Hence the equation has the property 2°.

Let furthermore 7° hold. Then from (18) for $p=1$ it follows that

$$\lim_{t\to\infty} |y(t, t_0, x_0)| \psi^{-1}(t) = 0.$$

Therefore the solution $y=0$ is (ψ, L_p)—asymptotically stable and (14) has the property 3°.

Remark. Theorem 6 generalizes a similar result of Fodor [5] for $\psi(t)\equiv 1$ and $f(t,x)\equiv A(t)x$, $t\in J$.

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OSCILLATORY AND NONOSCILLATORY SOLUTIONS OF DIFFERENCE EQUATIONS

BY

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In the paper we consider the following two problems of the qualitative theory of finite difference equations:

when all of the solutions are oscillatory ones,

what kind of asymptotic behaviour have the nonoscillatory solutions ?

These problems have been considered both in finite difference and differential equations theory in many recently published papers. The results and methods of this paper are partly similar to those presented in [1] however our equations have more general forms than the ones which would be strictly analogical to these considered therein.

We shall use the following notations:

$\mathbf{N}$ — the set of positive integers,

$\mathbf{R}_+ := (0, \infty)$, $\mathbf{R}_0 := [0, \infty)$, $\mathbf{R} := (-\infty, \infty)$.

For an arbitrary function (sequence) $x: \mathbf{N}\to\mathbf{R}$ the forward difference operator Δ is defined as usually $\Delta x_n = x_{n+1} - x_n$, $n\in\mathbf{N}$.

We say that the sequence (x_n) is nonoscillatory if it is eventually of constant sign that is there exists $n_0\in N$ such that for all $n\geq n_0$ $x_n>0$ or for all $n\geq n_0$ $x_n<0$, othervise the sequence is called oscillatory.

In the papers [2], [3] J. Korczak and M Migda considered the problem when a given difference equation has one solution (or when all its solutions) approaches any given polynomial type sequence. The theorem we shall prove below gives an answer on the opposite question.

We start with the equation of the form

$$(\mathrm{E}_1) \qquad \Delta[x_n + p x_{n-s}] + q_n x_{n-s} = 0$$

$$n = n_0, n_0+1, \ldots \quad s, n_0 \in \mathbf{N}, \ s < n_0.$$

Theorem 1. *Let $p<-1$ and let exist a constant $q\in\mathrm{R}_+$ such that $q_n\geq q$ for all $n\geq n_0$. Then every nonoscillatory solution y of (E_1) has the property*

$$y_n/n^k \to \infty \qquad \text{as } n\to\infty$$

$$[\text{or } y_n/n^k \to -\infty \qquad \text{as } n\to\infty \textit{ in the case } y \textit{ is eventually negative}],$$

where k is an arbitrary nonnegative integer.

Proof. Let us see that if $u=\{u_n\}$ is a solution of (E_1) then $v=\{-u_n\}$ is a solution too.

Let $y=\{y_n\}_{n=1}^{\infty}$ be an arbitrary nonoscillatory solution of (E_i). Taking into account the remark we have made we assume $y_n>0$ for all $n\geq n_i$, $n_i\in\mathbf{N}$.

Let us denote

(1) $$z_n^{(0)}=y_n+py_{n-s} \quad n=n_0,\ n_0+1\ldots$$

Therefore

(2) $$\Delta z_n^{(0)}=\Delta(y_n+py_{n-s})=-q_ny_{n-s}.$$

Because of positivity of y we have

$$\Delta z_n^{(0)}<0 \quad \text{for} \quad n\geq n_i+s:=n_2$$

so the sequence $\{z_n^{(0)}\}$ is strictly decreasing for $n\geq n_2$.

