expressed in terms of $\mathfrak{U}(d)$ will be called a d—notion. This, combined with the choice of $\Omega(0)$, means, e.g., that an ascending sequence (x_n) is d-Cauchy whenever

$$d(x_n,\ x_m)\to 0\ \text{as}\ n,\ m\to\infty\ (n\leqslant m);$$

also, (x_n) is d—asymptotic, in case

$$d(x_n,\ x_{n+1})\to 0\ \text{as}\ n\to\infty.$$

(The remaining notions have similar interpretations). Now, for a number of reasons we shall explain further, it is desirable to have a property like (an assumption we shall admit in the sequel)

(P_1) $\cap\,\Omega(0)=\{0\}$.

(Of course, in such a case, the property of being d—maximal reads: $z\leqslant u\leqslant v$ implies $d(u,\ v)=0$). Let us add the fact that $\mathfrak{U}(d)$ is pseudo-metrizable when

(P_2) $\Omega(0)$ has a denumerable basis.

Note that (P_1)+(P_2) may be written as a single property

(P_3) $\{0\}$ is a G_δ—set.

The introduced Y—pseudometric d generates two (distinct in general) sequential *convergence structures* $\to$ and $\leftarrow$ over X, by the convention

$$x_n\to x(x\leftarrow x_n)\ \text{provided}\ d(x_n,\ x)\to 0\ (d(x,\ x_n)\to 0).$$

We have to remark that, by the general working hypotheses, no implication of the form

$$x_n\to x(x\leftarrow x_n)\ \text{implies}\ (x_n)\ \text{is}\ d\text{-Cauchy}$$

is to be retained. However, we say the ambient space X is d-*complete* (mod.$\to$) when each d-Cauchy sequence converges (modulo$\to$); it will be called $(d,\ \leqslant)$—complete (mod. $\to$) in case the property in question is valid for ascending sequences only. Also, the ambient quasi-ordering $\leqslant$ will be said to be d-*self-closed* (modulo $\to$) when the limit (mod. $\to$) of each ascending sequence (in X) is an upper bound of it. Similar definitions may be given for the reverse convergence structure $\leftarrow$; but, in the sequel, unless explicitly stated, we shall work with the direct convergence structure $\to$ (which, for simplicity, will be deleted as a component of these notions). Now, it is clear that, under (C_2) plus

(C_5) X is $(d,\ \leqslant)$—complete

each ascending sequence converges (in X); this, in conjunction with

(C_6) $\leqslant$ is d—self-closed

gives at once (C_1). Summing up, we deduced as a variant of Theorem A the following maximality result due to G a l v i n (personal communication, quoted in T u r i n i c i [22]):

Theorem B. *Let the quasi-ordered space* $(X, \leqslant)$ *fulfiling* (C_1) *and the* Y*—pseudometric* d *over* X *with* Y *satisfying* (P_3), *be such that each ascending sequence is d-Cauchy (respectively, d-asymptotic). Then, for each* $x\in X$ *there is a d-maximal element* $z\in X$ *with* $x\leqslant z$. *In particular, a sufficient condition for* (C_1) *is the couple* (C_5)+(C_6).

(We must not forget that all these considerations are being made under the restrictive property (P_2). In the absence of this, the above result cannot be obtained without the Zorn maximality principle).

Now, a basic choice for the topological space in question is $Y=\mathbf{R}_+$. Then, Y—pseudometric will be simply referred to as *pseudometric*. And, the $\leqslant$—triangular property (C_3) reads

(C_3') for each $\varepsilon>0$, there exists $\delta>0$ such that, $x\leqslant y\leqslant z$ and $d(x, y)$, $d(x, z)\leqslant\delta$ imply $d(y, z)\leqslant\varepsilon$.

(In particular, this is clearly holding when d is a *semi-metric* over X, in the usual sense). The following variant of Theorem B is immediate by the above discussion: see K a n g [13] or T u r i n i c i [22] for details.

Theorem C. *Let the pseudometric* d *over* X *be such that either of the equivalent) conditions* (C_2)/(C_2') *is being fulfilled (modulo d). Then the quasi-ordering* $\leqslant$ *is necessarily normal (under* (C_1)). *If, in addition,* d *is* $\leqslant$*—triangular, then* (C_2)/(C_2') *has the same power as the couple* (C_1)/(C_4') *(written modulo d) and conclusion may be depicted as: for each* $x\in X$ *there is a* $z\in X$ *with* a) $x\leqslant z$, b) $z\leqslant w\Rightarrow d(z, w)=0$.

We may now return to the proof of Theorem A. Mutatis mutandis, a direct argument — adapted from the ones given in the above quoted papers — is immediately available in our general framework, to get the desired conclusion. However, it will be more useful to reduce the proof in question to that of Theorem C, because in such a case the already formulated results will become equivalent. To do this, note that, by the involved assumptions, the underlying pseudo-uniformity $\mathfrak{U}$ is equivalent with a denumerable one. So, without loss one may assume $\mathfrak{U}$ itself is denumerable; that is, $\mathfrak{U}=(U_n\,;\ n\in\mathbf{N})$, with $U_0=X^2$.

Denote

(D_3) $V_n=\cap\{U_i\,;\ i\leqslant n\}$, $n\in\mathbf{N}$; $\mathfrak{V}=(V_n\,;\ n\in\mathbf{N})$.

Clearly, $V_0=U_0$ and $n\to V_n$ is decreasing from $(\mathbf{N}, \leqslant)$ to $(X^2, \subseteq)$. Moreover as $\cap\mathfrak{U}=\cap\mathfrak{V}$, the notions of $\mathfrak{U}$—maximal and $\mathfrak{V}$—maximal are the same. Finally, note that, by the above definition (D_3), the notions of $\mathfrak{U}$—Cauchy and $\mathfrak{V}$—Cauchy are equivalent for an (ascending) sequence. Under these preliminaries, let us introduce a pseudometric $d=d(\mathfrak{V})$ over X by the convention.

$$d(x, y)=\inf\{2^{-n}\,;\ (x, y)\in V_n\},\ (x, y)\in X^2.$$

Denoting for simplicity

$$G_n=[0,2^{1-n}),\qquad n\in\mathbf{N},$$

one has (cf. a previous notation)

$$V_n=U(d, G_n),\ \text{for all}\quad n\in\mathbf{N}.$$

Therefore, $\mathfrak{V}=U(d, G_n)\,;\ n\in\mathbf{N})$ is equivalent with $\mathfrak{U}(d)=(U(d, G)\,;\ G\in\Omega(0))$. But in this case, conditions of Theorem C are fulfilled (modulo d). By the conclusion of that result we have promised, for each $x\in X$, a d-maximal element $z\in X$ with $x\leqslant z$. This, plus the fact that $\mathfrak{V}$—maximality equals d-maximality ends the argument.

An important consequence of Theorem C is the following. Let $(X, \leq)$ be a quasi-ordered space. Given the function $F:X\to\mathbf{R}$, call the point $z\in X$, F—*maximal*, provided $z\leq w$ implies $F(z)=F(w)$ (that is, F is constant on $X(z, \leqslant)=\{x\in X\,;\ z\leqslant x\}$). Clearly, F-maximal is equivalent with d-maximal, where the semimetric d over X is introduced as

$$(D_4)\qquad d(x, y)=|F(x)-F(y)|,\quad x, y\in X.$$

This, in the presence of an assumption like

(C_7) F is decreasing and bounded from below

makes Theorem C applicable. So, we arrived at the following 1976 ordering principle due to B r é z i s and B r o w d e r [6]:

Theorem D. *Under the assumptions* (C_1)+(C_7), *we have that, for each* $x\in X$ *there exists a F-maximal point* $z\in X$ *with* $x\leqslant z$.

Now, as results from Turinici [21], Theorem C is, in turn, reductible to the above statement (hence, equivalent with it). In other words, Theorem D is but a re-formulation of Theorem A (or Theorem B).

