By (10)

$$y^2+z^2+w^2 \leq D_1^{-1}V$$

and thus

$$\dot{V}-D_5|p(t)|V \leq D_4|p(t)|,$$

where $D_4=3D_3$ and $D_5=D_3/D_1$.

Therefore we obtain the result

$$V(t) \leq 1/x(t)(V(0)+D_4\int_0^t |p(s)|x(s)ds)$$

where $x(t)\equiv\exp(-D_5\int_0^t |p(s)|ds)$. Since $x(t)\leq 1$ for $t\geq 0$

$$V(t) \leq (V(0)+D_4A)e^{D_5A},$$

where $V(0)=V(x(0),\ y(0),\ z(0),\ w(0))$. This proves (6).

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ON THE STABILITY AND BIFURCATION OF SOLUTION OF A GENERAL SEIR EPIDEMIOLOGICAL MODEL

BY

M. M. A. EL-SHEIKH and A. A. AMMAR

1. Introduction. Epidemiological models has been considered by many authors (see [10], [11]). In 1983 Smith [12] studied the 3-dimensional SIR system

$$S'=b-bS-H(I,\ S),\ I'=H(I,\ S)-(\nu+b)I\quad R'=\nu I-bR. \tag{1.1}$$

where S, I and R denote susceptible, infections and recovered individuals, respectively, ν denotes recovery rate, the birth and death rates are assumed equal and represented by constant b. The author considered the special case $H(I,\ S)=\beta(1+\delta\cos 2\pi t)IS$. In 1985, Schwartz [11] have generalized this SIR model and considered the SEIR model

$$S'=b-bS-\beta(t)IS,\ \ E'=\beta(t)IS-(b+\nu)E,\ I'=\alpha E-(b+\nu)I,\ R'=bI-bR, \tag{1.2}$$

where E denotes the fraction of population which is exposed and the probability of still remaining in the exposed class at time τ after initial contact is $e^{-\alpha\tau}$. In 1986, Liu et al [6] studied another generalization of (1.1) however, they neglected the exposed population. They considered the 3-dimensional system

$$S'=-IH(I,\ s)-bs+\gamma R+\beta(N),\ I'=IH(I,\ s)-(b+\nu)I,\ R'=\nu I-(b+\gamma)R \tag{1.3}$$

where $H(I,\ s)$ is the incidence rate per infective individual. In the present paper we introduce an SEIR model which is a partial generalization of [6] and [11]. We consider the 4-dimensional system

$$\begin{gathered} S'=\beta(N)-bs-IH(I,\ s)+\gamma R,\ E'=IH(I,\ s)-(b+\alpha)E,\ I'=\alpha E-(b+\nu)I, \\ R'=\nu I-(b+\gamma)R. \end{gathered} \tag{1.4}$$

where γ as in (1.3) represents the per capita rate of loss of immunity. In section 2, we introduce a general 4-dimension model (1.4). We study the stability of steady states. Then we consider a special case of the incidence rate $H(I,\ s)$ discussed by [6]. In section, 3, we discuss the bifurcation of periodic solutions. We use a very similar techniques to [8], [2], [3], in applying Hopf bifurcation theory and to [1], [5] in applying center manifold theory.

2. On stability of a generalized SEIR model. We consider the generalized SEIR epidemic model

$$S' = B(N) - bs - IH(I, S) + \gamma R,\ E' = IH(I, S) - (b+\alpha)E,$$
(1.4)
$$I' = \alpha E - (b+\nu)I,\ R' = \nu I - (b+\gamma)R$$

Following [6], [11], we restrict our attention for stable equilibrium population N_0 satisfying

$$N_0 = S + E + I + R > 0$$

and $B(N_0) = bN_0$ Thus (1.4) can be reduced to the 3-dimensional system

$$E' = IH(I, N_0 - E - I - R) - (b+\alpha)E = F_1(E,\ I,\ R),$$
(2.1)
$$I' = \alpha E - (b+\nu)I = F_2(E,\ I,\ R),\quad R' = \nu I - (b+\gamma)R = F_3(E,\ I,\ R)$$

