ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL. I. CUZA" IAŞI
Tomul XXXVII, s. I. a. Matematică, 1991, f. 4

OSCILLATORY PROPERTIES OF CERTAIN ODD ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

BY

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1. Introduction. Dahiya [2] studied and obtained conditions for the oscillatory properties of even order nonlinear delay differential equations of the form

$$(1)\qquad \left(r(t)x'(t)\right)^{2n-1} + \sum_{i=1}^{k} f_i(t)F_i(x_{\tau_i}(t), x'_{\sigma_i}(t), \dots, x_{\sigma_i}^{2n-1}(t)) + G(x_{a_1}(t), x'_{a_2}(t), \dots, x_{a_{2n}}^{2n-1}(t)) = 0$$

where

$$x_{\tau_i}(t) = x(t - \tau_i(t)), \quad i = 1, 2, \dots, k,$$
$$x_{\sigma_i}^{(j)}(t) = x^{(j)}(t - \sigma_i(t)), \quad j = 1, 2, \dots, 2n-1,$$
$$x_{a_m}^{(r)}(t) = x^{(r)}(t - a_m(t)), \quad r = 0, 1, \dots, 2n-1, \quad m = 1, 2, \dots, 2n.$$

The term j denotes the order of differentiation with respect to t and delays $\tau_i(t)$, $\sigma_i(t)$ and $a_m(t)$ are bounded by a common constant M non–negative, non–decreasing and continuous real valued functions of t

$f_i : \mathbf{R} \to \mathbf{R}$ are continuous for each i,

$F_i : \mathbf{R}^{2n} \to \mathbf{R}$ are continuous for each i,

$G : \mathbf{R}^{2n} \to \mathbf{R}$ is continuous.

Besides, sufficient smoothness condition is assumed without mention to ensure the existence and uniqueness of solutions of (1) in $[t_0, +\infty)$.

In this paper, we extend the results of Dahiya [2] to certain odd order nonlinear delay differential equations of the form

$$(2)\qquad \left(r(t)x'(t)\right)^{2n-2} - \sum_{i=1}^{k} f_i(t)F_i(x_{\tau_i}(t), x'_{\sigma_i}(t), \dots, x'^{\,2n-2}_{\sigma_i}(t)) = 0$$

where

$$x_{\tau_i}(t) = x(t - \tau_i(t)), \quad i = 1, 2, \dots, k,$$
$$x_{\sigma_i}^{(j)}(t) = x^{(j)}(t - \sigma_i(t)), \quad j = 1, 2, \dots, 2n-2.$$

The term j denotes the order of differentiation with respect to t and delays $\tau_i(t)$, $\sigma_i(t)$ are bounded by a common constant M, non-negative, nondecreasing, and continuous real valued functions of t.

$$\begin{aligned} &f_i : \mathbf{R} \to \mathbf{R} && \text{are continuous for each } i \\ &F_i : \mathbf{R}^{2n-1} \to \mathbf{R} && \text{are continuous for each } i \end{aligned} \tag{3}$$

2. Main Results. We need the following two lemmas for our main results.

Lemma 1. *(Kiguradge [8]). If $x(t) > 0$, $x'(t) > 0$, $x''(t) < 0$, and $x(t)$ is real, then for sufficiently large t, there exists a constant $L > 0$ such that*

$$x'(t) \,/\, x(t) \le L/t. \tag{4}$$

Lemma 2. *(Starkos and Petsoulas [6]). Under the hypothesis of Lemma 1, there exist constants $L_i > 0$, $i = 1, 2, \ldots, n$ such that*

$$x(t - \tau_i(t) \,/\, x(t) \ge L_i \tag{5}$$

and

$$\lim_{t\to\infty} \left(x(t - \tau_i(t)) \,/\, x(t)\right) = 1.$$

Theorem 1. *Suppose the following conditions hold in addition to (3)*

(i) $f_i(t) \ge 0$ for every $t \in [t_o, +\infty)$,

(ii) $r(t) \in C^{2n-2}[t_o, +\infty)$, $r(t)$ is bounded and satisfies $r(t) > 0$, $r'(t) \ge 0$, $(-1)^{j+1} r^{(j)}(t) \ge 0$, $j = 2, 3, \ldots, 2n-2$.