This yields two possibilities

(a) $$z_n^{(0)}>0 \quad \text{for all} \quad n\geq n_2$$

(b) there exists n_3, $n_3\geq n_2$ such that

$$z_n^{(0)}<0 \quad \text{for all} \quad n\geq n_3.$$

In order to prove the case (b), we suppose contrary that the case (a) stands. Therefore by (1) we have

$$0<y_{n-s}<\left(\frac{1}{-p}\right)y_n \quad n=n_2,\quad n_2+1,\ldots$$

hence

(3) $$0<y_{n-s}<\left(\frac{1}{-p}\right)^{i+1} y_{n+is}$$

for all $n\geq n_2$, $i=0, 1, 2, \ldots$ Notice that $\lim\limits_{i\to\infty}\left(\frac{1}{-p}\right)^{i+1}=0$, so by virtue of (3) it does not lead us to a contradiction if $\lim\limits_{i\to\infty}y_{n+is}=\infty$ for all $n\geq n_2$, and particularly for $n=n_2$, n_2+1, n_2+s-1. Each term of the sequence $\{y_n\}$ beongs to some of the subsequences $\{y_{n+is}\}_{i=0}^{\infty}=0$, $n=n_2,\ldots,n_2+s-1$. Therefore $y_n\to\infty$ as $n\to\infty$. Hence by (2) we get

$$\lim_{n\to\infty}\Delta z_n^{(0)}\leq\lim_{n\to\infty}(-q)y_{n-s}=-\infty$$

The above given relation is impossible if the case (a) holds. This contradiction informs that the case (b) holds. The sequence $\{z_n^{(0)}\}$ is strictly increasing and eventually negative. Thereby we have two possibilities :

(i_0) $$\lim_{n\to\infty}z_n^{(0)}=g \qquad (g\neq-\infty,\ g<0),$$

(ii_0) $$\lim_{n\to\infty}z_n^{(0)}=-\infty.$$

Let us suppose that (i_0) holds. Then by (2) we get

$$z_n^{(0)}-z_{n_2}^{(0)}=-\sum_{j=n_2}^{n-1}q_jy_{j-s}$$

from there

$$\sum_{j=n_2}^{\infty}q_jy_{j-s}=z_{n_2}^{(0)}-g.$$

This means that the series $\sum\limits_{j=n_2}^{\infty}q_jy_{j-s}$ is convergent.

Therefore by the Weierstrass test of convergence we find that the series $\sum\limits_{j=n_2}^{n}qy_{j-s}$ is convergent and consequently also the series $\sum\limits_{j=n_2}^{\infty}y_{j-s}$, and $\sum\limits_{j=n_2}^{\infty}y_j$. Hence by (1) we obtain

$$\sum_{j=n_2}^{n}|z_j^{(0)}|<\sum_{j=n_2}^{\infty}y_j+|p|\sum_{j=n_2}^{\infty}y_{j-s}<\infty.$$

Thus the sequence $\left\{\sum\limits_{j=n_2}^{n}|z_j^{(0)}|\right\}$ being increasing and bounded from above approaches some finite constant. Hence $\lim\limits_{n\to\infty}z_n^{(0)}=0$ but this gives a contradiction to (i_0).

Consequently we have $\lim\limits_{n\to\infty}z_n^{(0)}=-\infty$. Again, by (1) we obtain $y_{n-s}>\frac{1}{p}z_n^{(0)}$ from there

$$\lim_{n\to\infty}y_{n-s}\geq\frac{1}{p}\lim_{n\to\infty}z_n^{(0)}=\infty$$

So, the theorem holds for $k=0$. The rest of the proof follows by induction argument. Suppose the theorem is true for some $k\geq0$ i.e.

(4) $$\lim_{n\to\infty}y_n/n^k=\infty.$$

Let us denote

(5) $$z_n^{(k+1)}=\frac{1}{n^{k+1}}[y_n+py_{n-s}].$$

For an arbitrary constant $M\in\mathbf{R}_+$ we can find, by (4), an integer n_M such that for all $n\geq n_M$ $y_n\geq Mn^k$. Therefore $n\geqslant n_M+s$ we get

$$\Delta[n^{k+1}z_n^{(k+1)}]=-q_ny_{n-s}\leq-qM(n-s)^k.$$

After suitable summation we obtain.

