$$\liminf \sum_{j=1}^{s-t} \sum_{i=1}^{k} q_{i,n+j} > 1+p$$

then every solution of (E3) oscillates.

The proof of this theorem is similar to the previous one and will be omitted.

Finally we prove a theorem which treats the asymptotic behaviour of nonoscillatory solutions of (E 3).

Theorem 5. *Let $p\in \mathbf{R}_+$, $t, s_i \in \mathbf{N}$, $q_i : \mathbf{N}\to \mathbf{R}_0$ $i=1, \ldots, k$ and let exist an index $r\in\{1, \ldots, k\}$ and a constat $q\in \mathbf{R}_+$ such that $q_{r,n}\geq q$ for all $n\in \mathbf{N}$. Then every nonoscillatory solution of (E 3) tends to 0 as $n\to\infty$.*

Proof. Suppose (y_n) is nonoscillatory solution of (E 3) which is eventually positive (for $n\geq n_1$). Let us denote

(32) $$z_n = y_n + p y_{n-t} \quad \text{for} \quad n>t.$$

Take $n\geq n_2 := n_1 + \max (t, \max\limits_{1\leqslant i\leqslant k} s_i)$. We obtain

(33) $$\Delta z_n = -\sum_{i=1}^{k} q_{i,n} y_{n-s_i} < 0.$$

Therefore the sequence $\{z_n\}$ is strictly decreasing and by (32) is positive for $n \geqslant n_2$. Thus $\lim\limits_{n\to\infty} z_n = g$ exists, is finite and $g\geq 0$. Then from (33) we obtain

$$z_n - z_{n_2} = -\sum_{j=n_2}^{n-1} \sum_{i=1}^{k} q_{i,j} y_{j-s_i}$$

and therefore

$$z_{n_2} - g = \sum_{j=n_2}^{\infty} \sum_{i=1}^{k} q_{i,j} y_{j-s_i}.$$

Hence the series $\sum\limits_{j=n_2}^{\infty} \sum\limits_{i=1}^{k} q_{i,j} y_{j-s_i}$ is convergent. It is easy to check that from convergence of the above series we can get the convergence of the series $\sum\limits_{j=n_2}^{\infty} y_j$ and $\sum\limits_{j=n_2}^{\infty} y_{j-t}$ and consequently by (32) convergence of the series $\sum\limits_{j=n_2}^{\infty} z_j$ so it must be $\lim\limits_{n\to\infty} z_n = 0$ which by (32) implies $\lim\limits_{n\to\infty} y_n = 0$. The proof is complete.

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ON THE BOUNDEDNESS AND THE STABILITY OF SOME DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER

BY

HÜSEYIN BEREKETOGLU

1. Introduction. We shall be concerned here with the differential equations

$$x^{(4)}+f_1(\dot x, \ddot x, \dddot x)\dddot x+f_2(\dot x, \ddot x)+g(\dot x)+h(x)=0 \tag{1}$$

and

$$x^{(4)}+f_1(\dot x, \ddot x, \dddot x)\dddot x+f_2(\dot x, \ddot x)+g(\dot x)+h(x)=p(t) \tag{2}$$

in which f_1, f_2, g, h and p are continuous functions depending only on the arguments shown and $\frac{\partial f_1}{\partial x}$, $\frac{\partial f_1}{\partial \ddot x}$, $\frac{\partial f_2}{\partial \dot x}$, $\frac{dg}{d\dot x}$, $\frac{dh}{dx}$ exist and are continuous for all values of their arguments. Moreover, the existence and the uniqueness of the solutions of (2) will be assumed. Let $x(t)$ be a solution. Therefore we may write

$$\frac{dx}{dt}=\dot x, \quad \frac{d^2x}{dt^2}=\ddot x, \quad \frac{d^3x}{dt^3}=\dddot x, \quad \frac{d^4x}{dt^4}=x^{(4)}.$$

In what follows, we use the folowing system, which is equivalent to (2):

$$\dot x=y,\ \dot y=z,\ \dot z=w,\ \dot w=-f_1(y, z, w)w-f_2(y, z)-g(y)-h(x)+p(t). \tag{3}$$