(iii) Sgn $F_i(x_1, x_2, \ldots, x_{2n-1})$ =sgn x_1 and $F_i(-x_1, -x_2, \ldots, -x_{2n-1}) = -F_i(x_1, x_2, \ldots, x_{2n-1})$ for all i.

(iv) There exists some index j such that $F_j(\lambda x_1, \lambda x_2, \ldots, \lambda x_{2n-1}) = \lambda^{2p} F_j(x_1, x_2, \ldots, x_{2n-1})$ for all $(x_1, x_2, \ldots, x_{2n-1}) \in \mathbf{R}^{2n-1}$ real $\lambda \ne 0$ and some integer $p \ge 0$.

(v) $F_j(x_1, x_2, \ldots, x_{2n-1}) \to \infty$ as $x_1 \to \infty$ with some index j, $1 \le j \le k$; and

(vi) $\int_{t_o}^{\infty} f_j(t)\, dt = +\infty$.

Then all bounded continuous solutions of equation (2) are oscillatory.

Proof. We assume the existence of a nonoscillatory solution $x(\tau) \ne 0$ of equation (2). Conditions of Theorem 1 imply that $-x(\tau)$ is again a solution of (2). Therefore without loss of generality, we can assume $x(t) > 0$ eventually.

Suppose for $t \ge t_o \ge 0$, $x(t)$ and $x(t - \tau_i(t))$ are positive for all i. Choose t_1 so large that $x(t)$, $x(t - \tau_i(t))$ and $f_i(t)$ are all positive in $[t_1, \infty)$. Due to sign condition on F_i, it follows from equation (2) that

$$\left(r(t)x'(t)\right)^{2n-2} - f_i F_i(x_{\tau_i}(t), x'_{\sigma_i}(t), \ldots, x_{\sigma_i}^{2n-2}(t)) > 0 \tag{7}$$

or

$$\left(r(t)x'(t)\right)^{2n-2} > 0 \quad \text{for } t \in [t_1, \infty). \tag{8}$$

This in turn implies that $\left(r(t)x'(t)\right)^{2n-3}$ is increasing and must eventually have a constant sign and since $r(t) > 0$, it implies that $x'(t)$ must have a constant sign. Hence there exists a conveniently large $t_2 \ge t_1$ such that for $t \ge t_2$, $x'(t)$ is either positive or negative.

Case I. $\quad x(t) > 0, \quad x'(t) < 0, \quad t \in [t_2, \infty).$

Now,

$$\left(r(t)x'(t)\right)^{2n-2} > 0 \quad \text{and} \quad r(t)x'(t) < 0.$$

We claim that

$$\left(r(t)x'(t)\right)' \le 0 \quad \text{for} \quad t \in [t_3, \infty), t_3 \ge t_2. \tag{9}$$

Eventually, if $\left(r(t)x'(t)\right)' > 0$, then $\left(r(t)x'(t)\right)''$ being monotonic must be nonpositive because if $\left(r(t)x'(t)\right)'' > 0$, then $x'(t)r(t)$ being concave up and increasing will eventually be positive a contradiction. Proceeding this way and remembering that all derivatives of $r(t)x'(t)$ are monotonic, we find that $\left(r(t)x'(t)\right)^{2n-2} < 0$, a contradiction to (8). Hence (9) holds.

Integrating (9) between t_3 and t_1 we obtain

$$r(t)x'(t) \le r(t_3)x'(t_3) < 0, \quad x'(t) \le \frac{r(t_3)x'(t_3)}{r(t)}. \tag{10}$$

Then from (10), we get

$$x(t) \le x(t_3) + r(t_3)x'(t_3) \int_{t_3}^{t} \frac{1}{r(s)}\, ds. \tag{11}$$

Now as $t \to \infty$, the right hand side of (11) tends to negative infinity which is a contradiction since $x(t) > 0$ in $[t_3, \infty)$ and $r(t)$ is bounded. Hence either $x(t)$ is oscillatory or the following case holds.

