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ON (ψ, L_p) — STABILITY OF NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

BY

JAROSLAW MORCHALO

Strauss introduced in [6] a notion of L_p-stability of trivial solution of the differential system and proved several sufficient conditions.

In [5] Fodor utilized the results from [6] and proved sufficient conditions for L_p-stability of perturbed system independence on L_p-stability of nonperturbed system.

We consider the nonlinear system

$$x' = f(t, x) \tag{1}$$

along with its associated variational system

$$y' = f_x(t, 0)y, \tag{2}$$

$$z' = f_x(t, x(t, t_0, x_0))z, \tag{3}$$

where $f \in C[J \times \mathbf{R}^n, \mathbf{R}^n]$ and $f_x = \dfrac{\partial f}{\partial x}$ exists and is continuous on $J \times \mathbf{R}^n$, $J = \langle t_0, \infty)$, $I = (0, \infty)$, $\mathbf{R}_+ = \langle 0, \infty)$, $(t_0 \geq 0)$, $f(t, 0) = 0$ for $t \in J$ and $x(t, t_0, x_0)$ is the unique solution of (1) passing through the point (t_0, x_0). The matrix $\Phi(t, t_0, 0)$ given by

$$\Phi(t, t_0, 0) = \frac{\partial}{\partial x_0}(x(t, t_0, 0)) \tag{4}$$

is the fundamental matrix solution of the variational system (2) [2] which is the identity matrix at $t = t_0$, and the matrix

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0}(x(t, t_0, x_0)) \tag{5}$$

is the fundamental matrix solution of the variational system (3) which is the identity matrix at $t = t_0$.

Let $\psi : R_+ \to I$ be a continuous function.

In this paper notions of ψ-uniform stability, ψ-stability and ψ-asymptotic stability, (ψ, L_p)-stability and (ψ, L_p)-asymptotic stability of trivial solution of (1) are introduced and several new sufficient conditions for mentioned types of stability are proved. Furthermore, sufficient conditions are given for the uniform Lipschitz stability and L_p-stability of the perturbed system $x' = f(t, x) + g(t, x)$.

Before giving further details, we give some of the main definitions that we need in the sequel.

Definition 1. *The equation* (1) *will be called ψ-uniformly stable on* $\mathbf{R}_+$ *if there exists a constant $N>0$ such that for any solution x of* (1) *and all $t\geqslant s\geqslant 0$*

$$|\psi^{-1}(t)x(t)|\leqslant Nx(s)$$

hold.

Definition 2. *The zero solution of* (1) *will be called ψ-stable on J, if for any $\varepsilon>0$ there esists a $\delta>0$ such that $|x_0|<\delta$ implies $|\psi^{-1}(t)x(t)|<\varepsilon$ for all $t\in J$. If, moreover $\lim\limits_{t\to\infty}|\psi^{-1}(t)x(t)|=0$ the zero solution of* (1) *will be called ψ-asymptotically stable.*

Definition 3. *The zero solution of* (1) *will be called (ψ, L_p) — stable $(p\geqslant 1)$ on J if*

1° *the zero solution of* (1) *is ψ-stable on J,*

2° *there exists $\delta=\delta(t_0)$ such that if $|x_0|<\delta$, then*

$$\int_{t_0}^{\infty}|\psi^{-1}(t)x(t)|^p dt<\infty.$$

If the zero solution of (1) *is ψ-asymptotically stable and 2° holds, the zero solution of* (1) *will be called (ψ, L_p)-asymptotically stable.*

Definition 4 [4]. *The zero solution of* (1) *will be called uniformly Lipschitz stable if there exists $M>0$ and $\delta>0$ such that*

$$|x(t, t_0, x_0)|\leqslant M|x_0|, \text{ whenever } |x_0|<\delta \text{ and } t\in J.$$

Definition 5 [4]. *The zero solution of* (1) *will be called uniformly Lipschitz stable in variation if there exists $M>0$, and $\delta>0$ such that $|\Phi(t, t_0, x_0)|\leqslant M$ for all $|x_0|<\delta$ and $t\in J$.*