The problem in this paper was established by Ezeilo [3] for the equation

$$x^{(4)}+f(\ddot x)\dddot x+\alpha_2\ddot x+g(\dot x)+\alpha_4 x=p(t).$$

Harrow [5], [6], [7] derived interesting results for the problem

$$x^{(4)}+a\dddot x+f(\ddot x)+g(\dot x)+h(x)=p(t).$$

Later, Lalli and Skrapek [8] established similar results for

$$x^{(4)}+f_1(\ddot x)\dddot x+f_2(\dot x, \ddot x)+g(\dot x)+h(x, \dot x)=p(t).$$

Abou-El-Ela [1] investigated the boundedness results for

$$x^{(4)}+f(\dot x, \ddot x)\dddot x+\alpha_2\ddot x+g(\dot x)+\alpha_4 x=p(t).$$

Objectively, I will like to show similar results for the equations (1) and (2).

Theorem 1. *Suppose that*

(i) *there is a finite constant* $\alpha_1>0$ *such that*

$$f_1(y, z, w)\geqq\alpha_1 \text{ for all } y, z, w;$$

(ii) $f_2(y, 0)=g(0)=h(0)=0$, *and there are finite constants* $\alpha_2>0$, $\alpha_3>0$, $\beta>0$ *such that*

$$f_2(y, z)/z\geqq\alpha_2, \text{ for all } y, z\neq0,$$

$$g(y)/y\geqq\alpha_3 \text{ for all } y\neq0,$$

$$h(x)/x\geqq\beta \text{ for all } x\neq0;$$

(iii) $\alpha_4-\alpha_1\Delta_0/4\alpha_3)<h'(x)\leqq\alpha_4$ *for all* x, *where* α_4 *is a positive constant and* Δ_0 *is a positive constant such that*:

(iv) $\{\alpha_1\alpha_2-g(y)/y\}\alpha_3-\alpha_1\alpha_4 f_1(y, z, 0)\geqq\Delta_0$ *for all* $y\neq0$, z;

(v) $g'(y)-g(y)/y\leqq\delta_1$ *for all* $y\neq0$, *where* $\delta_1<2\alpha_4\Delta_0/(\alpha_1\alpha_3^2)$;

(vi) $\dfrac{1}{z}\displaystyle\int_0^z f_1(y, s, 0)ds-f_1(y, z, 0)\leqq\delta_2$ *for all* $z\neq0$, *wghere* $\delta_2<2\Delta_0/(\alpha_1^2\alpha_4)$;

(vii) $y\dfrac{\partial f_1(y, z, 0)}{\partial y}\leqq0$, $z\dfrac{\partial f_1(y, z, 0)}{\partial y}\leqq0$ *and* $\dfrac{\partial f_2(y, z)}{\partial y}\leqq0$ *for all* y, z;

$$z\frac{\partial f_1(y, z, w)}{\partial w}+d_2 y\frac{\partial f_1(x, z, w)}{\partial w}\geqq0 \text{ for all } y, z, w, \text{ where } d_2=\varepsilon+\alpha_4\alpha_3^{-1};$$

(viii) $f_2(y, z)/z-\alpha_2\leqslant\alpha_3^3/\alpha^2\varepsilon_0$ *for all* $z\neq0$, *where* ε_0

$$0<\varepsilon_0<\varepsilon\leqq\min\left\{\frac{1}{\alpha_1}, \frac{\alpha_4}{\alpha_3}, \frac{\Delta_0}{2\alpha_1\alpha_3D_0}, \frac{\alpha_3}{4\alpha_4D_0}\left(\frac{2\alpha_4\Delta_0}{\alpha_1\alpha_3^2}-\delta_1\right), \frac{\alpha_4}{4D_0}\left(\frac{2\Delta_0}{\alpha_1^3\alpha_4}-\delta_2\right)\right\}. \quad (4)$$

$$D_0=\alpha_1\alpha_2+\alpha_2\alpha_3\alpha_4^{-1}.$$

Then every solution $x(t)$ *of* (1) *satisfies*

$$x\to0,\ \dot{x}\to0,\ \ddot{x}\to0,\ \dddot{x}\to0 \text{ as } t\to\infty. \quad (5)$$