We assume that all coefficients in (1.4) and the partial derivatives $\frac{\partial H}{\partial I}$ and $\frac{\partial H}{\partial S}$ are positive. Now the linearized problem corresponding to (2.1) is

(2.2) $$Y' = MY$$

where $Y = \begin{vmatrix} Y_1 \\ Y_2 \\ Y_3 \end{vmatrix}$, $(Y_1, Y_2, Y_3) \in R^3$

and

(2.3) $$M = \begin{vmatrix} -I\frac{\partial H}{\partial S} - (b+\alpha) & \frac{(b+\alpha)(b+\nu)}{\alpha} + I\left(\frac{\partial H}{\partial I} - \frac{\partial H}{\partial S}\right) & -I\frac{\partial H}{\partial S} \\ \alpha & -(b+\nu) & 0 \\ 0 & \nu & -(b+\gamma) \end{vmatrix}$$

It is clear that (2.1) has $p_0(0, 0, 0)$ as an equilibrium point and the variational matrix of (2.1) at p_0 is

(2.3) $$M_0 = \begin{vmatrix} -\beta_1 & KN_0 & 0 \\ \alpha & -\beta_3 & 0 \\ 0 & \nu & \beta_2 \end{vmatrix}$$

where $\beta_1 = b+\alpha$, $\beta_2 = b+\gamma$, $\beta_3 = b+\nu$. In this case the eigenvalues of M_0 are

$$\lambda_1 = -\beta_2 < 0$$

and the other two roots $\lambda_{2,3}$ satisfy the equation

(2.4) $$\lambda^2 + (\beta_1 + \beta_3)\lambda - \alpha KN_0 = 0$$

Hence since

(2.5) $$\lambda_2 + \lambda_3 = -(\beta_1 + \beta_3) < 0,\ \lambda_2\lambda_3 = -\alpha KN_0 < 0$$

one root λ_2 (or λ_3) is negative and the other is positive. Thus $p_0(0, 0, 0)$ is unstable.

Now the nontrivial equilibrium point is $p(E_0, I_0, R_0)$ where,

$$E_0 = \left(\frac{b+\nu}{\alpha}\right) I_0 \quad\text{and}\quad R_0 = \left(\frac{\nu}{b+\gamma}\right) I_0.$$

The variational matrix of (2.1) at p is

(2.6) $$M = \begin{vmatrix} -I_0Hs_0 - \beta_1 & \beta_1\beta_3/\alpha + I_0(HI_0 - Hs_0) & -I_0Hs_0 \\ \alpha & -\beta_3 & 0 \\ 0 & \nu & -\beta_2 \end{vmatrix}$$

and the characteristic equation is

(2.7) $$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where,

$$a_1 = \beta_1 + \beta_2 + \beta_3 + I_0H_{s_0},$$
$$a_2 = \beta_2(\beta_1 + \beta_3) + (\beta_2 + \beta_3)I_0H_{s_0} + I_0(H_{s_0} - H_{I_0}),$$

and

(2.8) $$a_3 = [\beta_2(\beta_3 + \alpha) + \alpha)]I_0H_{s_0} - \alpha\beta_2I_0H_{I_0}.$$

Since $p(E_0, I_0, R_0)$ is asymptotically stable if $a_1 > 0$, $a_3 > 0$ snd $a_1a_2 - a_3 > 0$, then $p(E_0, I_0, R_0)$ is asymptotically stable if the two conditions

(i) $$H_{I_0} < H_{s_0}$$

and

(ii) $$[(\beta_1 + \beta_2 + \beta_3) + I_0H_{s_0}][\beta_3(\beta_1 + \beta_3) + (\beta_2 + \beta_3)I_0H_{s_0} +$$
(2.9) $$+ \alpha I_0(H_{s_0} - H_{I_0})] > (\beta_2\beta_3 + \nu)I_0H_{s_0} + \alpha\beta_2I_0(H_{s_0} - H_{I_0})$$

are satisfied.