Theorem 1. *Let $Y(t, t_0)$ be the fundamental matrix of the equation*

$$x'=A(t)x. \tag{6}$$

If there exist continuous positive functions $\psi(t)$ and $\varphi(t)$, $t\in J$, such that

(7)

$$1°\ \Big(\int_{t_0}^{t}|\psi^{-1}(t)Y(t,s)\varphi(s)|^p ds\Big)^{\frac{1}{p}}\leqslant K=\text{const}, \quad \textit{for } t\in J,$$

$$2°\ \psi(t)\exp\left(-\frac{K^{-p}}{p}\int_{t_1}^{t}\varphi^p(s)\psi^{-p}(s)\right)ds\leqslant K_1=\text{const}. \textit{ for } t\geqslant t_1>t_0.$$

then the zero solution of (6) *is uniformly Lipschitz stable.*

Proof. Let $h(t)=\varphi^p(t)|Y(t, t_0)|^{-p}$.

We consider the identity

$$Y(t, t_0)\int_{t_0}^{t}h(s)ds=\int_{t_0}^{t}h(s)Y(t, s)Y(s, t_0)ds=\int_{t_0}^{t}h(s)Y(t, s)\varphi(s)\varphi^{-1}(s)Y(s, t_0)ds.$$

Hence

$$|Y(t, t_0)|\int_{t_0}^{t}h(s)ds\leqslant\int_{t_0}^{t}h^{(p-1)p^{-1}}(s)|Y(t, s)|\psi(s)ds.$$

Using the Hölder inequality we get

$$|Y(t, t_0)|\Big(\int_{t_0}^{t}h(s)ds\Big)^{\frac{1}{p}}\leqslant\Big(\int_{t_0}^{t}(|Y(t, s)|\psi(s))^p ds\Big)^{\frac{1}{p}}$$

and with respect to (7) we have

$$|\psi^{-1}(t)Y(t, t_0)|\leqslant K\Big(\int_{t_0}^{t}h(s)ds\Big)^{-\frac{1}{p}}. \tag{8}$$

Denote $\mu(t)=\int_{t_0}^{t}h(s)ds$. Then (8) yields the inequality

$$\mu^{\frac{1}{p}}(t)|Y(t, t_0)\varphi^{-1}(t)|\leqslant K\varphi^{-1}(t)\psi(t)$$

and since

$$\frac{d\mu}{dt}=h(t)=|Y(t, t_0)|^{-p}\varphi^p(t),$$

we get

$$\frac{d\mu}{dt}\geqslant K^{-p}\varphi^p(t)\psi^{-p}(t)\mu(t).$$

Integrating from $t_1(t_1>t_0)$ to t we obtain the inequality (using Gronwall Lemma)

$$\mu(t)\geqslant\mu(t_1)\exp\Big(K^{-p}\int_{t_1}^{t}\varphi^p(s)\psi^{-p}(s)ds\Big). \tag{9}$$

From (8) and equality

$$|Y(t, t_0)|=\varphi(t)\,(h^{\frac{1}{p}}(t))^{-1}$$

we have

$$\psi^{-1}(t)\varphi(t)\,(h^{\frac{1}{p}}(t))^{-1}\mu^{\frac{1}{p}}(t)\leqslant K.$$

Hence

$$h^{\frac{1}{p}}(t)\geqslant K^{-1}\psi^{-1}(t)\varphi(t)\mu^{\frac{1}{p}}(t),$$

and

$$|Y(t, t_0)|\leqslant K\psi(t)\,(\mu(t))^{-\frac{1}{p}}.$$

Since

$$\mu^{\frac{1}{p}}(t)\geqslant\mu^{\frac{1}{p}}(t_1)\exp\Big(p^{-1}K^{-p}\int_{t_1}^{t}\varphi^p(s)\psi^{-p}(s)ds\Big),$$

then

$$|Y(t, t_0)|\leqslant K\mu^{-\frac{1}{p}}(t_1)\psi(t)\exp\Big(-p^{-1}K^{-p}\int_{t_1}^{t}\varphi^p(s)\psi^{-p}(s)ds\Big). \tag{10}$$

If we put $N\geqslant KK_1\mu^{-\frac{1}{p}}(t_1)$, we obtain $|Y(t, t_0)|\leqslant N$.