Remark. When $f_1(\dot{x}, \ddot{x}, \dddot{x})\equiv\alpha_1$, $f_2(\dot{x}, \ddot{x})=\alpha_2\ddot{x}$, $g(\dot{x})=\alpha_3\dot{x}$, $h(x)=\alpha_4x$, then the hypotheses (i)–(viii) reduce to

$$\alpha_1>0,\ \alpha_2>0,\ \alpha_3>0,\ \alpha_4>0,\ \{\alpha_1\alpha_2\alpha_3-\alpha_3^2\}\alpha_4-\alpha_1^2\alpha_4>0,$$

which is the Routh-Hurwitz criterion for the asymptotic stability in the large of trivial solution of the equation

$$x^{(4)}+\alpha_1\dddot{x}+\alpha_2\ddot{x}+\alpha_3\dot{x}+\alpha_4x=0.$$

This theorem is a natural extension of Ezeilo's Theorem 1 [3]. When $f_1(y, z, w)$ is replaced by $f(z)$, $f_2(y, z)$ by α_2z, $g(y)$ kept the same, $h(x)$ by α_4x, then the Lyapunov functions and the hypotheses reduce completely to Ezeilo's theorem.

Also, the hypotheses in Theorem 1 reduce to the hypotheses of the relevant theorems by Lalli and Skrapek [8] and Harrow [5], up to very small differences. These differences are due to the fact that the Lyat Lyapunov functions are not identical.

Theorem 2. *If the conditions in hypotheses* (i)–(viii) *of Theorem 1 hold, and if further*

$$\int_0^t |p(s)|\,ds\leqq A<\infty$$

for all $t\geqq0$, *where* A *is some positive number, then given any finite numbers* x_0, y_0, z_0, w_0 *there is a finite constant* $D=D(x_0, y_0, z_0, w_0)$ *such that the unique solution* $x(t)$ *of* (2) *which is determined by the initial conditions*

$$x(0)=x_0,\ \dot{x}(0)=y_0,\ \ddot{x}(0)=z_0,\ \dddot{x}(0)=w_0$$

satisfies

$$|x(t)|\leqq D,\ |\dot{x}(t)|\leqq D,\ |\ddot{x}(t)|\leqq D,\ |\dddot{x}(t)|\leqq D \quad (6)$$

for all $t\geqq0$.

2. The function V (x, y, z, w). The main tool in the proofs of (5) and (6) is the differentiable function $V(x, y, z, w)$ defined by

$$2V(x, y, z, w)=2d_2\int_0^x h(s)ds+(\alpha_2d_2-\alpha_4d_1)y^2+2\int_0^y g(s)ds+$$

$$+2\int_0^z(d_1f_2(y, s)-d_2s)ds+2\int_0^z sf_1(y, s, 0)ds+d_1w^2+$$

$$+2h(x)y+2d_1h(x)z+2d_2y\int_0^z f_1(y, s, 0)ds$$

$$+2d_1zg(y)+2d_2yw+2zw$$

where

$$d_1=\varepsilon+\alpha_1^{1},\quad d_2=\varepsilon+\alpha_4\alpha_3^{-1}, \quad (7)$$

ε being the constant in (4).

We show that V is a Lyapunov function for the system (3) with $p(t)\equiv0$. The proof depends on two lemmas.

Now, let

$$\Phi(y, z, 0)=\begin{cases}\dfrac{1}{z}\displaystyle\int_0^z f_1(y, s, 0)ds & \text{for } z\neq0\\ f(y, 0, 0) & \text{for } z=0\end{cases}$$

it follows from (i) and (vi) that

$$\Phi(y, z, 0)\geqq\alpha_1 \text{ for all } y, z, \quad (8)$$

$$\Phi(y, z, 0)-f_1(y, z, 0)\leqq\delta_2 \text{ for all } y, z. \quad (9)$$

Lemma 1. *Under the hypotheses* (i)–(viii) *of Theorem* 1, $V(0,0,0,0)=0$ *and there is a positive constant* D_1 *such that*

$$V(x,y,z,w)\geqq D_1(x^2+y^2+z^2+w^2) \tag{10}$$

for all x, y, z, w.