$$n^{k+1}z_n^{(k+1)}-(n_M+s)^{k+1}z_{n_M+s}^{(k+1)}\leq-qM\sum_{j=n_M+s}^{n-1}(j-s)^k.$$

Hence

(6) $$z_n^{(k+1)}\leq(n_M+s)^{k+1}z_{n_M+s}^{(k+1)}/n^{k+1}-qM\left(\sum_{j=n_M+s}^{n-1}(j-s)^k\right)/n^{k+1}.$$

By the Stolz Theorem we have

$$\lim_{n\to\infty}\left(\sum_{j=n_M+s}^{n-1}(j-s)^k\right)\Big/n^{k+1}=1/(k+1),$$

moreover

$$\lim_{n\to\infty}(n_M+s)^{k+1}z^{(k+1)}_{n_M+s}/n^{k+1}=0.$$

Hence the sequence on the right hand side of (6) tends to $-qM/(k+1)<0$. Therefore for sufficiently large n, say $n\geq n_3$ we get

(7) $$z_n^{(k+1)}<0.$$

Now, let us study the monotonicity of the sequence $\{z_n^{(k+1)}\}$. Using definition (5) we have

(8) $$\begin{aligned}\Delta z_n^{(k+1)}&=(n+1)^{-k-1}\Delta[y_n+py_{n-s}]+[y_n+py_{n-s}]\Delta n^{-k-1}=-\\&-n^{-k-1}(n+1)^{-k-1}\{[(n+1)^{k+1}-n^{k+1}]y_n+n^{k+1}q_ny_{n-s}+[(n+1)^{k+1}-\\&-n^{k+1}]py_{n-s}\}=-n^{-k-1}(n+1)^{-k-1}\left\{\left[\sum_{j=1}^{k+1}\binom{k+1}{j}n^{k+1-j}\right]y_n+\right.\\&\left.+\left[n^{k+1}q_n+p\sum_{j=1}^{k+1}\binom{k+1}{j}n^{k+1-j}\right]y_{n-s}\right\}.\end{aligned}$$

By the positivity of y we get

(9) $$\left[\sum_{j=1}^{k+1}\binom{k+1}{j}n^{k+1-j}\right]y_n>0 \quad \text{for all} \quad n\geq n_2.$$

Furthermore by

(10) $$\lim_{n\to\infty}n^{-k-1}\left(\sum_{j=1}^{k+1}\binom{k+1}{j}n^{k+1-j}\right)p=0$$

we obtain

$$\begin{aligned}&\lim_{n\to\infty}n^{k+1}\left[q_n+n^{-k-1}\left(\sum_{j=1}^{k+1}\binom{k+1}{j}n^{k+1-j}\right)p\right]\geqslant\\&\geqslant\lim_{n\to\infty}n^{k+1}\left[q+n^{-k-1}\left(\sum_{j=1}^{k+1}\binom{k+1}{j}n^{k+1-j}\right)p\right]=\infty.\end{aligned}$$

Therefore there exist $n_4, i_4\geq n_2$ such that for every $n\geq n_4$.

(11) $$\left[n^{k+1}q_n+p\sum_{j=1}^{k+1}\binom{k+1}{j}n^{k+1-j}\right]y_{n-s}>0.$$

Hence (8), (9) and (11) yield

(12) $$\Delta z_n^{(k+1)}<0$$

for sufficiently large $n\geq n_4$. This means that the sequence $\{z_n^{(k+1)}\}$ is strictly decreasing at least for n_4. By (7) and (12) we have

(13) $$z_n^{(k+1)}<0,\quad \Delta z_n^{(k+1)}<0 \quad \text{for all} \quad n\geq n_5:=\max[n_3, n_4].$$

Regarding (13) we find that there are two possibilities (similar as in the first part of the proof)

(i_{k+1}) $$\lim_{n\to\infty}z_n^{(k+1)}=g \qquad (g\neq-\infty,\ g<0)$$

or

(ii_{k+1}) $$\lim_{n\to\infty}z_n^{(k+1)}=-\infty.$$

Suppose (i_{k+1}) holds. Summing (8) and calculating the limit we get by (i_{k+1})

