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VARIANTS OF THE WELL ORDERING THEOREM AND THE AXIOMS OF CHOICE AND MULTIPLE CHOICE

BY

S.M. KIM

One new direction in set theory research concerns axiomatic investigation with atomic elements. Atomic elements are pre-defined elements, including predefined information about these elements. The first independence proofs in set theory used Zermelo–Fraenkel Set Theory with urelements (ZFU). Urelements are atomic elements U such that $\forall A(A \in U \longleftrightarrow A \neq 0 \wedge \neg \exists B(B \in A))$.

Axiom systems with atomic elements are of interest because sets of atomic elements can contain information not available to an axiom system alone. For example, first–order Peano arithmetic is not categorical. This means that more than one model of the set of integers as defined by Peano's axioms exists. The set of integers can be defined using axiom systems such as Zermelo–Fraenkel Set Theory (ZF), and within a model of ZF such a set is unique. In a broader sense such a definition does not yield a unique set of integers. This is because many models of ZF are possible and the set of integers in one model need not be identical to the set of integers in another. In other words, the axioms of ZF do not contain all the information necessary to uniquely define the integers. One could consider however, the set of integers to be a set of atomic elements, then add this set to the axioms of ZF, and thus create a stronger axiom system. Note that $\text{con}(ZF) \leftrightarrow \text{con}(ZFU)$, where $\text{con}(ZF)$ stands for "ZF is consistent". In this paper we work within this ZFU to investigate the interdependence of variants of the axiom of choice (AC), the axiom of multiple choice (AMC) and the well ordering Theorem. Recall that the AMC states that for every family S of non–empty sets there exists a function f such that $f(x)$ is a finite non–empty subset of x, for each $x \in S$. It is well known that AC and AMC are equivalent in ZF. However Ulrich Felgner and Thomas J. Jech [1] proved a Theorem that in ZFU $(AC \rightleftharpoons AMC)$. Now it is natural to ask under what conditions imposed on AC or AMC the reverse implication above holds true in ZFU. Felgner [2] also showed via Cohen's forcing technique that in ZF a form of the well–ordering theorem does not imply a form of the AC (Descriptions of these forms will be given later). Here again there arises a question whether the converse of this nonimplication holds in ZFU. It is the aim of this paper to answer these questions. For the most part, our notation will be that commonly employed in set theory. In general, $\alpha, \beta, \gamma, \ldots$ denote ordinals. We refer the reader to [3] for any notion we do not cover. We start with:

Fact 1 (Felgner and Jech). $ZFU \vdash (AC \rightleftharpoons AMC)$.

Corollary 1. *If there is a choice function on any well ordered family of non-empty sets then the axiom of multiple choice holds for that well ordered family.*

Corollary 2. *If there is a choice function on any family of non-empty linearly ordered sets, then the axiom of multiple choice holds for that family.*

Corollary 3. *If there is a choice function on any well ordered family of non-empty linearly ordered sets, then the axiom of multiple choice holds for that family.*

Next we prove the converse of Corollary 3:

Theorem 1. *If the axiom of multiple choice holds for a well ordered family of non-empty linearly ordered sets, then there is a choice function on that family.*

Proof. Be given a well ordered family A of non-empty, linearly ordered sets. Denote each linear ordering by r. By hypothesis, there is a multiple choice function f on A. Then the set $\{x | \exists B \in A$ (x is the r-minimum of $f(B))\}$ is a choice set on A: Note that r on $f(B)$ is a well ordering. q.e.d.

We now present two Lemmas, but do not give the proofs as they are straightforward.

Lemma 1. *The axiom of multiple choice a linearly ordered family of linearly orderable, non-empty sets implies the same axiom for a well ordered family of linearly orderable, non-empty sets.*

Lemma 2. *The axiom of multiple choice for a well ordred family of non-empty linearly orderable sets implies the same axiom for a well ordered family of non-empty linearly ordered sets.*

Thus we have established:

Theorem 2. *The axiom of multiple choice for a linearly ordered family of linearly orderable, non-empty sets implies the axiom of choice for a well ordered family of linearly ordered non-empty sets.*

Proof. Apply Lemma 1, Lemma 2 and Theorem 1. q.e.d.

In [2] Felgner showed that the existance of well ordering for the union of a well ordered set of well orderable sets does not imply the axiom of choice for a family of well orderable, non-empty sets. Since the axiom of choice for a family of linearly orderable, non-empty sets implies the same axiom for a family of well orderable, non-empty sets, the above result by Felgner implies:

Theorem 3. *The existance of well ordering for the union of a well ordered set of well orderable sets does not imply the axiom of choice for a family of linearly orderable, non-empty sets.*

Now there arises a question: is the converse of Theorem 3 provable in ZFU? This is answered by:

Theorem 4. *The axiom of choice for a family of linearly orderable, non-empty sets implies the existence of well ordering for the union of a well ordered family of well orderable sets.*

Proof. Assume the axiom of choice for a family of linearly orderable, non-empty sets. Be given a well ordered set A of well orderable non-empty sets. Denote the set $\{C | \phi \neq C \subset B \in A\}$ by M. Note that $M \supset A$ and each C in M is well-orderable. So by hypothesis there exists a choice map f on M. Define a well ordering for any B in A by $g(\gamma) = f(B - g[\gamma])$ for γ with $B - g[\gamma] \neq \phi$, where $g(\gamma)$ means the γ-th element of the well ordering. We see that for any B in A a γ with $g[\gamma] = B$ exists.

Finally, well order $\cup A$ as follows. Given any C_1 and C_2 in $\bigcup A$, take B_i to be the smallest of A in which C_i is (i=1,2). Now C_1 is smaller than C_2 if and only if (B_1 is smaller than B_2 in the well ordering of A) or ($B_1 = B_2$ and C_1 is smaller than C_2 in the well ordering of B_1). q.e.d.

Open problem. In view Corollary 3 and Theorem 1 there is in ZFU an equivalence of the AC and the AMC for a well ordered family of non-empty linearly ordered sets, which, by Fact 1, is not true for every family of non-empty sets. Finally, we ask as an open problem whether in ZFU the axiom of choice for a family of lineraly orderable, non-empty sets implies the axiom of multiple choice for a well ordered or a linearly ordered family of non-empty sets.

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Department of Mathematics
Yonsei University
Seoul, Korea