Case II. $\quad x(t) > 0, \quad x'(t) > 0 \quad \text{for } t \in [t_3, \infty).$

From (8)

$$\left(r(t)x'(t)\right)^{2n-2} > 0 \quad \text{and} \quad r(t)x'(t) > 0 \quad \text{in } [t_3, \infty).$$

We must have

$$\left(r(t)x'(t)\right)^{2n-3} < 0 \tag{12}$$

eventually, for if $\left(r(t)x'(t)\right)^{2n-3} > 0$, then $\left(r(t)x'(t)\right)^{2n-3}$ is concave up, increasing and therefore ultimately positive. This will eventually make $x(t) < 0$, a contradiction.

We claim that

$$(-1)^j \big(r(t)x'(t)\big)^j \le 0, \quad j = 0, 1, \ldots, 2n-2, \tag{13}$$

where j denotes the order of differentiation. To see this, suppose that $x(t)$ is bounded. If $\big(r(t)x'(t)\big)^{2n-3} > 0$, eventually then because of (12), $\big(r(t)x'(t)\big)^{2n-4}$ will be positive, and tend to infinity. Proceeding this way, we find that $r(t)x'(t) \to \infty$ as $t \to \infty$ a contrdiction. Hence $\big(r(t)x'(t)\big)^{2n-3} \le 0$ eventually and the claim holds by continuation of this process.

Since $r(t) > 0$, $r'(t) \ge 0$, $r''(t) \le 0, \ldots, r^{2n-3}(t) \le 0$, we get from (13) that

$$x(t) > 0, \quad x'(t) > 0, \quad x''(t) \le 0, \quad x'''(t) \ge 0, \ldots, x^{2n-1}(t) \le 0 \tag{14}$$

and

$$\lim_{t \to \infty} x^{(j)}(t) = 0, \quad j = 2, 3, \ldots, 2n-2.$$

Now since $x(t) > 0$, $x'(t) > 0$ and $x''(t) \le 0$, we appeal to Lemma 1 to obtain that for large t

$$x'(t) \,/\, x(t) \le L \,/\, t \tag{15}$$

where L is some positive constant. Hence

$$0 < \frac{x'_{\sigma_i}(t)}{x_{\sigma_i}(t)} \le \frac{x'_{\sigma}(t - \sigma_i(t))}{x(t'\tau_i(t))} \le \frac{x'(t-m)}{x(t-m)}, \; t \in [t_1 + m, \infty)$$

and by (15) it follows that

$$\lim_{t \to \infty} \big(x'_{\sigma_i}(t) \,/\, x_{\sigma_i}(t)\big) = 0. \tag{16}$$

Following the same arguments as in Dahiya [2], we have

$$\lim_{t \to \infty} x_{\tau_i}(t) \,/\, x(t) = 1.$$

Now define

$$q(t) = -x(t) \,/\, \big(r(t)x'(t)\big)^{2n-3};$$

then

$$1/q(t) = -\big(r(t)x'(t)\big)^{2n-3} \,/\, x(t).$$

Differentiating and using (13) and (14),

(17)

$$\left[\frac{1}{q(t)}\right]' = \frac{-\big(r(t)x'(t)\big)^{2n-2}}{x(t)} + \frac{x'(t)\big(r(t)x'(t)\big)^{2n-3}}{x^2(t)}, \quad \left[\frac{1}{q(t)}\right]' < \frac{-\big(r(t)x'(t)\big)^{2n-2}}{x(t)},$$

since $\dfrac{x'(t)\big(r(t)x'(t)\big)^{2n-3}}{x^2(t)} < 0$.

From equations (2) and (17), we get

$$\left[\frac{1}{q(t)}\right]' < -\frac{1}{x(t)} \left[\sum_{i=1}^{k} f_i(t)F_i(x_{\tau_i}(t), x'_{\sigma_i}(t), \ldots, x_{\sigma_i}^{2n-2}(t))\right]. \tag{18}$$

If we appeal to the fact that for $t \in [t_i, +\infty)$, $f_iF_i > 0$ for all i this leads to the fact that

$$\left[\frac{1}{q(t)}\right]' < -\frac{1}{x(t)} f_j(t)F_j(x_{\tau_j}(t), x'_{\sigma_j}(t), \ldots, x_{\sigma_j}^{2n-2}(t)). \tag{19}$$