Among the variants of this result, the following one is to be noted. Let us call the point $z\in X$, $\leqslant$*-maximal*, when $z\leqslant w$ implies $z\geqslant w$. As usual, we are interested to find conditions under which any $x\in X$ be majorized by a $\leqslant$-maximal $z\in X$. (The ambient quasi-ordering $\leqslant$ will be referred to as a *Zorn quasi-ordering* in such a case). It must be noted that the appropriate framework of the problem is the pseudometric one. Letting d be such an object, we denote by $d(x, Y)$ (for $x\in X$ and $Y\subseteq X$) the infimum of all $d(x, y)$, $y\in Y$. The pseudometric in question will be termed *weak* $\leqslant$*-asymptotic* when, for each $x\in X$, $\varepsilon>0$, there exists $y=y(x, \varepsilon)\geq x$ with

$$d(u, X(v, \leq))<\varepsilon,\ \text{for}\ y\leqslant u\leqslant v.$$

To get sufficient conditions under which

(C_8) d is weak $\leqslant$-asymptotic

be fulfilled, call a sequence (x_n) in X *weak d-asymptotic* in case $\inf\{d(x_n, x_m);\ n<m\}=0$. Then, clearly,

(C_8') each ascending sequence is weak d-asymptotic

is one of these, as it can be directly seen. Let also $\leftarrow$ stand for the dual convergence structure over X, induced by d (under a previous convention). We say $\leqslant$ is *weak d-self-closed* when, for each $x\in X$ there exists $y\geq x$ with the property that each ascending sequence (y_n) in $X(y, \leqslant)$ with $x\leftarrow y_n$ admits y as an upper bound.

Of course, a sufficient condition for

(C_9) $\leqslant$ is weak d—self-closed

is (with $y=x$)

(C_9') $\leqslant$ is d-self-closed (mod. $\leftarrow$)

by the very definition of this property.

Theorem E. *Under the general boundedness assumption* (C_7), *let the specific conditions* (C_8)+(C_9) *be admitted. Then,* $\leqslant$ *is a Zorn quasi-ordering.*

Proof. Let the function $F: X \to R_+$ be defined as

$$F(x) = \sup\{d(u, X(v, \le));\ x \le u \le v\},\quad x \in X.$$

(That such a convention is meaningful, results from (C_8) plus the remark that, in all these situations, X may be substituted by $X(y, \le)$ where $y \ge x$ is that entering in the underlying condition). Clearly, F is decreasing and bounded from below; hence, by (C_1) conditions of Theorem D are fulfilled. Given the point $x \in X$, there exists a F-maximal point $y \in X$ with $x \le y$; note that, by (C_8) again, $F(y) = 0$. Let $z \le y$ be that introduced by (C_9): we claim z is $\le$-maximal. Indeed, letting $w \ge z$, we have, in view of

$$d(y, X(v, \le)) = 0, \text{ for all } v \le w.$$

that an ascending sequence (v_n) in $X(w, \le)$ may be determined so that

$$d(y, v_n) < 2^{-n},\ n \in N \text{ (hence } y \leftarrow v_n).$$

This, along with the choice of z, gives $v_n \le z$, $n \in N$, wherefrom $w \le z$. The proof is complete. q.e.d.

As already said, sufficient conditions for (C_8) and (C_9) are (C_8') and (C_9') respectively. It is not without interest to point out that, with $(C_8')+(C_9')$ in place of $(C_8)+(C_9)$, the basic boundedness condition (C_1) is equivalent with its (weaker in general) counterpart

(C_1') each ascending d-asymptotic sequence is bounded above.

In fact, let (x_n) be an ascending sequence in X. That the property below is holding

for each $\varepsilon < 0$, there exists $n(\varepsilon) \in N$ such that $n(\varepsilon) \le n \le m$ implies $d(x_n, \{x_r;\ m \le r\}) < \varepsilon$

is almost evident, by (C_8'); so, we omit the details. In this case, it is not hard to build up an ascending sequence of ranks $(k(n): n \in N)$ with

(P_4) $$k(n) \le p \le q \Rightarrow d(x_p, \{x_r;\ q \le r\}) < 2^{-n}.$$

Now, let $h(1) \in [k(1), k(2)]$ be a given rank. By (P_4) a rank $h(2) \in [k(2), \infty)$ may be found with $d(x_{h(1)}, x_{h(2)}) < 2^{-1}$. Without loss, one may suppose $h(2) < k(3)$; in such a situation, by (P_4) again, a rank $h(3) \in [k(3), \infty)$ may be chosed so that $d(x_{h(2)}, x_{h(3)}) < 2^{-2}$, etc.. Inductively, we get an ascending sequence of ranks $(h(n);\ n \in N)$ in N, with

(P_5) $$d(x_{h(n)}, x_{h(n+1)}) > 2^{-n}, \text{ for all } n \in N.$$

Hence, $(x_{h(n)})$ is an ascending d-asymptotic sequence which, by (C_1'), is bounded above. This, in conjunction with the choice of (x_n) plus the cofinality of $(h(n);\ n \in N)$ in $(N, \le)$, ends the argument. Observe, incidentally, that when d is a semi-metric over X, the sequence $(x_{h(n)})$ constructed above is, by (P_5), d-Cauchy; so, in this case, we may substitute in (C_1'), d-asymptotic with d-Cauchy, to produce the same conclusion. This semi-metric version of Theorem E has been obtained by T u r i n i c i [21], through a direct Zorn procedure. Returning to the initial framework, note that, by the argument followed, Theorem E appears as a consequence of Theorem D. But, the reciprocal of this implication is also valid; this will be proved in the next section.

The statements subsumed to Theorems C—D admit denumerable versions, as follows. Let $\mathfrak{D} = (d_i;\ i \in N)$ be a sequence of pseudo-metrics over X. Then, the pseudo-uniformity (over X)

$$(\mathfrak{U}(\mathfrak{D}) = \{U(d), [0, \varepsilon));\ d \in \mathfrak{D},\ \varepsilon > 0\}$$

is easily shown to be metrizable. In particular, the sequence $\mathfrak{D}$ may be generated by the convention (D_1), where $\mathfrak{F} = (F_i,\ i \in N)$ is a sequence of functions from X into R.

Theorem F. *Let the sequence of pseudo-metrics $\mathfrak{D}$ be such that, for each $d \in \mathfrak{D}$, condition (C_2) (or (C_2')) is holding (modulo d). Then, if in addition (C_1) takes place, it is the case that each $x \in X$ is majorized by a $\mathfrak{D}$-maximal element $z \in X$ (introduced as: $z \le u \le v$ implies $d(u, v) = 0$ for all $d \in \mathfrak{D}$).*

Corollary: *Let $\mathfrak{F}$ be a sequence of functions from X into R, fulfiling (C_2). Then (under (C_1)) for each $x \in X$ there exists $z \in X$ with a) $x \le z$, b) $z \le w$ implies $F(z) = F(w)$, $F \in \mathfrak{F}$.*

This result is clearly equivalent with the above ones. It has been obtained by the author in [20], through a direct method. As a matter of fact, a denumerable version of Theorem B is also available; but, we shall not explicitly consider this fact in the sequel. For the non-denumerable (general) case we refer to [22].

2. The main results: a maximal-constructive approach. Let X be a nonempty set and $\le$, a quasi-ordering over X. We also give a semi-metric d over X: note that, by the symmetry properties, the convergence structures $\to$ and $\leftarrow$ introduced in the above section will coincide. As a substitute for Theorem 1, the following variational principle is proposed.

Theorem 2. *Suppose X is $(d, \le)$-complete, $\le$ is d-self-closed, and $F: X \to R$ is decreasing and bounded from below. Then, for each $\varepsilon > 0$, $\lambda > 0$ and $u \in X(F, \varepsilon)$ there exists $v = v(\varepsilon, \lambda, u)$ in $X(F, \varepsilon)$ with*

a') $u \le v$, $d(u, v) < \lambda$, $F(u) \le F(v)$

b') $F(w) - F(v) + (\varepsilon/\lambda)d(v, w) > 0$, *for each* $w \in X(v, \le)$, $d(v, w) \ne 0$.

That such a statement includes Theorem 1 is relatively simply to verify. Indeed, letting X, d, F be as in that result, let us introduce a quasi-ordering $\le$ on X by the convention

$$x_1 \le x_2 \text{ if and only if } F(x_1) \le F(x_2).$$

Clearly conditions of Theorem 2 are fulfilled by the triplet $(X, \le, d)$ and the same function F. Hence, for the given $\varepsilon > 0$, $\lambda > 0$ and $u \in X(F, \varepsilon)$, there exists $v = v(\varepsilon, \lambda, u)$ in $X(F, \varepsilon)$ such that a') + b') above are true. From these, a) is clear. Moreover, let $w \in X$ be such that $d(v, w) \ne 0$ (hence > 0). The expected relation b) will be surely valid whenever $F(v) \le F(w)$. So, without loss one may assume $F(v) \le F(w)$, that is, $v \le w$. But, in this case, b) results directly from b'). Hence, anyway, b) is true and the conclusion follows.