(14) $$\begin{aligned}\sum_{j=n_4}^{\infty}\frac{1}{j^{k+1}(j+1)^{k+1}}&\left\{\left[\sum_{i=1}^{k+1}\binom{k+1}{j}j^{k+1-i}\right]y_j+\left[j^{k+1}q_j+\right.\right.\\&\left.\left.+p\sum_{i=1}^{k+1}\binom{k+1}{j}j^{k+1-i}\right]y_{j-s}\right\}=z_{n_4}^{(k+1)}-g.\end{aligned}$$

Taking into account (10) we can find such $n_5\geq n_4$ that for all $n\geq n_5$

$$n^{k+1}\left[q_n+n^{-k-1}\left(\sum_{j=1}^{k+1}\binom{k+1}{j}n^{k+1-j}\right)p\right]>\frac{n^{k+1}q}{2}.$$

From (14) it follows that the series on the left side of (14) is convergent. Therefore by the Weierstrass test of convergence and the above given inequality we can show in succession the convergence of the series

$$\sum_{j=n_5}^{\infty}\frac{1}{j^{k+1}(j+1)^{k+1}}\left\{\left[\sum_{i=1}^{k+1}\binom{k+1}{j}j^{k+1-i}\right]y_j+j^{k+1}\frac{1}{2}qy_{j-s}\right\},$$

$$\sum_{j=n_5}^{\infty}\frac{1}{(j+1)^{k+1}}y_{j-s},$$

and taking into account

(15) $$\lim_{n\to\infty}n^{k+1}/(n-s)^{k+1}=1$$

and Abel test of convergence further the series

$$\sum_{j=n_5}^{\infty}\frac{1}{j^{k+1}}y_{j-s},\qquad \sum_{j=n_5}^{\infty}\frac{1}{j^{k+1}}y_j.$$

Hence by (5) we obtain

$$\sum_{j=n_5}^{n}|z_j^{(k+1)}|<\sum_{j=n_5}^{\infty}\frac{1}{j^{k+1}}y_j+|p|\sum_{j=n_5}^{\infty}\frac{1}{j^{k+1}}y_{j-s}<\infty.$$

From this it follows that the series $\sum_{j=n_5}^{\infty}z_j^{(k+1)}$ is convergent and therefore $\lim_{n\to\infty}z_n^{(k+1)}=0$ what give a contradiction to our assumption that (i_{k+1}) holds. Hence $\lim_{n\to\infty}z_n^{(k+1)}=-\infty$. Again by (5) we have $z_n^{(k+1)}>py_{n-s}/n^{k+1}$ from there

$$\lim_{n\to\infty}\frac{1}{n^{k+1}}y_{n-t}>\lim_{n\to\infty}\frac{1}{p}z_n^{(k+1)}=\infty.$$

Now by (15) we obtain $\lim_{n\to\infty} y_n/n^{k+1}=\infty$. This means that the theorem remains true for $k+1$ and an inductive argument follows.

Remark 1. The theorem we have proved shows that under suitable assumptions equation (E1) does not possess any solution which could be asymptoticaly neighbouring any sequence of polynomial type $w_n=a_kn^k+\ldots+a_1n+a_0$, $a_0,\ldots,a_k$ — constants in such a meaning that there does not exist any constant C for which $\lim_{n\to\infty} y_n/w_n=C$. Compare Korczak, Migda [2], [3]. Furthermore we can observe that for an arbitrary constant $\alpha\in R_0$ and arbitrary nonoscillatory solution of (E 1)

$$\lim_{n\to\infty} y_n/n^\alpha=\infty \quad\text{or}\quad \lim_{n\to\infty} y_n/n^\alpha=-\infty$$

in the case y is eventually positive or negative respectively.