The solution of (6) through the point (t_0, x_0) can be written as

$$x(t, t_0, x_0)=Y(t, t_0)x_0$$

Hence $|x(t, t_0, x_0)|\leqslant|Y(t, t_0)||x_0|\leqslant N|x_0|$ for all $|x_0|<\delta$ and $t\in J$.

The proof of the Theorem is now complete.

Corollary 1. *If in the Theorem 1 we put $p=1$, then the conclusion of the Theorem 2.4.* [4] *follows.*

Corollary 2. *If in Theorem* 1 *the condition* 2° *is replaced by the condition*

$$\exp\left(-p^{-1}K^{-p}\int_{t_1}^{t}\varphi^p(s)\psi^{-p}(s)ds\right)\leqslant K_2<\infty,$$

then the zero solution of (6) *will be ψ-uniformly stable on J.*

Corollary 2. *If the zero solution of* (6) *is ψ-stable and*

$$\exp\left(-K^{-p}\int_{t_1}^{t}\varphi^p(s)\psi^{-p}(s)ds\right)\in L_p(<t_0,\infty)).$$

then the zero solution of (6) *is (ψ, L_p) stable on J.*

Theorem 2. *If there exist continuous positive functions $\varphi(t)$ and $\psi(t)$ for $t\in J$ such that*

$$1^\circ\quad \left(\int_{t_0}^{t}|\psi^{-1}(t)\Phi(t,s,x_0)\varphi(s)|^p\,ds\right)^{\frac{1}{p}}\leqslant K=\text{const.}\quad \text{for } t\in J$$

(11)

$$\psi(t)\exp\left(-p^{-1}K^{-p}\int_{t_1}^{t}\varphi^p(s)\psi^{-p}(s)ds\right)\leqslant K_1=\text{const.}$$

for $t\geqslant t_1>t_0$ then the zero solution of (1) *is uniformly Lipschitz stable.*

Proof. The proof is similar to that of Theorem 1.

Theorem 3. *Suppose that*

1° *there exist continuous nonsingular matrices $A(t)$ and $B(t)$ for $t\in J$, and a continuous real-valued function $\alpha(t)$ for $t\in J$ such that for every $\xi\in\mathbb{R}^n$ and every $s\geqslant 0$ we have*

$$|\Phi(t,s,\xi)A(s)B^{-1}(t)|\leqslant\exp\int_{s}^{t}\alpha(u)du,\qquad t\geqslant s\geqslant 0,$$

2°

$$\exp\left(\int_{t_0}^{t}\alpha(u)du\right)\in L_p(<t_0,\infty)),\ p\geqslant 1)),$$

then

(12) $$\int_{t_0}^{t}(\psi^{-1}(s)\,|x(s,t_0,x_0)|)^p\,ds\leqslant C\,|A^{-1}(t_0)x_0|^p$$

for all $t\in J$, where $\psi(t)=|B(t)|$.

Proof. For $0\leqslant t_0\leqslant t$ we have

$$x(t,t_0,x_0)=\left[\int_0^t\Phi(t,t_0,sx_0)ds\right]x_0.$$

Hence

(13) $$|x(t,t_0,x_0)|\leqslant|A^{-1}(t_0)x_0\|B(t)|\exp\left(\int_{t_0}^{t}\alpha(u)du\right).$$

If we raise (13) to the p-th power ($p\leqslant 1$) and integrate from t_0 to t, then

$$\int_{t_0}^{t}(\psi^{-1}(s)\,|x(s,t_0,x_0)|)^p\,ds\leqslant C\,|A^{-1}(t_0)x_0|^p,\ \text{for } t\in J.$$

This completes the proof.

The next theorem is a generalization of the Theorem 3.5 [4].

