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GREEN'S FUNCTIONS FOR LINEAR SECOND ORDER MATRIX SYSTEMS

BY

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1. Introduction. It is well known that for the scalar case, the solution of two-point boundary value problems for linear partial differential equations may be reduced to the solution of ordinary differential equations containing a parameter and subject to certain boundary value conditions. The boundary value problem

$$p_2(t)x^{(2)}(t) + p_1(t)x^{(1)}(t) + p_0(t)x(t) = 0$$
$$m_{i1}x(0) + n_{i1}x(b) + m_{i2}x^{(1)}(0) + n_{i2}x^{(1)}(b) = 0; \qquad i = 1,2$$

where m_{ij}, n_{ij}, for $1 \le i,j \le 2$, are constants and p_k are complex valued functions on the closed interval $[0,b]$ and $p_2(t) \neq 0$, on $[0,b]$, is completely studied in [1,6].

The aim of this paper is to construct Green's functions associated with the vectorial two-point boundary value problem

$$\begin{aligned} &x^{(2)}(t) + A_1x^{(1)}(t) + A_0x(t) = f(t), && 0 \le t \le b \\ &M_{i1}x(0) + N_{i1}x(b) + M_{i2}x^{(1)}(0) + N_{i2}x^{(1)}(b) = R_i; && i = 1,2 \end{aligned} \tag{1.1}$$

where for each t in $[0,b]$, the unknown $x(t)$, the value $f(t)$ of the continuous function f and all coefficients appearing in (1.1) are matrices in $\mathbf{C}^{n\times n}$. Boundary value matrix problems of the type (1.1), with simpler boundary conditions have been considered from the Green's functions point of view in several papers such as [15,16,9], but all of them have the inconvenience of reducing the second order problem to an extended first order equivalent problem. This fact involves the computation of the matrix exponential $\exp(tC)$, where

$$C = \begin{bmatrix} 0 & I \\ -A_0 & -A_1 \end{bmatrix}$$

In these approaches the Green's function is given in terms of $\exp(tC)$ which involves an increase of the computational cost of the problem and a lack of explicitness because the solution $x(t)$ of (1.1) is given in the form $x(t) = [I,0]y(t)$, where

$y(t)$ is the solution of the equivalent first order boundary value problem. In [5], the problem

$$x^{(2)}(t) + A_1 x^{(1)}(t) + A_0 x(t) = f(t),$$
$$y(0) = y(b) = 0, \qquad 0 \le t \le b$$

is studied avoiding the above reducing approach and the Green's function of the problem is explicit in terms of data dimension and appropriate solutions or the differential equation

$$x^{(2)}(t) + A_1 x^{(1)}(t) + A_0 x(t) = 0 \tag{1.2}$$

but although the existence of such solutions is clear, they are not constructed in an explicit way. In [11], an explicit construction of the Green's function associated with the boundary value problem is given in terms of a complete set of solutions $\{X_0, X_1\}$ of the algebraic matrix equation

$$X^2 + A_1 X + A_0 = 0. \tag{1.3}$$

In this paper we obtain an uniqueness condition for the solution of (1.1) as well as an explicit expression of the solution of the problem in terms of a Green's matrix function $G(t,s)$ constructed in terms of data dimension and only one solution X_0 of the matrix equation (1.3).

The paper is organized as follows. In section 2 an uniqueness condition for the boundary value problem (1.1) is given. Section 3 is concerned with the construction of the Green's function and in section 4 an explicit solution of the problem (1.1) is presented.

2. Uniqueness of solution. This paper is based on the existence of a solution X_0 of equation (1.3) which has been studied with different tecniques and by different authors, [2,3,4,6,10,12].

Lemma 1. *Let X_0 be a solution of (1.3) and let $U_1(t)$, $U_2(t)$ be the matrix functions defined by*

$$U_1(t) = \exp(tX_0); \quad U_2(t) = \int_0^t \exp[(t-s)X_0]\exp[-s(X_0+A_1)]ds \tag{2.1}$$

(i) the general solution of (1.2) is given by

$$x(t) = U_1(t)P + U_2(t)Q \tag{2.2}$$

where P, Q, are arbitrary matrices in $\mathbf{C}^{n\times n}$.

(ii) the block matrix function $W(t)$ defined by

$$W(t) = \begin{bmatrix} U_1(t) & U_2(t) \\ U_1^{(1)}(t) & U_2^{(1)}(t) \end{bmatrix} \tag{2.3}$$

is invertible for all real number t.