Remark. The special case $H(I, s) = H(s) = Ks$, was discussed by *Liu et al* [6] for a 2-dimension reduced system. Here we insert this special case in the 3-component system (2.1), we have

(2.10) $$E' = KI(N_0 - E - I - R) - \beta_1E) \qquad I' = \alpha E - \beta_3I, \qquad R' = \nu I - \beta_2R.$$

Thus the nontrivial equilibrium point is $p(E_0, I_0, R_0)$ where,

$$E_0 = \frac{\beta_2\beta_3}{\alpha}\,\theta,\ \ I_0 = \beta_2\theta \ \text{ and }\ R_0 = \nu\theta,$$

where

$$\theta = K^{-1}[\alpha KN_0 - \beta_1\beta_2].$$

Now, since E_0, I_0 and R_0 assumed to be positive, by (2.9), either $\beta_1\beta_2 < \alpha KN_0$ and $\alpha\nu < \beta_2(\beta_3 - \alpha)$ or $\beta_1\beta_2 > \alpha KN_0$ and $\alpha\nu > \beta_2(\beta_3 - \alpha)$. We note that the two conditions (i) and (ii) of (2.9) are satisfied. So $p(E_0, I_0, R_0)$ of (2.10) is an asymptotically stable equilibrium point.

3. Bifurcation of periodic solutions. It is well known that the characteristic roots of (2.2) obey the characteristic equation

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \tag{3.1}$$

where a_j's denote the sum of the principal minors of M of order j. By (2.3).

$$a_1 = \text{trace } M = -\left[\beta_1 + \beta_2 + \beta_3 + I\frac{\partial H}{\partial S}\right]$$

$$a_2 = F_{2I}(F_{1E} + F_{3R}) - F_{2E}F_{1I} + F_{1E}F_{3R} = \beta_3\left(\beta_2 + I\frac{\partial H}{\partial S}\right) - I\left(\frac{\partial H}{\partial I} - \frac{\partial H}{\partial S}\right) +$$

$$+ \beta_2\left(\beta_1 + I\frac{\partial H}{\partial S}\right)$$

and

$$a_3 = \det M = -\beta_2\beta_3 I\frac{\partial H}{\partial S} + \alpha\beta_2 I\left(\frac{\partial H}{\partial I} - \frac{\partial H}{\partial S}\right) - \alpha\nu I\frac{\partial H}{\partial S}. \tag{3.2}$$

Since the determinant of the fundamental matrix for (2.2) is the exponential of the integral of the trace, the product of the characteristic roots of any periodic orbit is less than one. This implies that at leas one characteristic root has norm less than one and we have the following result.

Lemma 3.1. (see Hartman [4] pp. 255). *For each periodic solution of (2.1) there exists a 4-dimensional stable manifold.*

It is clear also by (3.2) that if the coefficients α, ν, b, chosen so that α; ν large enough and b small then $a_1 < 0$, $a_2 > 0$ and $a_3 < 0$.

Now, choosing γ, (the per capita rate of loss of immunity) a bifurcation parameter for (2.1). let γ_C be the value of γ at which (3.1) has two pure imaginary eigenvalues $\lambda_{1,2}$. Since equation (3.1) should have at least one real root, say λ_* we may have the factorization

$$(\lambda - \lambda_*)[\lambda^2 + (\lambda_* - a_1)\lambda + (\lambda_*^2 - a_1\lambda_* + a_2)] = 0 \tag{3.3}$$

But since $\lambda_1 + \lambda_2 + \lambda_3 = a_1$, at $\gamma = \gamma_C$ we get

$$\lambda_* = a_1 \text{ and } \lambda_1 = \bar{\lambda}_2$$

$$\lambda_{1,2} = -1/2\{(\lambda_* - a_1) \pm \sqrt{(\lambda_* - a_1)^2 - 4(\lambda_*^2 - a_1\lambda_* + a_2)}\}. \tag{3.4}$$

Now at $\gamma = \gamma_C$ the equation (3.1) can be written in the form

$$F_\gamma(a_1) = a_1 a_2 - a_3 = 0 \tag{3.5}$$

Hence since $a_3 < 0$ and $a_2 > 0$ at $\gamma = \gamma_C$ we should have $\lambda_* = a_1 < 0$.