Theorem 4. *If*

$$|\Phi(t,t_0)|\exp\int_{t_0}^{t}|\Phi(s,t_0)\|\Phi(t_0,s)[f_x(s,x(s,t_0,x_0))-f_x(s,0)]|\,ds<\infty$$

for all $t_0\geqslant 0$, $t\in J$, $|x_0|\leqslant\delta$, $\delta>0$, where $\Phi(t,t_0)=\Phi(t,t_0,0)$, then the zero solution of (1) *is uniformly Lipschitz stable in variation.*

Proof. Let $z(t,t_0,z_0)$ be a solution of (3) such that $z(t_0,t_0,z_0)=z_0$. If we write (3) in the form

$$z'=f_x(t,0)z+[f_x(t,x(t,t_0,x_0))-f_x(t,0)]z,$$

then the classical formula of variation of constants gives

$$z(t,t_0,z_0)=\Phi(t,t_0)\{z_0+\int_{t_0}^{t}\Phi(t_0,s)[f_x(s,x(s,t_0,x_0))-f_x(s,0)]z(s,t_0,z_0)ds\}.$$

Using Lemma 1[7] we obtain

$$|z(t,t_0,z_0)|\leqslant|\Phi(t,t_0)\|z_0|\,.\exp\left(\int_{t_0}^{t}\Phi(s,t_0)\|\Phi(t_0,s)[f_x(s,x(s,t_0,x_0))-f_x(s,0)]|ds\right).$$

It follows from assumption of theorem that

$$|z(t,t_0,z_0)|\leqslant M\,|z_0|$$

for all $t_0\geqslant 0$, $t\in J$ and $|x_0|\leqslant\delta$.

Now

$$|\Phi(t,t_0,x_0)|=\sup_{|z_0|\leqslant 1}|\Phi(t,t_0,x_0)z_0|\leq\sup_{|z_0|\leqslant 1}(M\,|z_0|)\leq M\ \text{for}\ |x_0|\leq\delta.$$

Hence the zero solution of (1) is uniformly Lipschitz stable in variation

Theorem 5. *If in the equation*

(14) $$y'=f(t,y)+g(t,y)$$

1° $$|f(t,y)|\leq h_1(t)\Omega(|y|),\quad t\in J,$$

2° $$|g(t,y)|\leq h(t)w(|y|),\quad g(t,cy)=cg(t,y)\quad \text{for } t\in J \text{ and } c\in\mathbb{R}.$$

where

(i) $\Omega(u)$ *is a nondecreasing positive submultiplicative continuous function on* $(0,\infty)$

(ii) $w(u)>0$ *is nondecreasing and continuous for* $u>0$,

(iii) $h(t)$, $h_1(t)$ *are continuous on* $(0,\infty)$.

3° $$G(u)=\int_{u_0}^{u}\frac{ds}{w(s)},\qquad 0<u_0\leq u,\qquad G(\infty)=\infty,$$

G^{-1} is the inverse of G,

4° $$F(u)=\int_{u_0}^{u}\frac{ds}{\Omega(s)},\qquad 0<u_0\leq u,\qquad F(\infty)=\infty,$$

F^{-1} is the inverse of F,

5° $$G^{-1}\left(G(1)+\int_{t_1}^{t} h(s)ds\right)\cdot F^{-1}\left\{F(|x_0|)+\int_{t_1}^{t} h_1(s)\frac{\Omega(|x_0|)}{|x_0|}\left[G^{-1}\left(G(1)+\int_{t_1}^{s} h(z)dz\right)\right]ds\right\}<\infty,$$

for all $t_1\geq t_0\geq 0$,
then the zero solution of (14) is globally uniformly Lipschitz stable.

Proof. Since

$$y(t,t_0,x_0)=x_0+\int_{t_0}^{t}(f(s,y(s,t_0,x_0))+g(s,y(s,t_0,x_0)))ds$$

it follows that

$$\frac{|y(t,t_0,x_0)|}{|x_0|}\leqslant 1+\int_{t_0}^{t} h(s)w\left(\frac{|y(s,t_0,x_0)|}{|x_0|}\right)ds+\int_{t_0}^{t} h_1(s)\frac{\Omega(|x_0|)}{|x_0|}\Omega\left(\frac{|y(s,t_0,x_0)|}{|x_0|}\right)ds.$$

This implies by the Dhongade-Deo inequality [3] that:

$$|y(t,t_0,x_0)|\leq|x_0|\{G^{-1}(G(1)+\int_{t_0}^{t} h(s)ds)\}\cdot$$

$$F^{-1}\left\{F(|x_0|)+\int_{t_0}^{t} h_1(s)\frac{\Omega(|x_0|)}{|x_0|}\left[G^{-1}\left((G(1)+\int_{t_0}^{s} h(z)dz\right)\right]ds\right\}.$$

Hence $|y(t,t_0,x_0)|\leq M|x_0|$ *for* all $|x_0|<\infty$ and the conclusion of the theorem follows.