The implication between these results seems to be strict, in view of the fact that *lsc* and decreasing are not reductible to each other; for example, when $X = R$, the function

$$F(x) = \operatorname{sgn}(-x),\quad x \in X$$

is clearly decreasing but not *lsc*. However, as we shall see below, the converse implication is also valid (making these results be technically equivalent). This cannot be done without firstly checking the validity of Theorem 2. It is exactly our main aim to supply a proof of this result, via Theorem D. Further, we show that, actually, this last statement is a consequence of Theorem 1; therefore, we closed the circle between these results. The obtained equivalence is, perhaps, interesting in itself, but not very productive from a variational viewpoint;

this because the differential interpretation of b′) is rather different from that of b). These facts will be clarified in Section 3.

Proof of Th. 2 via Th. D. Let us introduce a new quasi-ordering $\leqq$ on X, by the convention

(D_1) $x \leqq y$, if and only if $x \leqslant y$, $(\varepsilon/\lambda)d(x, y) \leqslant F(x) - F(y)$.

Suppose (x_n) is an ascending (modulo $\leqq$) sequence in X, i.e.

(1) $$x_n \leqslant x_m, \ (\varepsilon/\lambda)d(x_n, x_m) \leqslant F(x_n) - F(x_m), \quad n \leqslant m.$$

It immediately follows by the boundedness property imposed upon F that (x_n) is an ascending (modulo $\leqslant$) d-Cauchy sequence in X. By the $(d, \leqslant)$-completeness property we admitted, $x_n \to x$ as $n \to \infty$, for some $x \in X$. This, combined with $\leqslant$ being d-self-closed, yields

(2) $$x_n \leqslant x \ (\text{hence } F(x_n) \leqslant F(x)), \text{ for all } n \in \mathbf{N}.$$

As a consequence, the second relation in (1) gives

$$(\varepsilon/\lambda)d(x_n, x_m) \leqslant F(x_n) - F(x_m), \quad n \leqslant m.$$

Now, take the limit as $m \to \infty$ in this relation. We get

(3) $$(\varepsilon/\lambda)d(x_n, x) \leqslant F(x_n) - F(x), \quad n \in \mathbf{N}.$$

which, combined with (2), gives

$$x_n \leqq x, \ \text{for all } n \in \mathbf{N};$$

in other words, (x_n) is bounded above (modulo $\leqq$). On the other hand, F is decreasing from $(X, \leqq)$ to R; hence, Theorem D is applicable. By the conclusion of that result we have that, for the given $u \in X(F, \varepsilon)$, a point $v \in X$ exists with

a) $u \leqq v$, b) $v \leqq w$ implies $F(v) = F(w)$.

The former of these relations reads

$$u \leqslant v, \ (\varepsilon/\lambda)d(u, v) \leqslant F(u) - F(v) \leqslant F(u) - \inf F(X) < \varepsilon;$$

and, from this, $d(u, v) < \lambda$. The latter may be written as

$$v \leq w, \ (\varepsilon/\lambda)d(v, w) \leq F(v) - F(w) \Rightarrow F(v) = F(w) \ (\text{hence, } d(v, w) = 0).$$

But, this is just b′). The proof is complete. q.e.d.

Perhaps it is not without importance to point out that, as a matter of fact, d need not be a semi-metric in Theorem 2, but merely a pseudo-metric satisfying the following kind of triangular property

(C_3'') $d(x, z) \leqslant d(x, y) + d(y, z)$, whenever $x \leqslant y \leqslant z$.

Indeed, if this condition is admitted, we have (in view of (2))

$$(\varepsilon/\lambda)d(x_n, x) \leqslant (\varepsilon/\lambda)d(x_n, x_m) + (\varepsilon/\lambda)d(x_m, x) \leqslant F(x_n) - F(x_m) + (\varepsilon/\lambda)d(x_m, x) \leqslant$$
$$\leqslant F(x_n) - F(x) + (\varepsilon/\lambda)d(x_m, x), \quad n \leqslant m.$$

Therefore, passing to limit as $m \to \infty$, ons re-obtains (3); this ends the argument. We preferred, however, the semi-metric language for a number of technical reasons of a variational nature.

So, far, Theorem 2 (hence Theorem 1) is a consequence of the Brezis-Browder ordering principle subsumed to Theorem D. Before answering to the converse question let us show the result under consideration is also a particular case of Theorem E. In fact, by the argument above, we have that each ascending (mod. $\leqq$) sequence in X is necessarily convergent (hence weak d-asymp-

totic). On the other hand, the limit of any ascending (modulo $\leqq$) sequence is, by the above discussion, an upper bound of it (mod. $\leqq$); hence $\leqq$ is d-delf closed. This shows Theorem E applies to our elements. And consequently, for each $x \in X$ there exists $z \in X$ with

a″) $x \leqq z$, b″) $z \leqq w$ implies $w \leqq z$.

Now, a″) gives the first assertion in Theorem 2. Moreover, b″) implies $F(z) = F(w)$; hence $d(z, w) = 0$, and the conclusion is clear.

We may now return to the above mentioned problem, which may be- re-formulated as: Is Theorem 2 a genuine particular case of the Brezis-Browder ordering principle? At a first glance, the answer must be positive, by the fact that, with respect to the semimetric d constructed as in (D_4), no completeness property is to be deduced for the associated structure (X, d), under the as- sumptions of Theorem D. But, if we note that, by a result in Brunner (see the introductory part) Theorem 1 is equivalent with the Principle of De- pendent Choices, a negative answer is more probable. It is exactly our main objective to show this is indeed the case, in the sense: Theorem D is a conse- quence of Theorem 1 (hence, a fortiori, of Theorem 2). This, combined with the preceding discussion, may be also depicted as: the variational principles subsumed to these theorems are equivalent with any of the maximality results in the preceding section.

Proof of Th. D via Th. 1. Suppose $(X, \leqslant)$ is a quasi-ordered space ful- filing (C_1), and let $F: X \to R$ be a function with the properties listed in (C_7). Given the arbitrary fixed point $x \in X$ we may construct, be the same procedure as in [20], an ascending sequence (x_n) in $X(x, \leqslant)$ with

$$F(x_n) - F(x) < 2^{-n}, \ \text{whenever } x_n \leqslant x, \ n \in \mathbf{N}.$$

In view of (C_1), (x_n) has upper bounds. Letting y be one of these, put $Y = X(y, \leqslant)$. It is clear, by the above properties, that (Y, d) is complete, where $d: X^2 \to [0, \infty)$ is the semi-metric attached to F under the model of (D_4). In addition, the (bounded from below) function F is trivially shown to be *lsc* over Y. Hence, Theorem 1 applies to the triplets (Y, d, F) and $(\varepsilon, \lambda, y)$, where $\varepsilon, \lambda > 0$ are taken in such a way that $0 < \varepsilon/\lambda < 1$. This yields a point $z \in Y$ with the property b) of that statement, namely

$$(\varepsilon/\lambda)|F(z) - F(w)| \leq F(z) - F(w) \Rightarrow F(z) = F(w).$$

But, evidently, the left relation of this implication is true whenever $z \leq w$; hence, z is F-maximal, as desired. q.e.d.

The following technical remark about the components of Theorem 2 should be retained. Suppose $\leqslant$ is the trivial quasi-ordering on X: then, the introduced function F is decreasing if and only if it is constant. In other words, for non-trivial quasi-orderings, Theorem 2 appears as a completion of Theorem 1. This raises the question of whether or not a common (formal) extension of these results may be formulated. The answer is positive and may be given along the following lines. We term the function $F: X \to \mathbf{R}$, $\leqslant$*-lower semi- continuous*, when

(x_n) ascending, $x_n \to x$ and $F(x_n) \leqslant \lambda$, for all $n \in \mathbf{N}$ imply $F(x) \leqslant \lambda$.

Clearly, lsc implies $\leqslant$-*lsc*; and, when $\leqslant$ is d-self-closed, the decreasing pro- perty also gives $\leqslant$-*lsc* (without implying *lsc*, in general, as underlined in the introductory part).

Theorem 3. *Assume that, in Theorem 2, decreasing is replaced by $\leqslant$ - lsc. Then, conclusions of that result are valid.*

A direct proof of this assertion may be given by the same procedure as that used in deducing Theorem 2 from Theorem 1). As a variant of this, let $\leqq$ be the quasi-ordering on X defined by (D_1'). Clearly, conditions of Theorem 2 are applicable to the triplet $(X, \leqq, d)$ and the same function F; this ends the argument.