Proof. Part (i) is a consequence of lemma 1 of [7]. Note that

$$W(t) = \tag{2.4}$$

$$= \begin{bmatrix} \exp(tX_0) & \int_0^t \exp[(t-s)X_0]\exp[-s(X_0+A_1)]ds \\ X_0\exp(tX_0) & X_0\int_0^t \exp[(t-s)X_0]\exp[-s(X_0+A_1)]ds + \exp[-t(X_0+A_1)] \end{bmatrix}$$

Now, from lemma 1 of [8] it follows that $W(t)$ is invertible and

$$W(t)^{-1} = \begin{bmatrix} * & -\int_0^t \exp(-sX_0)\exp[(t-s)(X_0+A_1)]ds \\ * & \exp[t(X_0+A_1)] \end{bmatrix}. \tag{2.5}$$

Remark 1. Note that from lemma 1 the unique solution of (1.2) satisfying $x(0) = c_0$, $x^{(1)}(0) = c_1$, is given by (2.2), where

$$\begin{bmatrix} P \\ Q \end{bmatrix} = W(0)^{-1}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 - X_0c_0 \end{bmatrix}$$

Let us consider the homogeneous problem associated to (1.1)

$$\begin{aligned} &x^{(2)}(t) + A_1x^{(1)}(t) + A_0x(t) = 0; && 0 \le t \le b \\ &M_{i1}x(0) + N_{i1}x(b) + M_{i2}x^{(1)}(0) + N_{i2}x^{(1)}(b) = 0; && i = 1,2. \end{aligned} \tag{2.6}$$

Now, we are going to show when the boundary value problem (2.6) has no nontrivial solution.

By imposing that the general solution $x(t)$ of (1.2) given by (2.2) satisfies the boundary value conditions of (2.6), it follows that

$$S\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.7}$$

where $S = (S_{ij})_{1\le i,j\le 2}$ is the block matrix defined by

$$S_{ij} = M_{i1}U_j(0) + N_{i1}U_j(b) + M_{i2}U_1^{(1)}(0) + N_{i2}U_j^{(1)}(b); \qquad 1 \le i,j \le 2. \tag{2.8}$$

Hence it follows that problem (2.6) has no nontrivial solution if and only if the matrix S defined by (2.8) is non singular.

Let $x_p(t)$ be a particular solution of

$$x^{(2)}(t) + A_1 x^{(1)}(t) + A_0 x(t) = f(t) \tag{2.9}$$

then, from lemma 1, the general solution of (2.9) is given by

$$x(t) = x_p(t) + U_1(t)P + U_2(t)Q. \tag{2.10}$$

Note that $x(t)$ given by (2.10) satisfies (1.1) if and only if matrices P, Q satisfy

$$S\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix},$$

where

$$K_1 = R_1 - M_{11}x_p(0) - N_{11}x_p(b) - M_{12}x_p^{(1)}(0) - N_{12}x_p^{(1)}(b),$$

$$K_2 = R_2 - M_{21}x_p(0) - N_{21}x_p(b) - M_{22}x_p^{(1)}(0) - N_{22}x_p^{(1)}(b).$$

Hence, problem (1.1) has a unique solution if matrix S is non-singular.

3. Green's functions in terms of data dimension. We begin this section with a definition of Green's matrix function analogous to the one of scalar case [6].

Definition 1. *Let $G(t,s)$ be a $\mathbb{C}^{n\times n}$ valued matrix function defined on the square $0 \le t, s \le b$. We say that $G(t,s)$ is a Green's function for the problem (2.6) if the following properties are satisfied:*
i) $G(t,s)$ is continuous in $(t,s) \in [0,b]\times[0,b]$. Moreover, $\partial G/\partial t$ is continuous in (t,s), for (t,s) on each of the triangles $0 \le t < s \le b$, $0 \le s < t \le b$.
ii) If I is the identity matrix in $\mathbb{C}^{n\times n}$, one gets the jump discontinuity:

$$\frac{\partial G}{\partial t}(s+0,s) - \frac{\partial G}{\partial t}(s-0,s) = I. \tag{3.1}$$

iii) As a function of t, $G(t,s)$ satisfies (2.6) in $[0,b]$, if $t \neq s$.

