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UNIFORM DISSIPATIVE SOLUTIONS FOR SOME FOURTH ORDER NONLINEAR DIFFERENTIAL EQUATIONS

BY

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1. Introduction. Consider the following fourth order nonlinear differerential equations with two nonlinear functions:

$$x^{(iv)} + f_1(x''') + f_2(x'') + cx' + dx = p(t, x, x', x'', x'''); \tag{1.1}$$

$$x^{(iv)} + f_1(x''') + bx'' + f_3(x') + dx = p(t, x, x', x'', x'''); \tag{1.2}$$

and

$$x^{(iv)} + f_1(x''') + bx'' + cx' + f_4(x) = p(t, x, x', x'', x'''); \tag{1.3}$$

The case for the equation

$$x^{(iv)} + ax''' + bx'' + f_3(x') + f_4(x) = p(t, x, x', x'', x'''). \tag{1.4}$$

was studied in [5].

We shall be concerned with the uniform dissipativity of solutions for these equations using the frequency domain method. The situations where one of the nonlinear functions in each of the equations (1.1) through (1.3) is linear was discussed in [7,8] by Barbalat and Halanay. Problems of these form have been studied earlier by Ezeilo and Tejumola [9], Harrow [11], amongst others, usind the Lyapunov theory. For a good listing the reader is referred to Ressig, Sansone and Conti [14], Chapter 6 and the references therein. However, the use of the frequency-domain method introduced by Popov, Kalman and Yacubovich and latter generalizations by Yacubovich [15,16,17], Barbalat and Halanay [8] continues to attract the attention of many researchers. The recent work of Afuwape [1,5], Yunger [18] with the references therein are examples. In [1], a duality principle was introduced and applied to some third-order nonlinear differential equation arising from the Vacuum tube circuit problem. In continuation of this appplication,we shall consider dual equations to equations (1.1) through (1.3) in the forms:

$$(1.5)\qquad y^{(iv)}+\frac{d^3}{dt^3}[ay+l_1(y)]+\frac{d^2}{dt^2}[by+l_2(y)]+cy'+dy=p(t,y,y',y'',y''');$$

$$(1.6)\qquad y^{(iv)}+\frac{d^3}{dt^3}[ay+l_1(y)]+by''+\frac{d}{dt}[cy+l_3(y)]+dy=p(t,y,y',y'',y''');$$

and

$$(1.7)\qquad y^{(iv)}+\frac{d^3}{dt^3}[ay+l_1(y)]+by''+cy'+[dy+l_4(y)]=p(t,y,y',y'',y''').$$

2. Basic assumptions and notations. We shall give in this section the basic assumptions on our functions f_j, $(j=1,2,3,4)$ and p, which appear in equations (1.1) through (1.5).

2.1. Assumptions.

(A_1) functions f_j, $(j=1,2,3,4)$ and p are continuous in their respective arguments, with $f_j(0)=0$;

(A_2) there exist constants $a>0$, $\mu_1\geq 0$ such that

$$a\leq f_1(u)/u\leq a+\mu_1,\quad (u\neq 0)\text{ and } a\leq f'_1(u)\leq a+\mu_1;$$

(A_3) there exist constants $b>0$, $\mu_2\geq 0$ such that

$$b\leq f_2(z)/z\leq b+\mu_2,\quad (z\neq 0)\text{ and } b\leq f'_2(z)\leq b+\mu_2;$$

(A_4) there exist constants $c>0$, $\mu_3\geq 0$ such that

$$c\leq f_3(y)/y\leq c+\mu_3,\quad (y\neq 0)\text{ and } c\leq f'_3(y)\leq c+\mu_3;$$

(A_5) there exist constants $d>0$, $\mu_4\geq 0$ such that

$$d\leq f_4(x)/x\leq d+\mu_4, (x\neq 0)\text{ and } d\leq f'_4(x)\leq d+\mu_4;$$

(A_6) constants a, b, c and d in (A_2)–(A_5) satisfy

$$A_4 := (ab-c)c-a^2d>0$$

with a consequence that $b^2-4d>0$;

(A_7) Recognising that the roots w_0^2, w_1^2 of the quadraic equation $w^4-bw^2+d=0$ are real, we assume that

$$0<w_0^2<\frac{c}{a}<w_1^2<b$$

and note the identies

$$\omega_0^2(c\omega_1^2-ad)=d(c-a\omega_0^2),\text{ and } (\omega_1^2-\omega_0^2)=\sqrt{(b^2-4d)}=(2\omega_1^2-b);$$

(A_8) constants μ_1, μ_2, μ_3, and μ_4 appearing in (A_2) through (A_5) satisfy

(i) $$0\leq\mu_1<d(c-a\omega_0^2)(\omega_1^2-\omega_0^2)/(c\omega_0^4);$$

(ii) $$0\leq\mu_2<(c\omega_1^2-ad)/d\equiv(c-a\omega_0^2)/\omega_0^2;$$

(iii) $$0\leq\mu_3<(c\mu_1^2-ad)/d;$$

(iv) $$\mu_4\geq\delta_4>0;$$

(v) $$\mu_1/\mu_2<cd(\omega_1^2-\omega_0^2);$$

(vi) $$\omega_0^2<\mu_1/\mu_3<(a\omega_1^2-c)(c-a\omega_0^2)^{-1}\omega_1^2;$$

and

(A_9) the function $p(t,x,x',x'',x''')$ is bounded for all t.

2.2. Basic results. For completeness, we recall the following results for general nonlinear diferential system of the form

$$(2.1)\qquad X'=AX-B\phi(\sigma)+P(t,X),\quad \sigma=C^*X,$$

where A is a stable $(n\times n)$–real matrix, B and C are $(n\times m)$–real matrices, (with C^* as the transpose of C), $\phi(\sigma)=col(\phi_j(\sigma_j)),(j=1,2,...,m)$ and $P(t,X)$ is an n–vector.