Formally, Theorem 2 appears now as an intermediary step towards Theorem 3. So, it is legitimate to ask about the reason of treating it separately in this series of results. The answer is the following: in addition to giving us a new form of Ekeland's principle, the specific assumptions in Theorem 2 allow — in a vectorial perspective — a more general treatment on the boundedness conditions imposed upon F, than those in Theorem 3. To be more specific, assume the co-domain, $\mathbf{R}$, of the function F, is to be replaced by a normed space S, ordered by a *cone*. We call the function $F: X \to S$, *semi-bounded below*, in case $F(X(u, \leqslant))$ is bounded from below for each $u \in X$. Of course, condition

(C_7) F is bounded below

is a particular case of

(C_7') F is semi-bounded below.

The former of these allows F be $\leqslant$-*lsc* or decreasing. But, the latter cannot be handled — to produce conclusion in Theorem 2, with $X(u, \leqslant)$ in place of X — without a decreasing assumption about F. Some further aspects of the problem (treated, in some particular cases, by N e m e t h [15]) will be discussed elsewhere.

Now, Theorem 3 is not only useful as a (formally) cumulative result, but also as a step in deriving a (quasi-) order version of the smooth variational principle subsumed to Theorem 1*. To do this, let the triplet $(X, \leqslant, d)$ be as in Theorem 3. For $p \geqslant 1$ we consider the clase $\Gamma_p(\leqslant)$ of all functions

$$G(x) = \sum_n \mu_n (d(x, v_n))^p, \qquad x \in X,$$

where (μ_n) is a sequence of positive numbers with $\sum_n \mu_n = 1$, and (v_n) is an ascending (mod. $\leqslant$) convergent sequence in X. The convergence of this functional series is uniform over bounded subsets of X (by the Weierstrass criterion). Hence, the (partial sum) functional sequence

$$G_n(x) = \sum_{k \leqslant n-1} \mu_k (d(x, v_k))^p, \ x \in X\,;\ n = 1, 2, \ldots$$

converges towards the function G, uniformly on bounded subsets of X. This in particular says G is continuous and, moreover

(P_6) $\qquad G_n(x_n) \to G(x)$ as $n \to \infty$, whenever $x_n \to x$.

Theorem 3'. *Under the above choice of the triplet $(X, \leqslant, d)$, suppose $F: X \to R$ is $\leqslant$-lsc and bounded from below. Then, given the constants $\varepsilon > 0$, $\lambda > 0$, $p \geqslant 1$ and the point $u \in X(F, \varepsilon)$, there may be determined a $G \in \Gamma_p(\leqslant)$ and a $v \in X(F, \varepsilon)$ with the properties a') of Theorem 2, plus*

c*) $F(w) - F(v) + (\varepsilon/\lambda^p)(G(w) - G(v)) \geqslant 0$, *for each* $w \geqslant v$.

In particular, a sufficient condition for the $\leqslant$-lsc is the decreasing property (upon F).

Proof. We shall follow the procedure in B o r w e i n and P r e i s s [1], with some modifications. Define the constants $(\varepsilon_1, \varepsilon_2)$ and (θ, μ, δ) by

$$F(u) - \inf F(X) < \varepsilon_2 < \varepsilon_1 < \varepsilon, \quad 0 < \mu < 1 - (\varepsilon_1/\varepsilon)$$
$$0 < \theta < \mu(1 - (\varepsilon_2/\varepsilon_1)^{1/p})^p, \quad \delta = (1 - \mu)(\varepsilon/\lambda^p).$$

Also, denote $F_0 = F$, $v_0 = u$. Suppose that $(F_i\,;\ 0 \leqslant i \leqslant n)$ and $(v_i;\ 0 \leqslant i \leqslant n)$ have been constructed. We put

(D_5) $\qquad F_{n+1}(x) = F_n(x) + \delta\mu^n (d(x, v_n))^p, \quad x \in X$

and choose $v_{n+1} \in X(v_n, \leqslant)$ so that

(D_6) $\qquad F_{n+1} \in (v_{n+1}) \leqslant \theta F_n(v_n) + (1 - \theta) \inf F_{n+1}(X(v_n, \leqslant)).$

To see the effectiveness of this last construction, we note that

$$F_n(v_n) = F_{n+1}(v_n) \geqslant \inf F_{n+1}(X(v_n, \leq)).$$

So, if the infimum is attained at v_n, one takes $v_{n+1} = v_n$; otherwise, use the definition of the infimum to get the desired point, $v_{n+1} \geqslant v_n$. Hence the iterative process $(D_5) + (D_6)$ is indefinitely constructible; it gives us a sequence (v_n) which is, until now, ascending. To establish its convergence, one may proceed as follows. Denote

$$\alpha_n = F_n(v_n), \quad \sigma_n = \inf F_n(X(v_n, \leqslant)), \quad n \in \mathbf{N}.$$

By the above definitions, one has

$$\sigma_n \leqslant \sigma_{n+1} \leqslant \alpha_{n+1} \leqslant \theta\alpha_n + (1 - \theta)\sigma_{n+1} \leqslant \alpha_n, \quad n \in \mathbf{N}.$$

In other words, (σ_n) is increasing, (α_n) in decreasing and $\sigma_n \leqslant \alpha_n$, for all $n \in \mathbf{N}$; moreover

$$\alpha_{n+1} - \sigma_{n+1} \leqslant \theta(\alpha_n - \sigma_n), \quad n \in \mathbf{N}.$$

From this, we deduce

$$\alpha_n - \sigma_n \leqslant \theta^n (\alpha_0 - \sigma_0), \quad n \in \mathbf{N},$$

which in particular says (σ_n) and (α_n) have the same limit. On the other hand,

$$\alpha_n \geqslant \alpha_{n+1} = F_n(v_{n+1}) + \delta\mu^n (d(v_n, v_{n+1}))^p \geqslant \sigma_n + \delta\mu^n (d(v_n, v_{n+1}))^p, \quad \text{for all} \quad n \in \mathbf{N},$$

wherefrom, combining with the relation we just obtained

(4) $\qquad d(v_n, v_{n+1}) \leqslant ((\alpha_n - \sigma_n)/\delta\mu^n)^{1/p} \leqslant (\theta/\mu)^{n/p} (\varepsilon_2/\delta)^{1/p}, \quad n \in \mathbf{N}.$

This shows (v_n) is an ascending d-Cauchy sequence. By the completeness hypothesis we admitted, (v_n) is thus convergent; and, in such a case, formula

(D_7) $\qquad G(x) = (1 - \mu) \sum_n \mu^n (d(x, v_n))^p, \quad x \in X,$

defines an element of $\Gamma_p(\leqslant)$. Let v stand for any of the points in $\lim_n (v_n)$. We claim (G, v) is the desired couple to satisfy the imposed requirements. Clearly, $u \leqslant v$ (as $\leqslant$ is d-self-closed). On the other hand, by (4) plus the relations introducing these constants

$$d(u, v) \leqslant \sum_n d(v_n, v_{n+1}) \leqslant (\varepsilon_2/\delta)^{1/p}/(1 - (\theta/\mu)^{1/p}) < (\varepsilon_1/\delta)^{1/p} < ((1-\mu)\varepsilon/\delta)^{1/p} = \lambda.$$

We also have (the sequence (G_n) being defined as in a previous place, with (μ_n) given by (D_7))

$$F_n = F + (\varepsilon/\lambda^p) G_n, \quad \text{for all} \quad n \geqslant 1.$$

This clearly gives

$$F(v_n)+(\varepsilon/\lambda^p)G_n(v_n)=\alpha_n\leqslant\alpha_0=F(u),\ n\geqslant 1.$$

Combining this with (P_6) plus the $\leqslant$-*lsc* property, one gets

$$F(v)\leqslant F(v)+(\varepsilon/\lambda^p)G(v)\leqslant\lim_n(\alpha_n)\ (\text{hence } F(v)\leqslant F(u)).$$

Finally, let $w\geqslant v$ be arbitrary fixed. We may write

$$F(w)+(\varepsilon/\lambda^p)G(w)=\lim_n F_n(w)\geqslant\lim_n\sigma_n=\lim_n\alpha_n.$$

This, plus the relation we just we derived, yields the conclusion c^*). The proof is complete. q.e.d.