In order to construct $G(t,s)$, let $\{U_1(.), U_2(.)\}$ be the pair of functions introduced in lemma 1. As $G(t,s)$ must satisfy (1.2) for $t \neq s$, then, from lemma 1, there exist matrices $P_i(s)$, $Q_i(s)$, $i = 1,2$, such that

$$\begin{aligned} G(t,s) &= U_1(t)P_1(s) + U_2(t)Q_1(s) \qquad 0 \le t < s, \\ G(t,s) &= U_1(t)P_2(s) + U_2(t)Q_2(s) \qquad s < t \le b. \end{aligned} \tag{3.2}$$

As $G(t,s)$ is continuous at $t = s$, one gets

$$[U_1(s)P_1 + U_2(s)Q_1] - [U_1(s)P_2 + U_2(s)Q_2] = 0. \tag{3.3}$$

From (3.1) it follows that

$$\left[U_1^{(1)}(s)P_2 + U_2^{(1)}(s)Q_2\right] - \left[U_1^{(1)}(s)P_1 + U_2^{(1)}(s)Q_1\right] = I. \tag{3.4}$$

Let us write

$$P_3 = P_2 - P_1, \qquad Q_3 = Q_2 - Q_1. \tag{3.5}$$

Then (3.3) and (3.4) take the form

$$U_1(s)P_3 + U_2(s)Q_3 = 0$$

$$U_1^{(1)}(s)P_3 + U_2^{(1)}(s)Q_3 = I$$

or

$$W(s)\begin{bmatrix} P_3 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{3.6}$$

where $W(s)$ is the invertible matrix defined by (2.3). Thus matrices P_3, Q_3, are uniquely determined. Now, from the expression of $(W(s))^{-1}$ given by (2.5), the unique solution of (3.6) is given by

$$\begin{aligned} P_3(s) &= -\textstyle\int_0^s \exp(-uX_0)\exp\left[(s-u)(X_0 + A_1)\right] du \\ Q_3(s) &= \exp\left[s(X_0 + A_1)\right]. \end{aligned} \tag{3.7}$$

Note that as matrices $P_3(s)$, $Q_3(s)$, are given by (3.7), from (3.5) it follows the Green's matrix function $G(t,s)$ defined by (3.2) is determined if we find matrices $P_2(s)$, $Q_2(s)$. For this we require that $G(t,s)$ defined by (3.2) satisfies the boundary value conditions of (2.6). Thus $P_2(s)$, $Q_2(s)$ must satisfy the algebraic system

$$S\begin{bmatrix} P_2(s) \\ Q_2(s) \end{bmatrix} = \begin{bmatrix} L_1(s) \\ L_2(s) \end{bmatrix}, \tag{3.8}$$

where S is given by (2.8) and

$$L_1(s) = M_{11}\left[U_1(0)P_3(s) + U_2(0)Q_3(s)\right] + M_{12}\left[U_1^{(1)}(0)P_3(s) + U_2^{(1)}(0)Q_3(s)\right]$$

$$L_2(s) = M_{21}\left[U_1(0)P_3(s) + U_2(0)Q_3(s)\right] + M_{22}\left[U_1^{(1)}(0)P_3(s) + U_2^{(1)}(0)Q_3(s)\right].$$

Taking into account (2.1), it follows that

$$\begin{aligned} L_1(s) &= (M_{11} + M_{12}X_0)P_3(s) + M_{12}Q_3(s), \\ L_2(s) &= (M_{21} + M_{22}X_0)P_3(s) + M_{22}Q_3(s), \end{aligned} \tag{3.9}$$

where $P_3(s)$, $Q_3(s)$ are given by (3.7).

Let T be the block inverse of the matrix S, $T = (T_{ij})_{1\le i,j\le 2}$, where T_{ij} is a matrix in $\mathbb{C}^{n\times n}$ for $1 \le i,j \le 2$. Then from (3.8)-(3.9) we have

$$\begin{aligned} P_2(s) &= T_{11}L_1(s) + T_{12}L_2(s) = T_{11}\left[(M_{11} + M_{12}X_0)P_3(s) + M_{12}Q_3(s)\right] + \\ &\quad + T_{12}\left[(M_{21} + M_{22}X_0)P_3(s) + M_{22}Q_3(s)\right], \\ Q_2(s) &= T_{21}L_1(s) + T_{22}L_2(s) = T_{21}\left[(M_{11} + M_{12}X_0)P_3(s) + M_{12}Q_3(s)\right] + \\ &\quad + T_{12}\left[(M_{21} + M_{22}X_0)P_3(s) + M_{22}Q_3(s)\right], \end{aligned} \tag{3.10}$$

where $P_3(s)$, $Q_3(s)$, are given by (3.7). Thus the following result has been established:

Theorem 1. *Let us assume that equation (1.3) has a solution X_0 and let U_1, U_2 be defined by (2.1). If the matrix S defined by (2.8) is non singular, then the boundary value matrix problem (2.6) has a unique Green's matrix function defined by (3.2), where $P_2(s)$, $Q_2(s)$ are defined by (3.10), $P_3(s)$, $Q_3(s)$ are determined by (3.7), and $P_1(s) = P_2(s) - P_3(s)$, $Q_1(s) = Q_2(s) - Q_3(s)$.*