Also, the critical value $\gamma = \gamma_C > 0$ is a solution of (3.5), which is the quadratic equation

$$-c_1\gamma^2 - c_2\gamma + c_3 = 0, \tag{3.6}$$

where

$$c_1 = \beta_3 + \beta_2\left(\beta_1 + I\frac{\partial H}{\partial S}\right) > 0,$$

$$c_2 = \alpha I\left(\frac{\partial H}{\partial I} - \frac{\partial H}{\partial S}\right) - \left(\beta_1 + I\frac{\partial H}{\partial S}\right)\delta - \beta_3\delta = \alpha I\left(\frac{\partial H}{\partial I} - \frac{\partial H}{\partial S}\right) - \delta^2$$

and

$$c_3 = I\frac{\partial H}{\partial S}\,|\,\alpha\nu - \beta_3\delta + \alpha I\left(\frac{\partial H}{\partial I} - \frac{\partial H}{\partial S}\right)|\;\delta - 1\;|.$$

$$\text{where } \delta = \left(\beta_1 + \beta_3 + I\frac{\partial H}{\partial S}\right).$$

Conversely, knowing that $a_3 < 0$ and $a_1 < 0$, $\gamma > 0$, if we can solve equation (3.6) for $\gamma_C > 0$, we then know that $a_2 > 0$ and λ_1, λ_2 are conjugate imaginary. Since by (3.6) for large α, ν, small be and $\frac{\alpha(1+\nu)}{\alpha+\beta^2} > \delta > \alpha I\left(\frac{\partial H}{\partial I} - \frac{\partial H}{\partial S}\right) > 1$, c_1 and c_2 are negative and c_3 is positive. $F_0(a_1) = -c_3 > 0$ and $\lim_{\gamma \to \pm\infty} F_\gamma(a_1) = -\infty$.

Thus γ_C is uniquely determined. Suppose that $\gamma > \gamma_C$. Then, from equation (3.6) we conclude that $F_\gamma(a_1) < 0$. On the other hand, if $\gamma < \gamma_C$ then $F_\gamma(a_1) > 0$.

Now, since by equation (3.3), $\lambda_* < 0$ and

$$F_\gamma(a_1) = a_1 a_2 - a_3 = (a_1 - \lambda_*)(\lambda_1\lambda_2 - a_1\lambda_*),$$

$$\text{Sgn } F_\gamma(a_1) = \text{Sgn } (a_1 - \lambda_*).$$

Consequently, if $\gamma > \gamma_C$, then $\text{Re}\,\lambda_{1,2} = 1/2(a_1 - \lambda_*) < 0$ and for $\gamma > \gamma_C$, $R_e\lambda_{1,2} > 0$. Thus we have the following result.

Theorem 1. *If the parameters, ν and b chosen so that $\frac{\alpha(1+\nu)}{\alpha+\beta^3} > \delta > \alpha I\left(\frac{\partial H}{\partial I} - \frac{\partial H}{\partial S}\right) > 1$, then at $\gamma = \gamma_C$, there exists an one parameter family of periodic solutions bifurcating from the critical point $p(E_0, I_0, R_0)$ with period T, where $T \to T_0$ as $\gamma \to \gamma_C$, $T_0 = 2\pi/\omega_0 = 2\pi/\sqrt{a_2}$ and a_2 as given in (3.2).*

Proof. By the above discussion it is clear that as γ is increased through γ_C, there exists a pair of complex conjugate eigenvalues $\lambda_{1,2}$ of the characteristic equation (3.1).

Since at $\gamma = \gamma_C$, $\lambda_* = a_1$, $\lambda_{1,2} = \pm i\sqrt{a_2} = i\,\omega_0$, $\omega_0 > 0$.

Now, since by assumption $\lambda_1 = \bar{\lambda}_2$, at $\gamma = \gamma_C$, $\text{Re}\,\lambda_2 = 1/2(\lambda_2 + \bar{\lambda}_2) = 0$.