Now, by the non-expansiveness of the functions in $\Gamma_1(\leqslant)$ — see the introductory part — an immediate consequence of c^*) is

b'') $F(w)-F(v)+(\varepsilon/\lambda)d(v,w)\geqslant 0$, for all $w\geqslant v$.

Concerning this fact, we note that, in the context of Theorem 2, the property b'') is nothing but an equivalent form of b'). Indeed, by Theorem D have that the point v appearing in this conclusion is F — maximal. Hence

$$F(v)=F(w),\ \text{as soon as } v\leqq w\,;$$

and, from this, plus the decreasing property of the function F, the assertion follows under an assumption like $d(v,w)=0$. The remaining situation ($d(v,w)\neq 0$) is supplied by b'); hence the conclusion. In the context of Theorem 3, this argument is no longer working, because F is not, in general, decreasing. But, in compensation to this, it is not hard to see that, as an easy consequence of the $\leqslant$-*lsc* property admitted,

$$F(x)=F(y)\ \text{whenever}\ d(x,y)=0\,;$$

therefore, b'') is again deductible (in this larger context). In particular, this tells us Theorem 3* above includes Theorem 3. It also includes Theorem 1* (to which it reduces when $\leqslant$ is the trivial quasi-ordering over X).

A technical question to be solved is that concerning the stationarity of the sequence (v_n) appearing in the construction above. Note that, as underlined in the above quoted paper (see also P h e l p s [18, ch. IV]) one may arrange the proof in Theorem 1* so that (in the metric context of that result)

(P_7) $$v_n\neq v,\ \text{for all } n\in\mathbf{N}.$$

It is natural then to ask whether a similar property is valid for Theorem 3*. The answer is affirmative, but under the stronger assumption.

$$\leqslant \text{ is an ordering over } X.$$

To verify this, two cases must be taken into account.

Case 1. The process (D_5)+(D_6) gives a sequence (v_n) in X with

$$v_n\neq v_{n+1},\ \text{for all } n\in\mathbf{N}.$$

We have, in such a situation

$$v_n<v_{n+1}\leqslant v_m,\ \text{for } n<m.$$

Hence, passing to limit as $m\to\infty$, one obtain (P_7), if we remember that $\leqslant$ is d-self-closed.

Case 2. The sequence (v_n) in X given by (D_5)+(D_6) fulfils

$$v_m=v_{m+1},\ \text{for at least one } m\in\mathbf{N}.$$

Let m be the first index with such a property. By the accepted conventions, this may occur if and only if F_{m+1} attains its infimum over $X(v_m,\leqslant)$ at v_m. In view of (D_5) (written for $n=m$) it is clear that F_{m+2} attains its infimum over $X(v_{m+1},\leqslant)$ at v_{m+1}. This, again by the above conventions, says $v_{m+1}=v_{m+2}$, and so on. Summing up, (v_n) is constant for $n\geqslant m$; namely

(P_7') $$v_0<\ldots<v_m=v_{m+1}=\ldots=v,$$

which is the concrete interpretation of (P_7) in this case. Of course, under such a circumstance, the functional series defining G is actually finite

$$G(x)=\sum_{n<m}\delta\mu^n(d(x,v_n))^p+\Big(\sum_{n\geqslant m}\delta\mu^n\Big)(d(x,v_m))^p,\quad x\in X.$$

We also note that, by the very constructive character of its proof, a vector extension of Theorem 3* is to be obtained in a more difficult way than the one described in a previous place for Theorem 3. At the same, the question of considering Theorem 3* beyond the semi-metric framework cannot be positively solved, in view of the essential role assumed by the triangle inequality in the deduction of the key property. (4).

§3. A variational approach (the non-constrained case). The information contained in the main results cannot be successfully exploited in the absence of an appropiate variational (differential) interpretation. It is our main objective in the following to establish some facts of this type. These are especially related to Theorem 2 and consist in a characterization, in terms of Gâteaux derivatives, of the almost minimum points for the non-constrained minimization problem (attached to the function $F:X\to R$)

(MP) $$\min\{F(x)\,;\,x\in X\}.$$

(The constrained version of (MP) will be discussed elsewhere).

To begin with, let X be a normed space. We shall denote by X' the *algebraic dual* of X and by X^* the *topological dual* of the same. Given a function $F:X\to R$, we shall term it *Gâteaux differentiable* (in the *algebraic* sense) over X when, for each $x\in X$, the number

(D_8) $$F'(x)(a)=\lim_{t\to 0+}t^{-1}(F(x+ta)-F(x))$$

exists for each $a\in X$ and the function $a\to F'(x)(a)$ is an element of X'. It will be termed *Gâteaux differentiable* — in the *topological* sense — over X when the above assertion is valid with X' replaced by X^*. Now, technically speaking, the variational characterizations we are looking for emerge from the conclusion b') in Theorem 2. This, however, cannot be handled in the traditional way encountered in the amorph case (of Theorem 1) because only points in $X(v,\leqslant)$ are involved. We therefore need some supplementary structural hypotheses about the space X, related, especially, to its order properties. To describe them, let us take $\leqslant$ as an ordering over X, induced by a (*convex*) *cone* X_+. We firstly admit that

(C_{10}) $\sup(x,y)$ (hence $\inf(x,y)$) exists, for all $x,y\in X$,

the ambient space X being termed a *vector lattice*, under the B i r k h o f f terminology [2, ch. XV, §9]. Denote

$$|x|=\sup(x,-x),\quad x\in X.$$

The basic facts about the mapping $x \to |x|$ are to be found, e. g., in C r i s t e s c u [10, ch. VII, § 1]. Here we only recall that it possesses the well known properties of a norm (with values in X_+). Now, the second assumption about our elements is that the ambient norm of X is a *solid* one, in the sense

(C$_{11}$) $$|x_1| \leqslant |x_2| \Rightarrow \|x_1\| \leqslant \|x_2\|.$$

The obtained structure X is being termed a *normed lattice*, under the imposed conditions; for locally convex extensions we refer to P e r e s s i n i [17, ch. II, § 4]. Immediate consequences of these assumptions are

(P$_8$) x and $|x|$ have the same norm (for each $x \in X$)

(P$_9$) X_+ is a closed cone in X (or, equivalently, $\leqslant$ is a *closed ordering*, in the sense of N a c h b i n [14, Appendix]);

for details, we refer to C r i s t e s c u (op. cit.).

Now, for al element $f \in X'$, increasing is identical with positive (in the sense)

(D$_9$) $$f(x) \geqslant 0, \text{ for all } x \in X_+).$$

The class of all such functionals will be denoted by X'_+. The following characterization of continuity for these objects will be in effect for us.

Lemma 1. *Let $f \in X'$ be positive. Then, f is continuous, if and only if*

$$L(f) = \sup\{f(x) ;\ \|x\| \leqslant 1,\ x \in X_+\}$$

is finite. Moreover, $\|f\| = L(f)$.

Proof. Let $x \in X$ be arbitrary fixed. We have $-x, x \leqslant |x|$; so, by positivity, $|f(x)| \leqslant f(|x|)$. This, in conjunction with (P$_8$), gives

$$|f(x)| \leq |L(f), \quad x \in X, \quad \|x\| \leq 1,$$

wherefrom $\|f\| \leqslant L(f)$. The converse relation being evident, we have $\|f\| = L(f)$. This ends the proof. q. e. d.

In the following, we also need a completeness (modulo $\leq$) property of X with respect to the ambient norm; that is

(C$_{12}$) each ascending (mod. $\leqslant$) norm-Cauchy sequence in X converges (to an element of X).

Of course, any *Banach lattice* — introduced as in B i r k h o f f (see a previous reference) — represents an example of such a structure.

We are now in position to formulate the first main result of the present section.

Theorem 4. *Let the conditions (C$_{10}$)—(C$_{12}$) be fulfilled and the function $F : X \to R$ be decreasing, bounded from below and Gâteaux differentiable over X in the algebraic sense. Then, for each $\varepsilon > 0$, $\lambda > 0$, and $u \in X(F, \varepsilon)$ there exists $v = v(\varepsilon, \lambda, u)$ in $X(F, \varepsilon)$ with the properties a') of Theorem 2, plus*

b') $$-F'(v) \in X^*_+, \quad \|F'(v)\| \leq \varepsilon/\lambda.$$

Proof. Let ε, λ and u be as above. By Theorem 2, we have promised a $v \in X(F, \varepsilon)$ with the properties a')+b') given there. The second one may be put in the form

$$0 \leqslant F(v) - F(w) \leq (\varepsilon/\lambda)\|v - w\|, \text{ for all } w \geqslant v;$$

In particular, for $w = v + ta$, $t > 0$, $a \in X_+$, one gets

$$0 \leqslant t^{-1}(F(v) - F(v + ta)) \leqslant (\varepsilon/\lambda)\|a\|, \quad t > 0, \quad a \in X_+$$

So, passing to limit as $t \to 0+$

$$0 \leqslant -F'(v)(a) \leqslant (\varepsilon/\lambda)\|a\|, \qquad a \in X_+.$$

This, in conjunction with Lemma 1, ends the argument. q. e. d.