Example 1. Consider the equation

$$\Delta[x_n-3x_{n-1}]+x_{n-1}=0.$$

It is evident that the hypotheses of Theorem 1 hold. Hence for an arbitrary nonoscillatory solution we have

$$y_n/n^k\to\infty \quad\text{or}\quad y_n/n^k\to-\infty$$

what is easy checked as the result, the general solution of this equation is $x_n=C_12^n+C_2n2^n$. In the second part of the paper we give the conditions for the oscillation of all solutions of some difference equations. We shall need the following Lemma.

Lemma. *Let $k\in\mathbf{N}$, $P:\mathbf{N}\to\mathbf{R}_+$. If*

$$\liminf_{n\to\infty}\sum_{j=0}^{k-1}p_{n+j}>1$$

then

(i) *the inequality $\Delta x_n-p_nx_{n+k}\le 0$, $n\in\mathbf{N}$ has no eventually negative solutions;*
(ii) *the inequality $\Delta x_n-p_nx_{n+k}\ge 0$ $n\in\mathbf{N}$ has no eventually positive solutions.*

If

$$\liminf_{n\to\infty}\sum_{j=0}^{k}p_{n+j}>1$$

then

(iii) *the inequality $\Delta x_n+p_nx_{n-k}\le 0$, $n>k$ has no eventually positive solutions;*
(iv) *the inequality $\Delta x_n+p_nx_{n-k}\ge 0$, $n>k$ has no eventually negative solutions.*

Proof. We shall prove the case (i). The rest of these cases can be proved in a very similar way.

Suppose in contrary to (i) that inequality

$$\Delta x_n-p_nx_{n+k}\le 0 \quad n\in\mathbf{N}$$

has a solution y which is eventually negative say $y_n<0$ for $n\ge n_1$.

By the hypotesis there exists n_2 such that for all $n\ge n_2$

$$\sum_{j=0}^{k-1}p_{n+j}>1. \tag{16}$$

For $n\ge n_3:=\max[n_1,n_2]$ we have

$$\Delta y_n\le p_ny_{n+k}<0$$

this means that the sequence (y_n) is strictly decreasing, thus

$$y_{n+k}>y_{n+k+1}>\ldots>y_{n+2k-1}.$$

Hence

$$p_{n+i}y_{n+k}>p_{n+i}y_{n+k+i}\ge\Delta y_{n+i} \qquad i=0,1,\ldots$$

Summing the above inequality with respect to i we obtain

$$y_{n+k}\sum_{i=0}^{k-1}p_{n+i}>y_{n+k}-y_n.$$

So,

$$y_{n+k}\left(\sum_{i=0}^{k-1}p_{n+i}-1\right)+y_n>0.$$

The above given inequality cannot be satisfied by (16) and our assumption that y is eventually negative is false. The case (i) is therefore proved.

Remark 2. The Lemma can be useful in studying of the oscillatory behaviour of solutions of some difference equations. Let us consider the equation

$$x_{n+2}-bx_{n+1}+cx_n=0, \qquad b,c\in\mathbf{R}_+. \tag{E 2}$$

If $c>b^2/2$ then every solution of (E 2) is oscillatory. This follows from the Lemma. Indeed dividing (E 2) by b^{n+2} and substituting $z_n=x_{n+1}/b^{n+1}$ we obtain the equation

$$\Delta z_n+cb^{-2}z_{n-1}=0. \tag{17}$$

Since $c>b^2/2$ then $c/b^2+c/b^2>1$. Therefore by (iii) equation (17) does not possess any solution which is eventually positive, while by (iv) any solution which is eventually negative. Hence all solutions of (17) must be oscillatory. But if $(z_n)=\{x_{n+1}/b^{n+1}\}$ oscillates the same occures with the sequence (x_n) by $b>0$.