Proof. Define $g(y)/y=g'(0)$ for $y=0$.
For $x\neq 0$, $2V(x,y,z,w)$ may be rewritten in the form

$$2V(x,y,z,w)=V_1+V_2+V_3+\frac{1}{\Phi(y,z,0)}\{w+z\Phi(y,z,0)+d_2y\Phi(y,z,0)\}^2+$$

$$+\frac{y}{g(y)}\left\{x\frac{h(x)}{x}+y\frac{g(y)}{y}+d_1\frac{g(y)}{y}z\right\}^2+$$

$$+\left\{d_1-\frac{1}{\Phi(y,z,0)}\right\}w^2+\left\{2\int_0^z sf_1(y,s,0)ds-z^2\Phi(y,z,0)\right\}$$

where

$$V_1=\{\alpha_2d_2-\alpha_4d_1-d_2^2\Phi(y,z,0)\}y^2+2\int_0^y g(s)ds-yg(y).$$

$$V_2=2d_1\int_0^z f_2(y,s)ds-\alpha_2d_1z^2+\left\{\alpha_2d_1-d_2-d_1^2\frac{g(y)}{y}\right\}z^2$$

$$=2d_1\int_0^z\left\{\frac{f_2(y,s)}{s}-\alpha_2\right\}sds+\left\{\alpha_2d_1-d_2-d_1^2\frac{g(y)}{y}\right\}z^2,$$

$$V_3=2d_2\int_0^x h(s)ds-\frac{y}{g(y)}\left[\frac{h(x)}{x}\right]^2x^2=$$

$$=2\int_0^x\frac{h(s)}{s}\left[\frac{\alpha_4}{\alpha_3}-\frac{y}{g(y)}h'(s)\right]sds+2\varepsilon\int_0^x h(s)ds.$$

$$\alpha_2d_2-\alpha_4d_1-d_2^2\Phi(y,z,0)=d_2\left\{\alpha_2-d_1\frac{g(y)}{y}d_2\Phi(y,z,0)\right\}+d_1\left\{d_2\frac{g(y)}{y}-\alpha_4\right\}>$$

$$>d_2\left\{\alpha_2-d_1\frac{g(y)}{y}-d_2\Phi(y,z,0)\right\}\text{by (7), (ii).}$$

Since $\Phi(y,z,0)=f_1(y,\tilde z,0)$, where $\tilde z=\theta z(0<\theta<1)$, we obtain

$$\alpha_2-d_1\frac{g(y)}{y}-d_2f_1(y,\tilde z,0)=\left\{\alpha_2-\frac{1}{\alpha_1}\frac{g(y)}{y}-\frac{\alpha_1}{\alpha_3}f_1(y,\tilde z,0)\right\}-\varepsilon\left\{\frac{g(y)}{y}+f_1(y,\tilde z,0)\right\}$$

$$\geqq\Delta_0/(\alpha_1\alpha_3)-\varepsilon(\alpha_1\alpha_2+\alpha_2\alpha_3\alpha_4^{-1})\text{ by (iv)}$$

$$=\Delta_0/(\alpha_1\alpha_3)-\varepsilon D_0. \tag{11}$$

But we note that

$$\int_0^y sg'(s)ds=yg(y)-\int_0^y g(s)ds.$$

By hypotheses (v),

$$2\int_0^y g(s)ds-yg(y)=\int_0^y g(s)ds-\int_0^y sg'(s)ds=\int_0^y\left\{\frac{g(s)}{s}-g'(s)\right\}sds\geqq-\frac{\delta_1}{2}y^2.$$

Therefore

$$V_1\geqq\frac{1}{2}\left\{\frac{2\alpha_4\Delta_0}{\alpha_1\alpha_3^2}-\frac{2\alpha_4D_0}{\alpha_3}\varepsilon-\delta_1\right\}y^2.$$

Since $0<2\alpha_4\Delta_0/(\alpha_1\alpha_3^2)-\delta_1$, we have

$$V_1\geqq 1/4\{2\alpha_4\Delta_0/(\alpha_1\alpha_3^2-\delta_1\}y^2\geqq 0\quad\text{by (4).}$$

By (7), (i) and (11)

$$\alpha_2d_1-d_2-d_1^2g(y)/y=d_1\{\alpha_2-d_1g(y)/y-d_2f_1(y,z,0)\}+d_2\{y,z,0)-1\}\geqq$$

$$\geqq d_1\{\alpha_2-d_1g(y)/y-d_2f_1(y,z,0)\}\geqq d_1\{\Delta_0/(\alpha_1\alpha_3)-\varepsilon D_0\}.$$