It is interesting to note that, in the absence of a decreasing property for F, this result is not in general valid, because $-F'(v)$ is not a positive (linear) functional. This reveals the essential role of such an assumption, for the argument above. For example, in the setting of Theorem 3, the conclusion b$^\circ$) cannot be retained. (That is, the result in question does not admit a variational interpretation. This motivates — from a „practical" perspective as well — the position of Theorem 2 within the series of results in the above section).

Now, as in the amorph case (E k e l a n d [12]), let us consider the problem of determining sufficient conditions for the existence of an exact minimum point for (MP). Unfortunately, the conditions in question are quite restrictive. To describe them, we need, however, some preliminary facts. Let the ordered normed space X be taken as above. For $a, b \in X$, $a \neq b$, we denote by $[a, b]$ (the *linear segment* between a and b) the image of the (real) interval $[0, 1]$ through the mapping $t \mapsto a + t(b - a)$. Of course, the restriction to $[a, b]$ of the normed structure of X is equivalent (via this mapping) to that of $[0, 1]$; this is also valid for the ordering structure, provided $a < b$. Let $F : X \to R$ be a function which is supposed to be Gâteaux differentiable (in the topological sense). It is clear that, in such a case, F is *segment-continuous* its restriction to any linear segment of X is continuous), but not, in general, continuous; this, however, will suffice for our purposes. Concerning its Gâteaux derivetive, we must admit

(C$_{13}$) F' is *segment-continuous* (mod. $\leqslant$) (that is, its restriction to any linear segment $[u, v]$ of X with $u < v$, is continuous).

This appears to be a strong hypothesis. For example, (C$_{13}$), is fulfilled when

(C$_{13}'$) F' is continuous (from X to X^*).

Note that, in such a case, F must be Fréchet differentiable (hence, in particular, continuous) and its Fréchet derivative is identical with F'; see the above quoted Ekeland's paper for details. The following simple fact is to be noted.

Lemma 2. *Under the above precised assumptions, let $a, b \in X$, $a < b$, be such that $F(a) > F(b)$, $F'(a) = 0$. Then, for each $\varepsilon > 0$, a point $x = x(\varepsilon)$ in $]a, b[$ will exist with $|F(x) - F(a)| < \varepsilon$, $0 < \|F'(x)\| < \varepsilon$.*

Proof. Denote, for simplicity

$$c = \sup\{s \in [a, b] ;\ F'(t) = 0,\ a \leqslant t \leqslant s\}.$$

By (C$_{13}$), we clearly have

$$F'(t) = 0, \text{ for all } t \text{ in } [a, c].$$

This in turn gives, by the mean value theorem in A u m a n n [1, ch. VII, § 1] that $F(a) = F(c)$. Hence $c < b$; of course, the case $c = a$ is not excluded. Now, in any interval $]c, d[$, $c < d < b$, there must be a point x_d for which $F'(x_d) \neq 0$. Hence, we may exhibit a decreasing sequence (x_n) in $]c, b[$ with

$$x_n \to c \text{ (hence } F(x_n) \to F(c) = F(a)) \text{ as } n \to \infty ;\ F'(x_n) \neq 0, \text{ for all } n \in \mathbf{N}.$$

On the other hand, by (C_{13}) once again $F'(x_n) \to F'(c) = 0$, as $n \to \infty$. This ends the argument. q. e. d.

We need one more convention. Let $F: X \to \mathbf{R}$ be a Gâteaux differentiable (in the topological sense) function. We say F satisfies a *Palais-Smale condition* (mod. $\leqslant$) in case

(C_{14}) each ascending sequence (x_n) with $F'(x_n) \neq 0$, $n \in N$, $F'(x_n) \to 0$ as $n \to \infty$ and $(F(x_n))$ = bounded (in $\mathbf{R}$) sequence, has a cluster point (in X).

This is related with the so-called „condition (C)" (obtained from (C_{14}) by deleting the word „ascending") used by the above quoted authors in their 1964 paper [16] devoted to global analysis. As a matter of fact, we shall apply (C_{14}) only to functions $F: X \to R$ which are bounded from below and to ascending sequences (x_n) with

(x_n) is contained in $X(F, \eta)$, for some $\eta > 0$.

But, in the following, we shall not effectively take into consideration this aspect. To establish a concrete circumstance under which (C_{14}) is valid, let us call the mapping $E: X \to X^*$, F — *compact* (modulo $\leqslant$) when

(x_n) ascending in X, and $(F(x_n))$ = bounded imply $(E(x_n))$ is relatively compact (in X^*).

Also, we shall say $G: X \to X^*$ is *F-admissible* (mod. $\leqslant$) in case

(x_n) ascending in X, $(F(x_n))$ = bounded and $(G(x_n))$ = convergent (in X^*) imply (x_n) has a convergent subsequence.

The following structural hypothesis about F' is now a particular case of the introduced condition

(C_{14}') F' may be represented as the sum $E + G$ where E is F-compact (mod. $\leqslant$) and G id F-admissible (mod. $\leqslant$).

The proof. of this assertion is almost evident; so, we omit the details. The idea of using this structural hypothesis goes back to P a l a i s and S m a l e (op. cit.); see also Z e i d l e r [25, ch. 38, §10].

We are now prepared for an appropriate answer to the problem we already formulated. The conditions in Theorem 4 prevail.

Theorem 5. *Let the function* $F: X \to R$ *be decreasing, bounded from below and Gâteaux differentiable (in the topological sense) over* X. *Suppose also that* $(C_{13})-(C_{14})$ *are holding. Then, the non-constrained minimization problem* (MP) *has an exact solution* $v \in X$ *with* $F'(v) = 0$.

Proof. By the admitted conditions $(C_{10})-(C_{12})$, we have that $(X, \leqslant)$ is directed. This, plus the decreasing property of the function F, yields

$$\inf F(X) = \inf F(X(x, \leqslant)), \text{ for all } x \in X.$$

Hence, by Theorem 4, for each $x \in X$ and $\varepsilon > 0$ there exists $y = y(x, \varepsilon) \geq x$ with the properties $F(y) \leqslant \inf F(X) + \varepsilon^2$, $\|F'(y)\| \leqslant \varepsilon$.

An immediate consequence is that, an ascending sequence (u_n) in X may be found with $F(u_n) \to \inf F(X)$, $F'(u_n) \to 0$, as $n \to \infty$.

Of course, one may assume without any loss that $(F(u_n))$ is strictly descending (hence (u_n) is strictly ascending). By Lemma 2, we may associate to any u_n, a point $v_n \in [u_n, u_{n+1}]$ with

$$|F(u_n) - F(v_n)| < 2^{-n}, \quad 0 < \|F'(v_n)\| < 2^{-n}, \quad n \in \mathbf{N}.$$

The sequence (v_n) is (strictly) ascending too and satisfies the premises of (C_{14}); so, it admits a cluster point, say v. By the assumptions involved, it is

now clear that $F(v) = \inf F(X)$. Indeed, by the closedness properties of the ambient ordering, we have

$$v_n \leqslant v \text{ (hence } F(v_n) \geqslant F(v)\text{), for all } n \in \mathbf{N};$$

this. combined with $F(v_n) \to \inf F(X)$ yields the desired conclusion. The last assertion in the statement is clear. q. e. d.

We are now going to the variational aspects of Theorem 3*. Let X be a normed space. By a *bornology* over X, denoted $(\#)$, we shall mean, as in B o r w e i n and P r e i s s [4], any (nonempty)) family of (norm) bounded subsets of X, whose union is the whole space (X). Among these, two are of particular interest, namely

(γ) = the Gâteaux bornology, consisting of all finite subsets of X

(φ) = the Fréchet bornology, consisting of all bounded subsets of X.

These may be considered as the „extremal" bornologies. For an intermediary member, we quote, e. g.,

(χ) = the Hadamard bornology, consisting of all compact subsets of X.