Let us consider the equation

$$\Delta[x_n+px_{n-t}]+\sum_{i=1}^{k}q_{i,n}x_{n-s_i}=0, \qquad n>t. \tag{E3}$$

Theorem 2. *Let $p<-1$, $t>s:=\max_{1\le i\le k}s_i$, $s_i\in\mathbf{N}$, $i=1,\ldots,k$; $q_i:\mathbf{N}\to\mathbf{R}_0$. Let furthermore there exists an index $r\in\{1,\ldots,k)$ and a constant $q\in\mathbf{R}_+$ such that $q_{r,n}\ge q$ for all $n\in\mathbf{N}$. If*

$$\liminf_{n\to\infty}\sum_{j=0}^{t-s-1}\sum_{i=1}^{k}q_{i,n+j}>-p \tag{18}$$

then every solution of (E3) oscillates.

Proof. Let us suppose that the Theorem does not hold and similary as in the proof of Theorem 1 take y which is eventually positive ($y_n>0$ for $n\geq n_1$). Let us denote

$$z_n=y_n+py_{n-t} \qquad n>t \tag{19}$$

and take $n\geq n_2:=n_1+t$. So, we have

$$\Delta z_n=-\sum_{i=0}^{k} q_{i,n}y_{n-s_i}<0. \tag{20}$$

Following the same way as in the proof of Theorem 1 we can prove that there exists n_3 such that for all $n\geq n_3$ $z_n<0$. By (19) we get

$$z_n>py_{n-t}, \quad n>t. \tag{21}$$

From (21) it follows

$$z_{n+(t-s_i)}>py_{n-s_i} \quad \text{for all} \quad i\in\{1,\ldots,k\}.$$

Since $t-s\leq t-s_i$ and the sequence $\{z_n\}$ is decreasing it is easy to observe that $z_{n+t-s}\geq z_{n+t-s_i}$.

Therefore $z_{n+(t-s)}>py_{n-s_i}, \quad i=1,\ldots,k.$

Thus

$$\sum_{i=1}^{k}\left(\frac{1}{-p}\right)q_{i,n}z_{n+(t-s)}>-\sum_{i=1}^{k}q_{i,n}y_{n-s_i}. \tag{22}$$

By (20) and (22) for the sequence z defined by (19) we get

$$\Delta z_n=-\sum_{i=1}^{k}q_{i,n}y_{n-s_i}<\left(-\frac{1}{p}\sum_{i=1}^{k}q_{i,n}\right)z_{n+(t-s)}.$$

Therefore $\{z_n\}$ satisfies the inequality

$$\Delta z_n-\left(\frac{1}{-p}\sum_{i=1}^{k}q_{i,n}\right)z_{n+(t-s)}<0. \tag{23}$$

From (18) it follows that hypothesis of the Lemma (i) is satisfied and that the eventually negative sequence z satisfies inequality (23) of the type contained in part (i). But this is impossible, so the proof is complete.

Theorem 3. *Let* $p<-1$, $t>s:=\max\limits_{1\leq i\leq k} s_i$, $s_i\in\mathbf{N}$, $q_i:\mathbf{N}\to\mathbf{R}_0$ *be periodic with period* t, $i=1,\ldots,k$ *and let exist an index* $r\in\{1,\ldots,k\}$ *and a constant* $q\in\mathbf{R}_+$ *such that* $q_{r,n}\geq q$ *for all* $n\in\mathbf{N}$. *If*

$$\liminf_{n\to\infty}\sum_{j=0}^{t-s-1}\sum_{i=1}^{k}q_{i,n+j}>-(1+p) \tag{24}$$

then every solution of (E 3) oscillates.

Proof. Suppose, for contradiction that there is an eventually positive solution y of (E 3). Let $y_n>0$ for $n\geq n_1$.