Applying condition *(iv)*, it follows that

$$\left[\frac{1}{q(t)}\right]' < \frac{-x_{\tau_i}(t)}{x(t)} \cdot x_{\tau_i}^{2p-1} f_j(t)F_j\Big(1, \frac{x'_{\sigma_j}(t)}{x_{\tau_i}(t)}, \ldots, \frac{x_{\sigma_j}^{2n-2}(t)}{x_{\tau_i}(t)}\Big). \tag{20}$$

Now $\displaystyle\lim_{t \to \infty} \big(x_{\tau_i}(t)/x(t)\big) = 1$ by Lemma 2. Clearly,

$$\lim_{t \to \infty} F_j\Big(1, \frac{x'_{\sigma_j}(t)}{x_{\tau_i}(t)}, \ldots, \frac{x_{\sigma_j}^{2n-2}(t)}{x_{\tau_i}(t)}\Big) = F_j(1, 0, 0, \ldots, 0) > 0$$

for all j. Therefore it follows from condition *(vi)* that

$$\int_{t_1+m}^{\infty} \frac{x_{\tau_i}(t)}{x(t)} f_jF_j\Big(1, \frac{x'_{\sigma_j}(t)}{x_{\tau_i}(t)}, \ldots, \frac{x_{\sigma_j}^{2n-2}(t)}{x_{\tau_i}(t)}\Big)\, ds = +\infty. \tag{21}$$

Integrating (20), we have

$$\frac{1}{q(t)} - \frac{1}{q(t_1+m)} = \int_{t_1+m}^{\infty} \left[\frac{1}{q(t)}\right]' dt.$$

$$\left[\frac{1}{q(t)}\right] < -x_{\tau_j}^{2p-1}(t) \int_{t_1+m}^{t} f_j(s)\frac{x_{\tau_j}(s)}{x(s)} F_j\Big(1, \frac{x'_{\sigma_j}(s)}{x_{\tau_j}(s)}, \ldots, \frac{x_{\sigma_j}^{2n-2}(s)}{x_{\tau_j}(s)}\Big)\, ds$$

and

$$\lim_{t \to \infty} \Big[\frac{1}{q(t)} - \frac{1}{q(t_1+m)}\Big] < -\infty. \tag{22}$$

Since $\big[\frac{1}{q(t)}\big]$ is positive and decreasing due to condition (13), therefore $\lim_{t \to \infty}\big[\frac{1}{q(t)}\big]$ is finite. In that case $\lim_{t \to \infty}\big[\frac{1}{q(t)} - \frac{1}{q(t_1+m)}\big]$ is finite but this contradicts the conclusion in (22). Hence the proof is complete.

Theorem 2. *Suppose equation (1) satisfies the following conditions in addition to the first four conditions of Theorem 1.*

(vii) $F_j(\lambda x_1, \lambda x_2, \dots, \lambda x_{2n-1}) = \lambda F_j(x_1, x_2, \dots, x_{2n-1})$ *for every* $(x_1, x_2, \dots, x_{2n-1}) \in \mathbf{R}^{2n-1}$ *and* $\lambda \in \mathbf{R}$;

(viii) $I \neq \phi$, *where* I *denotes the set of all indices for which the function* $F_j(x_1, x_2, \dots, x_{2n-1})$ *is non-increasing with respect to each variable* $x_2, x_4, \dots, x_{2n-2}$ *separately and increasing with respect to* $x_1, x_3, \dots, x_{2n-3}$ *as well as the function* $F_j(x, 0, 0, \dots, 0]/x$ *which are nonincreasing on* $(0, \infty)$;

(ix) *There exists a positive and differentiable function* $\phi(t)$, $t \le t_1$ *for some* t_1 *such that* $\phi(t) \le 0$ *and*

$$\int_{t_0}^{+\infty} \left[\phi(t)\left\{\sum f_i(t)F_i(t, 0, 0, \dots, 0)\right\} - \frac{\phi'^2(t) \mid r^{2n-3}(t) \mid}{4\phi(t)}\right] dt = +\infty$$

for every $\mu \ge 1$.

Then all bounded solutions of equation (1) are oscillatory.