Theorem 6. *If*

1° *the hypotheses of Theorem 3 are satisfied.*

2° $g(t,u)\in C(J\times\mathbf{R}^n,\mathbf{R}^n)$, $g(t,0)=0$ *for all* $t\in J$.

3° *there is a continuous function* $\beta(t)$ *for* $t\in J$ *such that the inequality* $|A^{-1}(t)g(t,u)|\leq\beta(t)|u|$ *holds whenever* $t\in J$.

4° $\beta\in L_q(J,m)$ *that is*

$$\int_{t_0}^{\infty}|\beta(s)|^q\, ds\leq m=\text{constant}.$$

5° $$\int_{t_0}^{\infty}[p\alpha(t)+m_1 2^{p-1}\psi^p(t)]dt\leqslant m_2=\text{constant},$$

where $m_1>0$ *is some constant*

6° $$\int_{t_0}^{\infty}\exp[\int_{t_0}^{t}(p\alpha(s)+m_1 2^{p-1}\psi^p(s))ds]dt\leq M_1=\text{constant}$$

7° $$\lim_{t\to\infty}\exp\int_{t_0}^{t}(\alpha(s)+m_1\psi(s))ds=0.$$

then the zero solution of equation (14) *is:*

1° ψ — *uniformly stable on* J, *if the conditions* 1°—5° *hold.*
2° ψ—*uniformly stable on* J *and for any* $\varepsilon>0$ *there exists* $\delta>0$ *such that for arbitrary solution* y *of* (14) *fulfilling* $|x_0|<\delta$ *there holds* $(\psi^{-1}(t)y(t,t_0,x_0))\in L_p(J,\varepsilon)$ *if the conditions* 1°—6° *hold.*
3° ψ—*uniformly stable on* J, $(\psi^{-1}(t)y(t,t_0,x_0))\in L_p(J,\varepsilon)$, *and the solution* $y(t)=0$, $t\in J$ *is* (ψ,L_p)—*asymptotically stable, if the conditions* 1°—7° *hold.*

Proof. Let y be an arbitrary solution of (14) such that $|x_0|$ is sufficiently small. Let $\langle t_0,t_1)$ be the existence interval of solution y. Using the nonlinear variation of constants formula in Alekseev [1], the solutions of (1) and the perturbed system (14) with the same initial values are related by

(*) $$y(t,s,x(s))=x(t,s,x(s))+\int_{s}^{t}\Phi(t,z,y(z,s,x(s)))g(z,y(z,s,x(s)))dz$$

for any $s\in\langle t_0,t_1)$ and $t\geq s$.
Hence from our assumptions of Φ and g we obtain

(15) $$|y(t,t_0,x_0)\psi^{-1}(t)|\exp(-\int_{t_0}^{t}\alpha(u)du)\leq|A^{-1}(t_0)x_0|+$$
$$+\int_{t_0}^{t}\beta(s)\psi(s)\exp(-\int_{t_0}^{s}\alpha(u)du)|y(s,t_0,x_0)\psi^{-1}(s)|ds$$

for $t\in\langle t_0,t_1)$. If we raise (15) to the p-th power ($p\geq 1$) and apply the Bellman Lemma, we conclude that

(16) $$(|y(t,t_0,x_0)|\psi^{-1}(t))^p\leq 2^{p-1}|A^{-1}(t_0)x_0|^p e^{m_2}.$$

From (16) it follows that there exists a constant $M>0$ such that

$$|y(t,t_0,x_0)|\leq\psi(t)M^{\frac{1}{p}}\leq\bar{M}\text{ for }t\in\langle t_0,t_1).$$

Hence we have that $t_1\to\infty$ and the solution y is defined on J. From the estimate (16) it follows that the solution $y=0$ is ψ—stable on J. Moreover from (*) for any $t_1\in J$ and $t\geq t_1$ the following estimate holds

(17) $$|y(t,t_0,x_0)|\psi^{-1}(t)\leq 2^{\frac{p-1}{p}}|A^{-1}(t_1)|e^{\frac{m_2}{p}}|x(t_1)|.$$

From (17) it follows that the equation (14) is ψ—uniformly stable. Thus (14) has the property 1°.