Let us denote

$$z_n=y_n+py_{n-t}, \qquad n>t \tag{25}$$

and

$$w_n=z_n+pz_{n-t}, \qquad n>2t. \tag{26}$$

It can be easy proved by virtue of the periodicity of $\{q_{i,n}\}$ that sequences $\{z_n\}$, $\{w_n\}$ are the solutions of (E 3) for suitable n. It is not difficult to define z_n, w_n for $n\leq t$ and $n\leq 2t$, of these sequences to satisfy (E 3) for all $n\in\mathrm{MN}$. Take $n\geq n_2:=n_1+t$. We have

$$\Delta z_n=-\sum_{i=1}^{k}q_{i,n}y_{n-s_i}<0. \tag{27}$$

Proceeding in a similar way as in Theorem 1 we find $n_3\geq n_2$ such that for all $n\geq n_3$ $z_n<0$. Taking $n\geq n_4:=x_3+t$ we obtain

$$\Delta w_n=-\sum_{i=1}^{k}q_{i,n}z_{n-s_i}>0. \tag{28}$$

Similary we can show that there exists $n_5\geq n_4$ such that $w_n>0$ for all $n\geq n_5$. By (26) and (27) it follows

$$w_n<(1+p)z_{n-t}. \tag{29}$$

Set in (29) successively $n+(t-s_i)$, $i=1,\ldots,k$ instead of n we obtain

$$w_{n+(t-s_i)}<(1+p)z_{n-s_i}.$$

Since $t-s\leq t-s_i$, w is increasing, then

$$\sum_{i=1}^{k}\left(-\frac{1}{1+p}q_{i,n}\right)w_{n+(t-s)}<-\sum_{i=1}^{k}q_{i,n}z_{n-s_i}. \tag{30}$$

From (28) and (30) it follows that the sequence w defined by (26) satisfies inequality

$$\Delta w_n=-\sum_{i=1}^{k}q_{i,n}z_{n-s_i}>\left(-\frac{1}{1+p}\sum_{i=1}^{k}q_{i,n}\right)w_{n+(t-s)}$$

or in other form

$$\Delta w_n-\left(-\frac{1}{1+p}\sum_{i=1}^{k}q_{i,n}\right)w_{n+(t-s)}>0. \tag{31}$$

Inequality (31) is of the form considered in Lemma (ii). By (24) the hypothesis of the Lemma holds. But w is a evetually positive solution of (31), and this is impossible, so the proof of theorem is complete.

Example 2. Let us consider an algebraic equation

$$z^{t+1}-z^t+qz^{t-s}+pz-p=0$$

where $t>s$, $t,s\in\mathrm{N}$, $p<-1$, $q>-(1+p)$, $p,q\in\mathrm{R}$. From Theorem 3 it follows that this equation does not possess any root in the set R_0.

Indeed we have that all solutions of the difference equation

$$x_{n+1}-x_n+px_{n+1-t}-px_{n-t}+qx_{n-s}=0$$

with p,q,s,t defined above oscillates, but in this case all roots of the considered algebraic equation should be negative or complex numbers.

Theorem 4. *Let* $p\in\mathrm{R}_+$, $t<s:=\min\limits_{1\leq i\leq k} s_i$, $s_i\in\mathbf{N}$, $q_i:\mathbf{N}\to\mathbf{R}_0$ *be periodic with period* t, $i=1,\ldots,k$ *and let exist an index* $r\in\{1,\ldots,k\}$ *and a constant* $q\in\mathrm{R}_+$ *such that* $q_{r,n}\geq q$ *for all* $n\in\mathbf{N}$. *If*

$$\liminf \sum_{j=1}^{s-t} \sum_{i=1}^{k} q_{i,n+j} > 1+p$$

then every solution of (E3) oscillates.

The proof of this theorem is similar to the previous one and will be omitted.

Finally we prove a theorem which treats the asymptotic behaviour of nonoscillatory solutions of (E 3).