Also, by (ii)

$$V_2\geqq(\alpha_1)^{-1}\{\Delta_0/(\alpha_1\alpha_3)-\varepsilon D_0\}z^2.$$

Furthermore,

$$\int_0^z sf_1(y,s,0)ds=z\int_0^z f_1(y,s,0)ds-\int_0^z s\Phi(y,s,0)ds$$

we get

$$2\int_0^z sf_1(y,s,0)ds-z^2\Phi(y,z,0)=\int_0^z\{f_1(y,s,0)-\Phi(y,s,0)\}sds\geqq$$

$$\geqq-(\delta_2/2)z^2\text{ by (9).}$$

Since $0<(2\Delta_0/\alpha_1^2\alpha_3)-\delta_2$, and so by (4)

$$V_2+2\int_0^z sf_1(y,s,0)ds-z^2\Phi(y,z,0)\geqq 1/2\{2\Delta_0/(\alpha_1^2\alpha_3)-2D_0\varepsilon/\alpha_1-\delta_2\}z^2,\geqq$$

$$\geqq 1/4\{2\Delta_0/(\alpha_1^2\alpha_3)-\delta_2\}z^2\geqq 0.$$

By hypotheses (iii) $\alpha_4/\alpha_3-y/g(Y)h'(x)\geqq 0$; and by (ii)

$$V_3\geqq\varepsilon\beta x^2.$$

From the definition of d_1 and (8)

$$\{d_1-1/\Phi(y,z,0)\}w^2\geqq\varepsilon w^2.$$

Hence, we obtain

$$2V(x,y,z,w)\geqq\varepsilon\beta x^2+1/4\{2\alpha_4\Delta_0/(\alpha_1\alpha_3^2)-\delta_1\}Y^2+1/4\{2\Delta_0/(\alpha_1^2\alpha_3)-\delta_2\}z_2+\varepsilon w^2$$

which proves that there is a positive constant D_1 such that

$V\geqq D_1(x^2+y^2+z^2+w^2)$ where

$$D_1=\min\{\varepsilon\beta/2,\ 1/8(2\alpha_1\Delta_0/(\alpha_1\alpha_3^2)-\delta_1),\ 1/8(2\Delta_0/(\alpha_1^2\alpha_3)-\delta_2),\ \varepsilon/2\}.$$

The case $x=0$ is clear.

Lemma 2. *Under the hypotheses (i)–(viii) of Theorem 1 there exist, a positive constnt D_2 such that if (x, y, z, w) is any solution of (3) with $p(t)=0$ then*

$$\dot V=\frac{d}{dt}V(x,\ y,\ z,\ w)\leqq -D_2(y^2+z^2+w^2).$$

Proof. A straightforward gives

$$\dot V=d_1z\int_0^z\frac{\partial f_2(y,\ s)}{\partial y}ds+z\int_0^z s\frac{\partial f_1(y,\ s,\ 0)}{\partial y}ds+d_2z\int_0^z y\frac{\partial f_1(y,\ s,\ 0)}{\partial y}ds-$$
$$-d_1f_1(y,z,w)w^2+w^2+d_2z^2f_1(y,\tilde z,0)+d_1z^2g'(y)-\alpha_2z^2-\{-\alpha_2d_2yz+d_2yf_2(y,\ z)+$$
$$+zf_2(y,\ z)-\alpha_2z^2\}-\{f_1(y,\ z,\ w)zw-f_1(y,\ z,\ 0)zw+f_1(y,\ z,\ w)d_2yw-$$
$$-f_1(y,\ z,\ 0)d_2yw\}-\{d_2g(y)/y-h'(x)\}y^2-\{\alpha_4-h'(x)\}\tilde d_1yz,\ \ \tilde z=\theta z(0<\theta<1).$$

Because of (vii), it follows that

$$z\int_0^z\frac{\partial f_2(y,s)}{\partial y}ds\leqq 0,\ z\int_0^z s\frac{\partial f_1(y,s,0)}{\partial y}ds\leqq 0\text{ and }z\int_0^z\frac{\partial f_1(y,\ s,\ 0)}{\partial y}ds\leqq 0$$

and by (i), we obtain

$$\dot V\leqq -\{\alpha_2-d_1g'(y)-d_2f_1(y,\ \tilde z,\ 0)\}z^2-\{d_1\alpha_1-1\}w^2-V_4-V_5-\{d_2g(y)/y-$$
$$-h'(x)\}y^2-\{\alpha_4-h'(x)\}d_1yz.$$

By the same way as in (11), it follows that

$$\alpha_2-d_1g'(y)-d_2f_1(y,\ \tilde z,\ 0)\geqq\Delta_0/(\alpha_1\alpha_3)-\varepsilon D_0\geqq\Delta_0/(2\alpha_1\alpha_3)\ \text{ by (4)}.$$