Proof. We proceed as in Theorem 1 by assuming the existence of a non-oscillatory solution $x(t) > 0$ in $[t_1, \infty)$. Then claims (13) and (14) of Theorem 1 hold. Now define the transformation

$$z(t) = \frac{\big(r(t)x'(t)\big)^{2n-3}}{x(t-m)}\phi(t), \quad t \in [t_1, \infty). \tag{23}$$

Differentiating and making appropriate substitution from equations (1), we obtain

$$\begin{aligned} z'(t) = {} & \phi(t)\sum_{i=1}^{k} f_i(t)F_i(x_{\tau_i}(t), x'_{\sigma_i}(t), \dots, x_{\sigma_i}^{2n-2}(t)) - \\ & - \frac{x'(t-m)\big(r(t)x'(t)\big)^{2n-3}\phi(t)}{x^2(t-m)} + \frac{x(t-m)\big(r(t)x'(t)\big)^{2n-3}\phi(t)}{x^2(t-m)}. \end{aligned} \tag{24}$$

Proceeding as in Dahiya [2] we have that

$$z'(t) \ge \phi(t)\frac{\sum_{i=1}^{k} f_i(t)F_i(x(t-m), 0, 0, \dots, 0)}{x(t-m)} - \frac{\phi'^2(t) \mid r^{2n-3}(t) \mid}{4\,\phi(t)} \tag{25}$$

since $x_{\sigma_i}^{(j)}$ are decreasing and positive. Now,

$$x(t) \le x(t_1) + x'(t_1)(t - t_1) \qquad \text{for } t \in [t_1, +\infty) \text{ or}$$
$$x(t-m) \le x'(t_1)t + x(t_1) - (t_1 + m)x'(t_1)$$

and therefore exists $t_2 \ge t_1 + m$ and an appropriate constant $\mu \ge 1$ such that $x(t-m) \le \mu t$ for $t \in [t_2, +\infty)$.

Therefore

$$z'(t) \ge \phi(t)\sum_{i \in I}\frac{f_i(t)F_i(\mu t, 0, 0, \dots, 0)}{\mu t} - \frac{\phi'^2(t) \mid r^{2n-3}(t) \mid}{4\,\phi(t)}. \tag{26}$$

Clearly,

$$F_i(\mu\, t, 0, 0, \dots, 0) \ge F_i(t, 0, 0, \dots, 0) \quad \text{for } \mu \ge 1$$

and F_i are increasing in their arguments, we have from (26) that

$$z'(t) \ge \phi(t)\sum_{i \in I}\frac{f_i(t)F_i(t, 0, 0, \dots, 0)}{\mu t} - \frac{\phi'^2(t) \mid r^{2n-3}(t) \mid}{4\,\phi(t)} \qquad \text{or}$$

$$z(t) \ge z(t_1) + \int_{t_2+m}^{t_1}\left[\phi(s)\frac{\sum f_i(s)F_i(s, 0, 0, \dots, 0)}{\mu\, s} - \frac{\phi'^2(s) \mid r^{2n-3}(t) \mid}{4\,\phi(s)}\right] ds. \tag{27}$$

Now it is easy to see that $z(t)$ is eventually positive from (viii) which is a contradiction, since $z(t)$ is negative from (27).

Theorem 3. *Suppose equation (2) satisfy conditions (vi) and (vii) of Theorem 2 and the first four conditions of Theorem 1 along with the following condition*

(ix) *There exists a positive and differentiable function* $\phi(t)$, $t \ge t_1$ *for some* t_1 *such that* $\phi'(t) \le 0$ *and*

$$\int^{\infty} [\phi(t)\ \{\ \sum f_i(t)F_i(1, 0, 0, \dots, 0)\} - \frac{\phi'^2(t) \mid r^{2n-3}(t) \mid}{4\,\phi(t)}]\, dt = \infty.$$

Then, all bounded solutions of equation (1) are oscillatory.

Proof. Proceeding as in Dahiya [2] and making appropriate substitution, we have

$$z'(t) \ge \frac{x(t)}{x(t-m)}\phi(t)\left[\sum f_i(t)F_i\left(\frac{x(t-m)}{x(t)}, 0, 0, \dots, 0\right)\right] - \frac{\phi'^2(t) \mid r^{2n-3}(t) \mid}{4\,\phi(t)} \tag{28}$$

$$= \phi(t)\left[\sum f_i(t)F_i(1, 0, 0, \dots, 0)\right] - \frac{\phi'^2(t) \mid r^{2n-3}(t) \mid}{4\,\phi(t)}. \tag{29}$$

Clearly,

$$\frac{F_i(x(t-m)\ /\ x(t), 0, 0, \dots, 0)}{x(t-m)\ /\ x(t)} \ge F_i(1, 0, 0, \dots, 0)$$

by Lemma 2, $x(t-m)\ /\ x(t) \to 1$ as $t \to \infty$.