Now, let $\leqslant$ be a quasi-ordering over X generated by a *wedge* X_+ (in the P e r e s s i n i's terminology [17, ch. I, § 1]). Given the function $F: X \to R$, we say $x^* \in X^*$ is a $(X_+, (\#))$ — *subderivative* (*superderivative*) of it at the point $x \in X$ when, for each $\eta > 0$ and $S \in (\#)$, there exists $\delta > 0$, such that

$$\Lambda_x(t, F, h) \geqslant x^*(h) - \eta (\Lambda_x(t, F, h) \leqslant x^*(h) + \eta), \quad 0 < t < \delta, \quad h \in S \cap X_+.$$

Here, we denoted for simplicity

$$\Lambda_x(t, F, h) = t^{-1}(F(x + th) - F(x)), \; t > 0, \; h \in X.$$

The considered function is called $(X_+, (\#))$ — *subdifferentiable* (*superdifferentiable*) at $x \in X$ when the class $\partial^{\#}F(x)$ $(\partial_{\#}^{+} F(x))$ of all $(X_+, (\#))$ — subderivatives (superderivatives) is not empty. It will be said to be $(X_+, (\#))$ — *differentiable* at $x \in X$ when

$$\nabla^{\#}_{+}F(x) = \partial^{\#}_{+}F(x) \wedge \partial^{+}_{\#}F(x) \text{ is not empty.}$$

To give a partial description of this property — of a central importance in the sequel — we need some preliminary facts. Suppose $\leqslant$ is an ordering over X (induced by a cone X_+) with respect to which conditions $(C_{10})-(C_{12})$ be fulfilled. Let $(X_+)^{pa}$ be the class of all functions $f: X_+ \to R$ satisfying (D_9) plus

$$f(x, y) = f(x) + f(y), \; x, y \in X_+ \text{ (additivity).}$$

If, in addition, one assumes

$$f(x) \leqslant \mu \|x\|, \; x \in X_+, \text{ for some } \mu > 0 \text{ (boundedness),}$$

the obtained subclass of $(X_+)^{pa}$ will be denoted as $(X_+)^{pab}$. Now, by a well known result (Cristescu [10, ch. V, §3]) each element $f \in (X_+)^{pa}$ has a unique prolongation to X; that is, $f = g \mid X_+$, for a certain $g \in X'_+$ determined in a unique manner. Moreover, by Lemma 1, the assertion remains valid for the couple $((X_+)^{pab}, X^*_+)$ instead of $((X_+)^{pa}, X'_+)$. By convention, we call the function F, X_+ — *Gâteaux differentiable* in the *algebraic* sense at $x \in X$ when $F'(x)(a)$ (defined by (D_8)) exists for each $a \in X_+$, and the function $a \to F'(x)(a)$ is an element of $(X_+)^{pa}$. In this case, the unique prolongation of this function to the whole X, again denoted by $F'(x)$, will be termed the X_+ — *Gâteaux differential* of the considered function at the point x. Accordingly, F will be said to be X_+ — *Gâteaux differentiable* in the *topological* sense at $x \in X$, when the above convention is true with $(X_+)^{pab}$ in place of $(X_+)^{pa}$. The (unique) element $F'(x)$ in X_+^* whose restriction to X_+ is that given by (D_8) is easily shown to be the single point in $\nabla_+^{(\gamma)}F(x)$. This is also valid for $\nabla_+^{\#}F(x)$, but the interpretation is the following: for each S in (#).

(D_{10}) $\Lambda_x(t, F, h) \to F'(x)(h)$ as $t \to 0+$, uniformly for $h \in S \cap X_+$.

An important aspect of this construction is related to the case $F = \|\cdot\|$. We say that the norm in question is $(X_+, (\#))$ — *smooth* when $\nabla_+^{\#}(\|\cdot\|)(x)$ exists for all $x > 0$. This, by the above convention, ammounts to saying that $\|\cdot\|$ is X_+ — Gâteaux differentiable (in the topological sense) at any point of X_+ distinct from the origin, uniformly on members of $(\#) \cap X_+ = \{S \cap X_+ ; S \in (\#)\}$. Precisely, suppose $x > 0$ is arbitrary fixed. The function $t \to \|x + th\|$ is convex over its existence domain, for each $h \in X$. Hence, $t \to \Lambda_x(t, \|\cdot\|, h)$ is increasing and bounded from below (as $t \to 0+$); in other words, the function (from X_+ to R)

$$(D_{11}) \qquad x^*(h) = \lim_{t \to 0+} \Lambda_x(t, \|\cdot\|, h), \quad h \in X_+$$

is well defined and, by (C_{11}), positive (in the sense of (D_9)). Moreover, by the properties of the norm

$$x^*(h) \leqslant \|h\|, \text{ for all } h \in X_+,$$

which tells us this function is necessarily bounded. To have the required property we have to impose an additivity condition over X_+ for the function $h \to x^*(h)$. Then, its unique prolongation to X (also denoted by x^*) will appear as the unique X_+ — Gâteaux derivative of the function F at x. Note that by the above boundedness property, combined with Lemma 1, this derivative is an element of X_+^*; more exactly

$$(P_{10}) \qquad \|x^*\| = 1, \quad x^*(x) = \|x\|.$$

In fact, the evaluation $\|x^*\| \leqslant 1$ is clear; and, the second of these relations follows at once by definition. Hence the assertion is proved. Summing up, one may identify $\nabla_+^{\#}(\|\cdot\|)(x)$ with

$\nabla_+^{(\gamma)}(\|\cdot\|)(x)$ = the unique $x^* \in X_+^*$ with the properties (P_{10}).

But, we must retain the complete meaning of the introduced notion, precised by (D_{10}).

The linear functional $h \to x^*(h)$ we just introduced must not be confused with the function $h \to \lim_{t \to 0+} \Lambda_x(t, \|\cdot\|, h)$. This, however, may be done whenever this function is additive over X; that is, (D_{11}) will be valid over the whole X. In other words, the $(X_+, (\#))$ — smoothness property of the ambient norm cannot be reduced in general to the (#) — smoothness property of the same (defined as in Phelps [18, ch. IV]).

The following auxiliary statement is in effect for us. Remember that — in the precised setting — by $\Gamma_p(\leqslant)$ we denoted the class of all functions $G : X \to \mathbf{R}$ introduced as

$$G(x) = \sum_n \mu_n \|x - v_n\|^p, \quad x \in X,$$

where (μ_n) is a sequence of positive numbers with $\sum_n \mu_n = 1$ and (v_n) is an ascending (mod. $\leqslant$) convergent sequence in X. Precisely (cf. the remarks in the preceding section) the sequence in question is either a) ascending strictly:

$$v_0 < \ldots < v_n < \ldots < v = \lim_n v_n$$

or b) constant, in the sense

$$v_0 < \ldots < v_m = v_{m+1} = \ldots = v = \lim_n v_n \quad (\text{for some } m \in \mathbf{N}).$$

The (functional) series which defines G is effectively infinite in case a) only. It becomes a finite (functional) sum in case b); this, however, cannot prevent us from treating it as an infinite (functional) series.

Lemma 3. *Assume $p > 1$ and $\|\cdot\|$ is $(X_+, (\#))$ — smooth. Then, the function $G \in \Gamma_p(\leqslant)$ defined as above is $(X_+, (\#))$ — differentiable at each $x \in X$ with*

$$v_n \leqslant x, \text{ for all } n \in \mathbf{N}.$$

and

$$(5) \qquad \nabla_+^{\#}G(x) = p \sum_{n \in I(x)} \mu_n \|x - v_n\|^{p-1} \nabla_+^{(\gamma)}(\|\cdot\|)(x - v_n),$$

where, by convention, we put $I(x) = \{n \in \mathbf{N} ; v_n < x\}$. And consequently, we have the evaluation

$$(6) \qquad \|\nabla^{\#}G(x)\| \leqslant p(\lambda(x))^{p-1}, \text{ where } \lambda(x) = \sup\{\|x - v_n\| ; \ n \in \mathbf{N}\}.$$

Proof. Define for simplicity

$$F_n(z) = \|z - v_n\|^p, \quad z \in X, \ n \in \mathbf{N}.$$

By the $(X_+, (\#))$ — smoothness property of the ambient norm, it is clear that (the point $x \in X$ being taken as in the statement)

$$(5') \qquad \nabla_+^{\#}F_n(x) = 0, \text{ in case } v_n = x,$$
$$p\|x - v_n\|^{p-1} \nabla_+^{(\gamma)}(\|\cdot\|)(x - v_n), \text{ otherwise.}$$

(The hypothesis $p > 1$ was effectively used in the first of these relations only; we do not enter into further details, because the proof of this assertion is almost evident). Now, assume that $I(x) = \mathbf{N}$ (or, in other words, that the sequence (v_n) is taken as in case a) we just described). Let $\eta > 0$ and $S \in (\#)$ be arbitrary fixed. Denote $\mu(S) = \sup\{\|h\| ; \ h \in S \cap X_+\}$. By the obtained relation (5'), plus the previous discussion involving $\nabla_+^{(\gamma)}(\|\cdot\|)(x)$, it is clear that

$$\|\nabla_+^{\#}F_n(x)(h)\| \leqslant p(\lambda(x))^{p-1}\mu(S) \leqslant p(\lambda(x) + \mu(S))^p,$$

for all $h \in S \cap X_+$, and all $n \in \mathbf{N}$.