From (7)

$$d_1\alpha_1-1=\varepsilon\alpha_1.$$

Further, from (ii) $V_4=0$ when $z=0$, and again from (ii) if $z\neq 0$

$$V_4=(f_2(y,\ z)/z-\alpha_2)(z^2+d_2yz)\geqq -(f_2(y,\ z)/z-\alpha_2)d_2^2y^2/4.$$

Therefore, V_4 satisfies

$$V_4\geqq -(f_2(y,\ z)z-\alpha_2)d_2y^2/4\ \text{ for all } y,\ z$$

and so by (viii) and (7)

$$V_4\geqq -\varepsilon_0\alpha_3y^2.$$

Moreover, $V_5=0$ when $w=0$, and from (vii) if $w\neq 0$

$$V_5=\left\{\frac{f_1(y,\ z,\ w)-f_1(y,\ z,\ 0)}{w}\right\}zw^2+\left\{\frac{f_1(y,\ z,\ w)-f_1(y,\ z,\ 0)}{w}\right\}d_2yw^2=$$
$$=\left\{z\frac{\partial f_1(y,z,\mu w)}{\partial w}+d_2y\frac{\partial f^1(y,\ z,\ \mu w)}{\partial w}\right\}w^2\geqq 0\ \text{ where } 0<\mu<1.$$

By (7) and (ii)

$$\{d_2g(y)/y-h'(x)\}y^2\geqq\varepsilon\alpha_3y^2+(\alpha_4-h'(x))y^2$$

thus

$$\{d_2g(y)/y-h'(x)\}y^2+\{\alpha_4-h'(x)\}d_1yz\geqq\varepsilon\alpha_3y^2+(\alpha_4-h'(x)(y^2+d_1yz)\geqq$$
$$\geqq\varepsilon\alpha_3y^2-(\alpha_4-h'(x))d_1^2z^2/4\ by\ \text{(iii)}.$$

Now

$$\dot V\leqq -\Delta_0/(2\alpha_1\alpha_3)z^2-\varepsilon\alpha_1w^2+\varepsilon_0\alpha_3y^2-\varepsilon\alpha_3y^2+(\alpha_4-h'(x))d_1^2z^2/4=-(\varepsilon-$$
$$-\varepsilon_0)\alpha_3y^2-[\Delta_0/(2\alpha_1\alpha_3)-(\alpha_4-h'(x))\ d_1^2/4]\,z^2-\varepsilon\alpha_1w^2.$$

By (4) and hypotheses (iii)

$$\frac{\Delta_0}{2\alpha_1\alpha_3}-(\alpha_4-h'(x))\frac{d_1^2}{4}\geqq\frac{\Delta_0}{2\alpha_1\alpha_3}-(\alpha_4-h'(x))\frac{1}{\alpha_1^2}>\frac{\Delta_0}{4\alpha_1\alpha_3}.$$

Hence, we have

$$\dot V\leqq -(\varepsilon-\varepsilon_0)\alpha_3y^2-\Delta_0/(4\alpha_1\alpha_3)z^2-\varepsilon\alpha_1w^2\leqq -D_2(y_2+z^2+w^2)$$

where $D_2=\min\{(\varepsilon-\varepsilon_0)\alpha_3,\ \Delta_0/(4\alpha_1\alpha_3),\ \varepsilon\alpha_1\}$.

3. Proof of Theorem 1 By Lemma 1

$$V(x,\ y,\ z,\ w)=0\ \text{if and only if}\ x^2+y^2+z^2+w^2=0$$
$$V(x,\ y,\ z,\ w)>0\ \text{if and only if}\ x^2+y^2+z^2+w^2>0,$$
$$V(x,\ y,\ z,\ w)\to 0\ \text{if and only if}\ x^2+y^2+z^2+w^2\to\infty,$$

Also let $(x(t),\ y(t),\ z(t), w(t))$ be the solution of (3) with $p(t)=0$, such that $x(0)=x_0$, $Y(0)=y_0$, $z(0)=z_0$, $w(0)=w_0$, where $(x_0,\ y_0,\ z_0,\ w_0)$ is an arbitrary point in the $x,\ y,\ z,\ w$—space. Consider the function $y((t)\equiv V(x,\ y,\ z,$ $w)$ corresponding to this solution. Then by Lemma 2

$$V(t)\leqq V(0)\ \text{ for }\ t\geqq 0.$$

Hence, the proof of (5) can be obtained using the method in [3].