Theorem 5. *Let $p \in \mathbf{R}_+$, $t, s_i \in \mathbf{X}$, $q_i : \mathbf{X} \to \mathbf{R}_0$ $i=1, \ldots, k$ and let exist an index $r \in \{1, \ldots, k\}$ and a constat $q \in \mathbf{R}_+$ such that $q_{r,n} \geq q$ for all $n \in \mathbf{X}$. Then every nonoscillatory solution of (E 3) tends to 0 as $n \to \infty$.*

Proof. Suppose (y_n) is nonoscillatory solution of (E 3) which is eventually positive (for $n \geq n_1$). Let us denote

$$z_n = y_n + p y_{n-t} \quad \text{for} \quad n > t. \tag{32}$$

Take $n \geq n_2 := n_1 + \max(t, \max_{1 \leq i \leq k} s_i)$. We obtain

$$\Delta z_n = -\sum_{i=1}^{k} q_{i,n} y_{n-s_i} < 0. \tag{33}$$

Therefore the sequence $\{z_n\}$ is strictly decreasing and by (32) is positive for $n \geq n_2$. Thus $\lim_{n\to\infty} z_n = g$ exists, is finite and $g \geq 0$. Then from (33) we obtain

$$z_n - z_{n_2} = -\sum_{j=n_2}^{n-1} \sum_{i=1}^{k} q_{i,j} y_{j-s_i}$$

and therefore

$$z_{n_2} - g = \sum_{j=n_2}^{\infty} \sum_{i=1}^{k} q_{i,j} y_{j-s_i}.$$

Hence the series $\sum_{j=n_2}^{\infty} \sum_{i=1}^{k} q_{i,j} y_{j-s_i}$ is convergent. It is easy to check that from convergence of the above series we can get the convergence of the series $\sum_{j=n_2}^{\infty} y_j$ and $\sum_{j=n_2}^{\infty} y_{j-t}$ and consequently by (32) convergence of the series $\sum_{j=n_2}^{\infty} z_j$ so it must be $\lim_{n\to\infty} z_n = 0$ which by (32) implies $\lim_{n\to\infty} y_n = 0$. The proof is complete.

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ON THE BOUNDEDNESS AND THE STABILITY OF SOME DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER

BY

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1. Introduction. We shall be concerned here with the differential equations

$$x^{(4)} + f_1(\dot{x}, \ddot{x}, \dddot{x})\dddot{x} + f_2(\dot{x}, \ddot{x}) + g(\dot{x}) + h(x) = 0 \tag{1}$$

and

$$x^{(4)} + f_1(\dot{x}, \ddot{x}, \dddot{x})\dddot{x} + f_2(\dot{x}, \ddot{x}) + g(\dot{x}) + h(x) = p(t) \tag{2}$$

in which f_1, f_2, g, h and p are continuous functions depending only on the arguments shown and $\frac{\partial f_1}{\partial \ddot{x}}, \frac{\partial f_1}{\partial \dddot{x}}, \frac{\partial f_2}{\partial \ddot{x}}, \frac{dg}{d\dot{x}}, \frac{dh}{dx}$ exist and are continuous for all values of their arguments. Moreover, the existence and the uniqueness of the solutions of (2) will be assumed. Let $x(t)$ be a solution. Therefore we may write

$$\frac{dx}{dt} = \dot{x}, \quad \frac{d^2x}{dt^2} = \ddot{x}, \quad \frac{d^3x}{dt^3} = \dddot{x}, \quad \frac{d^4x}{dt^4} = x^{(4)}.$$

In what follows, we use the folowing system, which is equivalent to (2):

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = w, \quad \dot{w} = -f_1(y, z, w)w - f_2(y, z) - g(y) - h(x) + p(t). \tag{3}$$

The problem in this paper was established by Ezeilo [3] for the equation

$$x^{(4)} + f(\ddot{x})\dddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + \alpha_4 x = p(t).$$

Harrow [5], [6], [7] derived interesting results for the problem

$$x^{(4)} + a\dddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t).$$

Later, Lalli and Skrapek [8] established similar results for

$$x^{(4)} + f_1(\ddot{x})\dddot{x} + f_2(\dot{x}, \ddot{x}) + g(\dot{x}) + h(x, \dot{x}) = p(t).$$

Abou-El-Ela [1] investigated the boundedness results for

$$x^{(4)} + f(\dot{x}, \ddot{x})\dddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + \alpha_4 x = p(t).$$

Objectively, I will like to show similar results for the equations (1) and (2).

Theorem 1. *Suppose that*