4. Solution of the non-homogeneous boundary value problem. Let us consider the intermediate boundary value problem

$$\begin{aligned} &x^{(2)}(t) + A_1 x^{(1)}(t) + A_0 x(t) = f(t), \qquad 0 \le t \le b \\ &M_{i1}x(0) + N_{i1}x(b) + M_{i2}x^{(1)}(0) + N_{i2}x^{(1)}(b) = 0; \qquad i = 1,2 \end{aligned} \tag{4.1}$$

It is an easy matter from section 2 that if problem (2.6) is well-posed, then when f is continuous, the boundary value problem (4.1) has a unique solution. Now, we are interested in obtaining such a solution in terms of the constructed Green's matrix function. Let $x(t)$ be defined by

$$x(t) = \int_0^b G(t,s) f(s) ds, \tag{4.2}$$

where $G(t,s)$ is the Green's function constructed in section 3. Let us rewrite (4.2) as:

$$x(t) = \int_0^t G(t,s)f(s)ds + \int_t^b G(t,s)f(s)ds.$$

Taking derivatives and using the Leibnitz' rule, we have

$$x^{(1)}(t) = \int_0^t \frac{\partial G(t,s)}{\partial t} f(s)ds + G(t,t)f(t) + \int_t^b \frac{\partial G(t,s)}{\partial t} f(s)ds - G(t,t)f(t)$$

$$\begin{aligned} x^{(2)}(t) = &\int_0^t \frac{\partial^2 G(t,s)}{\partial t^2} f(s)ds + \frac{\partial G(t,t-0)}{\partial t} f(t-0) + \int_t^b \frac{\partial^2 G(t,s)}{\partial t^2} f(s)ds - \\ &- \frac{\partial G(t,t+0)}{\partial t} f(t+0) = \int_0^b \frac{\partial^2 G(t,s)}{\partial t^2} f(s)ds + f(t) \end{aligned}$$

because of continuity of $f(t)$ and (3.1).

Then, from property (*iii*) of definition 1, it follows that

$$\begin{aligned} &x^{(2)}(t) + A_1 x^{(1)}(t) + A_0 x(t) = \\ &= \int_0^b \left[\frac{\partial^2 G(t,s)}{\partial t^2} + A_1 \frac{\partial G(t,s)}{\partial t} + A_0 G(t,s) \right] f(s)ds + f(t) = f(t) \end{aligned}$$

Furthermore, as $G(t,s)$ satisfies (2.6) it follows that

$$\begin{aligned} &M_{i1}x(0) + N_{i1}x(b) + M_{i2}x^{(1)}(0) + N_{i2}x^{(1)}(b) = \\ &= \int_0^b \left[M_{i1}G(0,s) + N_{i1}G(b,s) + M_{i2}\frac{\partial G}{\partial t}(0,s) + N_{i2}\frac{\partial G}{\partial t}(b,s) \right] f(s)ds = 0; \end{aligned}$$

$i = 1,2$.

Thus, the following result has been proved:

Theorem 2. *The boundary value problem (4.1) provided that problem (2.6) is well-posed and $f(t)$ is continuous, has a unique solution $x(t)$ given by (4.2).*

Now the solution of the problem (1.1) can be easily obtained.

Let us consider the non-homogeneous systems:

$$\begin{aligned} &x^{(2)}(t) + A_1 x^{(1)}(t) + A_0 x(t) = 0, \qquad 0 \le t \le b, \\ &M_{11}x(0) + N_{11}x(b) + M_{12}x^{(1)}(0) + N_{12}x^{(1)}(b) = I, \\ &M_{21}x(0) + N_{21}x(b) + M_{22}x^{(1)}(0) + N_{22}x^{(1)}(b) = 0 \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &x^{(2)}(t) + A_1 x^{(1)}(t) + A_0 x(t) = 0, \qquad 0 \le t \le b, \\ &M_{11}x(0) + N_{11}x(b) + M_{12}x^{(1)}(0) + N_{12}x^{(1)}(b) = 0, \\ &M_{21}x(0) + N_{21}x(b) + M_{22}x^{(1)}(0) + N_{22}x^{(1)}(b) = I. \end{aligned} \tag{4.4}$$

As above, they have a unique solution. Let us denote $G_1(t)$ and $G_2(t)$ the solution of problem (4.3) and (4.4) respectively. From theorem 2 the solution of (1.1) is given by

$$x(t) = G_1(t)R_1 + G_2(t)R_2 + \int_0^b G(t,s)f(s)ds. \tag{4.5}$$

Now, from lemma 1, we may obtain $G_1(t)$ and $G_2(t)$ by imposing the boundary value conditions (4.3) and (4.4) to the expression

$$x(t) = U_1(t)P + U_2(t)Q$$

This lead us to

$$\begin{bmatrix} P \\ Q \end{bmatrix} = S^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix}$$

for $G_1(t)$, and

$$\begin{bmatrix} P \\ Q \end{bmatrix} = S^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix}$$

for $G_2(t)$.