Thus by above.

$$\text{Re}\,\lambda_2 \gtrless 0 \text{ for } \gamma \gtrless \gamma_C$$

and

$$\frac{d}{d\gamma}(\mathrm{Re}\,\lambda_2)\Big|_{\gamma=\gamma_c} = -1/2\,\frac{d}{d\gamma}(\lambda_* - a_1)\Big|_{\gamma=\gamma_c} = \mathrm{Re}\left(\frac{d}{d\gamma}\lambda_2\right)\Big|_{\gamma=\gamma_c} < 0.$$

This completes the proof.

Remark. Using the technique of [9], the Jacobian matrix M can be written in the diagonal form

$$\begin{vmatrix} \dot y_1 \\ \dot y_2 \\ \dot y_3 \end{vmatrix} = \begin{vmatrix} k_{11} & k_{12} & 0 \\ k_{21} & k_{22} & 0 \\ 0 & 0 & a_2 \end{vmatrix}.$$

where $k_{12} = -k_{21} = -\sqrt{a_2}$.

Now, using the substitution $\rho = \gamma - \gamma_c$, then $k_{11} = k_{22}$ at $\rho = 0$ and we can write (2.2) in the form

$$\dot y = \Phi(y, \rho), \quad \dot\rho = 0.$$

Then, a center manifold C exists for this canonical suspended system at $(Y, \rho) = (0, 0) \in \mathbb{R}^3 \times \mathbb{R}^1$ (see [1], [5], [7]).

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THE C-PROJECTIVE CURVATURE OF SOME REAL HYPERSURFACES IN COMPLEX SPACE FORMS

BY

V. OPROIU

P. Matzeu and the present author have defined in [2] the C-pro-
jective curvature tensor field of a torsion free connection, adapted to a normal
almost contact structure on an odd-dimensional manifold. In the case of a
Sasakian manifold this C-curvature tensor field does vanish if ans only if the
considered Sasakian manifold has constant φ-sectional curvature. In the present
paper we study the C-projective curvature in the case of the normal almost
contact structures induced on certain real haypersurfaces of complex space
forms. Recall that a normal almost contact structure is induced on a real
hypersurface of a Kaehlerian manifold if and only if the Weingarten maopping
does commute with the linear operator defining the almost contact structure
on the hypersurface. In this case the Weingarten mapping has at most three
distinct eigenvalues (see [3], [6]). We show that the η-umbilical real hyper-
surfaces of the complex space forms are C-projectively flat i.e. their C-pro-
jective curvature tensor field vanishes. The normal almost contact real hyper-
surfaces of a complex space form for which the Weingarten mapping has exactiy
three distinct eigenvalues are never projectively flat. The manifolds and tensor
fields considered in this paper are C^∞. The usual notations from the diffe-
rential geometry are used throughout this paper.

1. Real hypersurfaces of complex space forms. Let $\bar M$ be a Kaehlerian
manifold of complex dimension $n+1$. Denote by J the almost complex struc-
ture tensor field of $\bar M$ and let $\bar g$ be the metric tensor field of $\bar M$. Then

$$J^2 = -I, \quad \bar g(JX, JY) = \bar g(X, Y); \quad X, Y \in \mathcal{X}(\bar M) \tag{1}$$

and if $\bar\nabla$ is the Levi Civita connection of $\bar g$ we have

$$(\bar\nabla_X J)Y = 0; \quad X, Y \in \mathcal{X}(\bar M), \tag{2}$$

If $\bar M = \bar M^{n+1}(c)$ is an $(n+1)$-dimensional complex space form of constant holomorphic sectional curvature $4c$, the curvature tensor field $\bar R$ of $\bar\nabla$ is given by

$$\bar R_{XY}Z = c\{\bar g(Y, Z)X - \bar g(X, Z)Y + \bar g(JY, Z)JX - \bar g(JX, Z)JY - -2\bar g(JX, Y)JZ\}. \tag{3}$$

Consider $M \subset \bar M^{n+1}(c)$ a $(2n+1)$-dimensional real hypersurface in the complex space form $\bar M^{n+1}(c)$. Suppose M is orientable and let N be the unit