Integrating (29), we get

$$z(t) \ge z(t_1) + \int_{t_1}^{t}\left\{\phi(s)[f_i(s)F_i(1, 0, 0, \dots, 0)] - \frac{\phi'^2(s) \mid r^{2n-3}(s) \mid}{4\,\phi(s)}\right\} ds. \tag{30}$$

The right hand side tends to $+\infty$ as $t \to \infty$ due to condition *(ix)* and this is a contradiction since $z(t)$ is negative as defined in (23). Hence the proof.

Theorem 4. *Suppose equation (2) satisfies conditions (i)–(iv), (vii), (viii) and the following condition*

(x) There exists a positive and differentiable function $\phi(t)$, $t \geq t_1$ for some t_1 such that $\phi'(t) \leq 0$ and

$$\int^{\infty} \phi(t) \sum f_i(t) F_i(t,0,0,0,\ldots,0) \,/\, \mu t)\, dt = \infty.$$

Then all bounded solutions of equation (1) are oscillatory.

Proof. Differentiating (23), we have

$$z'(t) > \frac{1}{z(t-m)} \phi(t) \sum_{i \in I} f_i(t) F_i(x_{\tau_i}(t), x'_{\sigma_i}(t), \ldots, x_{\sigma_i}^{2n-2}(t)). \tag{31}$$

Since

$$\begin{aligned} z'(t) &> \phi(t) \sum_{i \in I} f_i(t) F_i(x(t-m), 0, 0, \ldots, 0) \,/\, x(t-m) \\ &> \phi(t) \sum_{i \in I} f_i(t) F_i(t, 0, 0, 0, \ldots, 0) \,/\, \mu t. \end{aligned} \tag{32}$$

Integrating (32), we get

$$z(t) > z(t_1) + \int_{t_1+m}^{t} \left[\phi(s) \sum f_i(s) F_i(s,0,0,\ldots,0) \,/\, \mu s\right] ds.$$

The right handside goes to infinity as $t \to \infty$. This is a contradiction since $z(t)$ is negative.

Example. Consider the fifth order delay differential equation

$$x^{(iv)}(t) - x(t-\pi) = 0, \quad f(t) = 1 > 0, \quad \int_0^{\infty} f(t)\, dt = \infty, \quad r(t) = 1 > 0.$$

The solution $x(t) = \cos t - \sin t$ is oscillatory.

It is clear that all the conditions of our Theorem 1 are satisfied.

REFERENCES

1. Bradely, J. S. — *Oscillation theorems for a second order delay equation.* J. Differ. Equations 8 (1970), 397–403.
2. Dahiya, R. S. — *Oscillation criteria of even–order nonlinear delay differential equations.* J. Math. Anal. Appl. 54 (1976), 653–665.
3. Gollwitzer, H. E. — *On non-linear oscillations for a second order delay equation.* J. Math. Anal. Appl. 26 (1969), 385–389.
4. Ladas, G., Ladde, G., Papadakis, J. S. — *Oscillations of functional differential equations generated by delays.* J. Differ. Equations 12, (1972), 385–395.
5. Onose, H. — *Oscillatory properties of ordinary differential equations of arbitrary order.* J. Differ. Equations 7 (1970), 454–458.
6. Starkos, V. A. and Petsoulas, A. G. — *Some oscillations criteria for second order non-linear delay differential equations.* J. Math. Anal. Appl. 30 (1970), 695–701.
7. Klong, J. W. — *A note on second order non-linear oscillations.* SIAM Rewiew 10 (1968), 88–91.
8. Kiguradge, I. T. — *On the oscillation of solutions of the equation* $\frac{d^m u}{dt^m} + a(t) \mid u \mid^n \operatorname{sgn} u = 0$. Mat. Sb. 65 (1964), 172–187 (Russian).

Received 31.V.1990

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