On the other hand, we have, by a direct argument

$$|\Lambda_x(t, F_n, h)| \leqslant p(\lambda(x) + \mu(S))^{p-1}\mu(S) \leqslant p(\lambda(x) + \mu(S))^p,$$
$$0 < t < 1, \ h \in S \cap X_+, \ n \in \mathbf{N}.$$

Let $m = m(\eta, S)$ be a rank with the property $\sum_{n \geqslant m} \mu_n < \eta / 4p(\lambda(x) + \mu(S))^p$.

By the informations above, one obtains

$$|\Lambda_x(t, \sum_{n \geqslant m} \mu_n F_n, h) - \sum_{n \geqslant m} \mu_n \nabla_+^{\#}F_n(x)(h)| < \eta/2,$$

for all t in (0, 1) and all h in $S \cap X_+$.

On the other hand, by the definition of $\nabla_+^{\#}$ (see (D_{10}))

$$|\Lambda_x(t, \sum_{n<m} \mu_n F_n, h) \to \sum_{n<m} \mu_n \nabla_+ {}^{\#} F_n(x)(h) \quad \text{as} \quad t \to 0+,$$

uniformly with respect to $h \in S \cap X_+$.
Hence, for some $\delta = \delta(\eta, S)$ in $(0, 1)$,

$$|\Lambda_x(t, \sum_{n<m} \mu_n F_n, h) - \sum_{n<m} \mu_n \nabla_+ {}^{\#} F_n(x)(h)| < \eta/2.$$

whenever $0 < t < \delta$, $h \in S \cap X_+$.
This, added to the previous relation, proves (5). The remaining case $I(x) \neq N$ (that is, $I(x) = \{0, ..., m-1\}$, where m is the rank introduced by the case b) involving our sequence) is almost evident ; so, we omit the details. The proof is complete q. e. d.

We are now in position to give the announced variational completion of Theorem 3*. As already precised, assumptions $(C_{10})-(C_{12})$ prevail.

Theorem 4*. *Let the function $F : X \to R$ be $\leq$-lsc and bounded from below. Suppose also that the ambient norm is $(X_+, (\#))$ — smooth, for a certain bornology, $(\#)$, over X. Then, given the constants $\varepsilon > 0$, $\lambda > 0$, $p > 1$ and the point $u \in X(F, \varepsilon)$, there may be determined a $v \in X(F, \varepsilon)$ with the properties a') in Theorem 2, that is,*

$$u \leq v, \quad \|u - v\| < \lambda, \quad F(u) \geq F(v),$$

as well as

c) $\partial_+ {}^{\#} F(v) \cap p(\varepsilon/\lambda) B^* \neq \emptyset$ (B^* = *the dual unit ball*).

In particular, a sufficient condition for the $\leq$-lsc property of the underlying function is the decreasing one.

Proof. By Theorem 3*, we have promised, for the starting elements ε, λ, p and $u \in X(F, \varepsilon)$, a point $v \in X(F, \varepsilon)$ with the properties a') we just precised as well as with the property c*) in that result, which may be described as

$$\Lambda_v(t, F, h) + (\varepsilon/\lambda^p)\Lambda_v(t, G, h) \geq 0, \quad t > 0, \quad h \in X_+,$$

for some G in $\Gamma_p(\leq)$, introduced as in a previous place, for some sequence (μ_n) of positive numbers with $\sum\limits_n \mu_n = 1$, and some ascending (mod. $\leq$) convergent (to v) sequence (v_n) in X. By Lemma 3, G is $(X_+, (\#))$ — differentiable at v. Its $(X_+, (\#))$-derivative, $\nabla_+ {}^{\#} G(v) = \nabla_+^{(\gamma)} G(v)$ is given by (5) and satisfies the evaluation (6), in the sense

$\|\nabla_+ {}^{\#} G(v)\| \leq p\lambda^{p-1}$ (if we take (4) into account).
Let $\eta > 0$ and $S \in (\#)$ be arbitrary fixed. There exists $\delta = \delta(\eta, S) > 0$ with

$$\nabla_+ {}^{\#} G(v)(h) - \eta \leq \Lambda_v(t, G, h) \leq \nabla_+ {}^{\#} G(v)(h) + \eta, \quad 0 < t < \delta, \quad h \in S \cap X_+,$$

by the very definition of $\nabla_+ {}^{\#} G(v)$. This, combined with the above, yields

$$\Lambda_v(t, F, h) \geq -(\varepsilon/\lambda^p)\nabla_+ {}^{\#} G(v)(h) - (\varepsilon/\lambda^p)\eta, \quad 0 < t < \delta, \quad h \in S \cap X_+.$$

In other words, $v^* = -(\varepsilon/\lambda^p)\nabla_+ {}^{\#} G(v)$ is an element of $\partial_+ {}^{\#} F(v)$. Moreover, the evaluation we just obtained gives

$$\|v^*\| \leq (\varepsilon/\lambda^p) p\lambda^{p-1} = p(\varepsilon/\lambda) \quad (\text{that is, } v^* \in p(\varepsilon/\lambda)B^*)$$

The proof is complete. q. e. d.

When $p = 1$, this result is no longer valid ; but, in this case, a good substitute of it is the one described by Theorem 4. In other words, these two statements are, in a certain sense complementary to each other. The obtained facts may be directly used in deriving order counterparts of the well known (amorph) density properties obtained in the last decade (see, e. g., Phelps [18, chs. IV–V] for a quasi-complete reference). These aspect will be discussed elsewhere.

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ON (ψ, L_p) — STABILITY OF NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

BY

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Strauss introduced in [6] a notion of L_p-stability of trivial solution of the differential system and proved several sufficient conditions.

In [5] Fodor utilized the results from [6] and proved sufficient conditions for L_p-stability of perturbed system independence on L_p-stability of nonperturbed system.

We consider the nonlinear system

$$x'=f(t,x) \tag{1}$$

along with its associated variational system

$$y'=f_x(t,0)y, \tag{2}$$

$$z'=f_x(t,x(t,t_0,x_0))z, \tag{3}$$

where $f\in C[J\times R^n, R^n]$ and $f_x=\frac{\partial f}{\partial x}$ exists and is continuous on $J\times R^n$, $J=<t_0,\infty)$, $I=(0,\infty)$, $R_+=<0,\infty)$, $(t_0\geqslant 0)$, $f(t,0)=0$ for $t\in J$ and $x(t,t_0,x_0)$ is the unique solution of (1) passing through the point (t_0,x_0). The matrix $\Phi(t,t_0,0)$ given by

$$\Phi(t,t_0,0)=\frac{\partial}{\partial x_0}(x(t,t_0,0)) \tag{4}$$

is the fundamental matrix solution of the variational system (2) [2] which is the identity matrix at $t=t_0$, and the matrix

$$\Phi(t,t_0,x_0)=\frac{\partial}{\partial x_0}(x(t,t_0,x_0)) \tag{5}$$

is the fundamental matrix solution of the variational system (3) which is the identity matrix at $t=t_0$.

Let $\psi: R_+\to I$ be a continuous function.

In this paper notions of ψ-uniform stability, ψ-stability and ψ-asymptotic stability, (ψ, L_p)-stability and (ψ, L_p)-asymptotic stability of trivial solution of (1) are introduced and several new sufficient conditions for mentioned types of stability are proved. Furthermore, sufficient conditions are given for the uniform Lipschitz stability and L_p-stability of the perturbed system $x'=f(t,x)+g(t,x)$.