Let besides the condition 6° holds. Then by integrating

(18) $$(|y(t,t_0,x_0)|\psi^{-1}(t))^p\leq 2^{p-1}|A^{-1}(t_0)x_0|^p\exp\int_{t_0}^{t}(p\alpha(s)+m_1 2^{p-1}\psi^p(s))ds$$

from t_0 to t we get

$$\int_{t_0}^{t}(|y(s,t_0,x_0)|\psi^{-1}(s))^p\, ds\leq 2^{p-1}|A^{-1}(t_0)|^p|x_0|^p M_1.$$

Therefore $y=0$ is (ψ,L_p)—stable.

If $2^{p-1}|A^{-1}(t_0)|^p|x_0|^p M_1<\varepsilon$ we obtain that $(y(t,t_0,x_0)\psi^{-1}(t))\in L_p(J,\varepsilon)$

Hence the equation has the property 2°.

Let furthermore 7° hold. Then from (18) for $p=1$ it follows that

$$\lim_{t\to\infty} |y(t,\ t_0,\ x_0)|\,\psi^{-1}(t)=0.$$

Therefore the solution $y=0$ is $(\psi,\ L_p)$ — asymptotically stable and (14) has the property 3°.

Remark. Theorem 6 generalizes a similar result of Fodor [5] for $\psi(t)\equiv 1$ and $f(t,x)\equiv A(t)x,\ \ t\in J$.

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OSCILLATORY AND NONOSCILLATORY SOLUTIONS OF DIFFERENCE EQUATIONS

BY

J. KORCZAK and J. POPENDA

In the paper we consider the following two problems of the qualitative theory of finite difference equations:

when all of the solutions are oscillatory ones.

what kind of asymptotic behaviour have the nonoscillatory solutions ?

These problems have been considered both in finite difference and differential equations theory in many recently published papers. The results and methods of this paper are partly similar to those presented in [1] however our equations have more general forms than the ones which would be strictly analogical to these considered therein.

We shall use the following notations:

$\mathbf{N}$ — the set of positive integers,

$\mathbf{R}_+:=(0,\ \infty)$, $\mathbf{R}_0:=[0,\ \infty)$, $\mathbf{R}:=(-\infty,\ \infty)$.

For an arbitrary function (sequence) $x:\mathbf{N}\to\mathbf{R}$ the forward difference operator Δ is defined as usually $\Delta x_n=x_{n+1}-x_n$, $n\in\mathbf{N}$.

We say that the sequence (x_n) is nonoscillatory if it is eventually of constant sign that is there exists $n_0\in\mathbf{N}$ such that for all $n\geq n_0$ $x_n>0$ or for all $n\geq n_0$ $x_n<0$, othervise the sequence is called oscillatory.

In the papers [2], [3] J. Korczak and M Migda considered the problem when a given difference equation has one solution (or when all its solutions) approaches any given polynomial type sequence. The theorem we shall prove below gives an answer on the opposite question.

We start with the equation of the form

$$(\mathrm{E}_1)\qquad \Delta[x_n+px_{n-s}]+q_nx_{n-s}=0$$

$$n=n_0,\ n_0+1,\ldots\quad s,\ n_0\in\mathbf{N},\ \ s<n_0.$$

Theorem 1. *Let $p<-1$ and let exist a constant $q\in\mathrm{R}_+$ such that $q_n\geq q$ for all $n\geq n_0$. Then every nonoscillatory solution y of (E_1) has the property*

$$y_n/n^k\to\infty \qquad \text{as}\quad n\to\infty$$

$$[\text{or } y_n/n^k\to-\infty \qquad \text{as } n\to\infty \textit{ in the case } y \textit{ is eventually negative}],$$

where k is an arbitrary nonnegative integer.

Proof. Let us see that if $u=\{u_n\}$ is a solution of (E_1) then $v=\{-u_n\}$ is a solution too.