4. Proof of Theorem 2. The proof is based on the method devised by Antosiewicz [2]. Let $(x(t),\ y(t), z(t),\ w(t))$ be the solution of (3) with $x(0)=x_0$, $y(0)=y_0$, $z(0)=z_0$, $w(0)=w_0$ and consider the function $V(t)\equiv V(x(t),\ y(t),\ z(t),$ $w(t))$ where $V(x,\ y,\ z,\ w)$ is the function V used in the proof of Theorem 1. Using this function, we have

$$\dot V\leqq -D_2(y^2+z^2+w^2)+(d_1w+d_2y+z)p(t)\leqq|d_1w+d_2y+z|\,|p(t)|.$$

But

$$|d_1w+d_2y+z|\leqq D_3(|y|+|z|+|w|),$$

where $D_3=\max\{1,\ d_1,\ d_2\}$.

Since $|y|<1+y^2$, $|z|<1+z^2$, $|w|<1+w^2$ for all $y,\ z,\ w$ we get

$$\dot V\leqq D_3(3+y^2+z^2+w^2)|p(t)|.$$

By (10)

$$y^2+z^2+w^2 \leqq D_1^{-1}V$$

and thus

$$\dot{V}-D_5|p(t)|V \leqq D_4|p(t)|,$$

where $D_4=3D_3$ and $D_5=D_3/D_1$.

Therefore we obtain the result

$$V(t) \leqq 1/x(t)(V(0)+D_4\int_0^t|p(s)|x(s)ds)$$

where $x(t) \equiv \exp(-D_5\int_0^t|p(s)|ds)$. Since $x(t) \leqq 1$ for $t \geqq 0$

$$V(t) \leqq (V(0)+D_4A)e^{D_5A},$$

where $V(0)=V(x(0),\ y(0),\ z(0),\ w(0))$. This proves (6).

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ON THE STABILITY AND BIFURCATION OF SOLUTION OF A GENERAL SEIR EPIDEMIOLOGICAL MODEL

BY

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1. Introduction. Epidemiological models has been considered by many authors (see [10], [11]). In 1983 Smith [12] studied the 3-dimensional SIR system

$$S'=b-bS-H(I,\ S),\ I'=H(I,\ S)-(v+b)I\quad R'=vI-bR. \tag{1.1}$$

where S, I and R denote susceptible, infections and recovered individuals, respectively, v denotes recovery rate, the birth and death rates are assumed equal and represented by constant b. The author considered the special case $H(I,\ S)=\beta(1+\delta\cos 2\pi t)IS$. In 1985, Schwartz [11] have generalized this SIR model and considered the SEIR model

$$S'=b-bS-\beta(t)IS,\ E'=\beta(t)IS-(b+v)E,\ I'=\alpha E-(b+v)I,\ R'=bI-bR, \tag{1.2}$$

where E denotes the fraction of population which is exposed and the probability of still remaining in the exposed class at time τ after initial contact is $e^{-\alpha\tau}$. In 1986, Liu et al [6] studied another generalization of (1.1) however, they neglected the exposed population. They considered the 3-dimensional system

$$S'=-IH(I,\ s)-bs+\gamma R+\beta(N),\ I'=IH(I,\ s)-(b+v)I,\ R'=vI-(b+\gamma)R \tag{1.3}$$

where $H(I,\ s)$ is the incidence rate per infective individual. In the present paper we introduce an SEIR model which is a partial generalization of [6] and [11]. We consider the 4-dimensional system

$$\begin{gathered}S'=\beta(N)-bs-IH(I,\ s)+\gamma R,\ E'=IH(I,\ s)-(b+\alpha)E,\ I'=\alpha E-(b+v)I,\\ R'=vI-(b+\gamma)R,\end{gathered} \tag{1.4}$$

where γ as in (1.3) represents the per capita rate of loss of immunity. In section 2, we introduce a general 4-dimension model (1.4). We study the stability of steady states. Then we consider a special case of the incidence rate $H(I,\ s)$ discussed by [6]. In section, 3, we discuss the bifurcation of periodic solutions. We use a very similar techniques to [8], [2], [3], in applying Hopf bifurcation theory and to [1], [5] in applying center manifold theory.

2. On stability of a generalized SEIR model. We consider the generalized SEIR epidemic model