Thus, the following result has been proved:

Theorem 3. *Let us assume that equation (1.3) has a solution X_0 and let U_1, U_2 be defined by (2.1). If the matrix S defined by (2.8) is non-singular, $T = S^{-1} = (T_{ij})_{1 \le i,j \le 2}$, and $f(t)$ is continuous, then the boundary value problem (1.1) has a unique solution given by (4.5), where*

$$G_1(t) = U_1(t)T_{11} + U_2(t)T_{21}$$
$$G_2(t) = U_1(t)T_{12} + U_2(t)T_{22}$$

Remark 2. Note that the solution of problem (1.1) provided by theorem 3 in terms of the Green's matrix function $G(t,s)$, involves the computation of the matrix exponentials $\exp(tX_0)$ and $\exp(t(X_0+A_1))$. There are many different techniques and procedures for computing the matrix exponential [13,14].

It is important to note that our approach allows us to construct in an explicit way the Green's function $G(t,s)$ and it is obtained without increasing the dimension of data. However, our algebraic approach is not universal because the associated algebraic equations (1.3) may be unsolvable, see [7], for instance. An interesting fact of our approach is that when one replaces in the Green's function $G(t,s)$, the exact solution X_0 by an approximate solution Z_n, for example when one solves equations (1.3) by an iterative method, [4], then the corresponding approximate Green's function $G_n(t,s)$, is such that the approximate solution $x_n(t)$ of (1.1), replacing everywhere Z_n by X_0, is of the same order of the approximation z_n of x_0. The proof of this fact is straighforward and is omitted.

Finally, we remark that all conclusions of this paper remain valid when $f(t)$ and $x(t)$ takes values into $B(X)$ the algebra of all bounded linear operators on a Banach space X. In this case, methods for obtaining solutions of (1.3) in the infinite–dimensional context may be found in [4] and [7].

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REFERENCES

1. E. A. Coddington and N. Levinson — *Theory of Ordinary Differential Equations*, New York, Mc Graw–Hill 1955.
2. J.E. Dennis, Jr, J.F. Traub and R.P. Weber — *The algebraic theory of matrix polynomials*, SIAM J. Numer. Anal., 13(1976), 831–845.
3. J.E. Dennis, Jr, J.F. Traub and R.P. Weber — *Algorithms for solvents of matrix polynomials*, SIAM J. Numer. Anal. 15(1978), 523–533.
4. J.E. Eisenfeld — *Operator equations and nonlinear eigenparameters problems*, J. Funct. Anal., 12(1973), 475–490.
5. K.A. Heimes — *Green's functions for linear second order systems*, SIAM J. Math. Anal. 9(1978), 207–214.
6. E.L. Ince — *Ordinary Differential Equations*, Dover, 1927.
7. L. Jódar — *Boundary value problem and Cauchy problems for second order operator equations*, Linear Algebra and its Appl. 83(1986), 29–38.
8. L. Jódar — *Explicit expression for Sturm- Liouville operator problems*, Proc. of the Edinburgh Math. Soc., 30(1987), 301–309.
9. L. Jódar — *Explicit solutions for second order operator differential equations with two boundary value conditions*, Linear Algebra and Appl., 103(1988), 73–86.
10. L. Jódar and E. Navarro — *On complete set of solutions for polynomial matrix equations*, Applied Maths. Letters, Vol.3, No.1(1990), 15–18.

11. L. Jódar — *Boundary value problems and Green's operator functions*, Glasnick Matematicky 24(44)(1989) 507–518.
12. P. Lancaster — *Lambda Matrices and Vibrating Systems*, Pergamon, Oxford, 1966.
13. C. Moler and C. Van Loan — *Nineteen dubious ways to compute the exponential of a matrix*, SIAM Review, 20(1978), 801–836.
14. C. Van Loan — *Computing integrals involving the matrix exponential*, IEEE Trans. Autom. Contr., AC.-23, 3(1978), 395–404.
15. R. Conti — *Recent trends in the theory of boundary value problems for ordinary differential equations*, Boll. Un. Mat. Ital., 22(1967), 135–178.
16. W. T. Reid — *Ordinary Differential Equations*, John Wiley, New York, 1971.

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