Suppose that

(i) for $|\alpha|\geq\lambda_0$,

$$0\leq\alpha\phi_j(\alpha)\leq\mu_{0j}\alpha^2;\qquad -\alpha_{1j}\leq\phi_j'(\alpha)\leq\alpha_{2j}$$

with $\mu_{0j}\leq\alpha_{2j}$ and $\alpha_{1j}\geq 0$;

(ii) for all $(t,X)\in\mathbf{R}^{n+1}$, $|P(t,X)|\leq\rho_0$.

Let $M=\text{diag}(\mu_{0j})$; $D'=\text{diag}(\alpha_{1j})$; $D''=\text{diag}(\alpha_{2j})$ and for all $\omega\in\overline{\mathbf{R}}$, $G(i\omega)=C^*(i\omega I-A)^{-1}B$.

Theorem A [8]. *Suppose that for (2.1) there exist diagonal matrices $D_1 > 0$, $D_3 \geq 0$ and D_2 such that the frequency-domain matrix inequality*

$$D_1 + \mathrm{Re}\{[D_1 M + i\omega D_2]G(i\omega)\}+ \tag{2.2}$$

$$+\omega^2 \mathrm{Re}\{D_3[I + (D'' - D')G(i\omega)] - G^*(-i\omega)D_3 D' D'' G(i\omega)\} > 0$$

holds for all $\omega \in \overline{\mathbf{R}}$, and such that the supplementary condition

$$\lim_{|\lambda| \to \infty} D_2 \left(\frac{1}{\lambda_2}\right) \mathrm{diag} \left[\int_0^\lambda \phi_j(\alpha) d\alpha - \frac{\lambda}{2}\phi_j(\lambda)\right] \geq 0 \tag{2.3}$$

holds, then system (2.1) has uniformly dissipative solutions.

Definition [1]. *The system*

$$Y' = A_1 Y - B_1 \Phi(\tilde{\sigma}) + P_1(t, Y), \qquad \tilde{\sigma} = C^* Y$$

is called the dual of system (2.1) if $A_1 = A^*$, $B_1 = C$, $C_1 = B$ and $P_1 = TP$, with T a nonsingular matrix transformation.

Theorem B [1]. *If in system (2.1), (A, B) and (A^*, C) are completely controllable, then the frequency domain conditions for its dual systems are equivalent.*

2.3. Notations and initial results. In the process of using the generalized theorems (Theorems A and B), on equations (1.1) through (1.3), the following notations and results shall be used:

Let $s = \omega^4 - b\omega^2 + d$; $t = c - a\omega^2$ and $|\Delta|^2 = s^2 + \omega^2 t^2$, $\omega \in \overline{\mathbf{R}}$. By (A_7), the roots of $s = 0$ are ω_0^2 and ω_1^2. Since $0 < \omega_0^2 < c/a < \omega_1^2 < b$, s and t cannot be simultaneously zero.

Lemma 1. *For some constant $\delta_1 < 0$,*

$$T_1(\mu, \omega^2) \equiv \frac{1}{\mu} - \frac{\omega^2(s + \omega^2 \delta_1 t)}{|\Delta|^2} > 0,$$

for all $\omega \in \overline{\mathbf{R}}$ whenever

$$0 \leq \mu < \frac{d(c - a\omega_0^2)(\omega_1^2 - \omega_0^2)}{c\omega_0^4}.$$

Proof of Lemma 1. Choose

$$\delta_1 = \frac{c\omega_0^2}{d(\omega_0^2 - \omega_1^2)}.$$

Lemma 2. *For some constant $\delta_2 > 0$,*

$$T_2(\mu, \omega^2) \equiv \frac{1}{\mu} - \frac{\omega^4(t - \delta_2 s)}{|\Delta|^2} > 0,$$

for all $\omega \in \overline{\mathbf{R}}$, whenever

$$0 \leq \mu < \left(\frac{c - a\omega_0^2}{\omega_0^2}\right) \equiv \left(\frac{c\omega_1^2 - ad}{d}\right).$$

Proof of Lemma 2. Choose

$$\delta_2 = \frac{A_4 \omega_0^2}{c^2(c - a\omega_0^2)}.$$

Lemma 3. *For some $\delta_3 > 0$*

$$T_3(\mu, \omega^2) \equiv \frac{1}{\mu} - \frac{\omega^2(\delta_3 s - t)}{|\Delta|^2} > 0,$$

for all $\omega \in \overline{\mathbf{R}}$ whenever

$$0 \leq \mu < (a\omega_1^2 - c).$$

Proof of Lemma 3. Choose δ_3 to be any positeve quantity.

Lemma 4. *For some constant $\delta_4 > 0$,*

$$T_4(\mu, \omega^2) \equiv \frac{1}{\mu} + \frac{s + \omega^2 \delta_4 t}{|\Delta|^2} > 0,$$

for all $\omega \in \overline{\mathbf{R}}$ whenever $0 \leq \mu < (A_4/a^2)$.

Proof of Lemma 4. Choose

$$\delta_4 = \frac{a^2(a\omega_1^2 - c)}{A_4}.$$

We also note that for some negative values of δ_4 we have $0 \leq \mu < A_4/a^2$. For example $\delta_4 = -a^2(c - a\omega_0^2)/A_4$.

3. Results on equation (1.1) and its dual. Consider the equation

$$x^{(iv)} + f_1(x''') + f_2(x'') + cx' + dx = p(t, x, x', x'', x'''). \tag{3.1}$$

Theorem 3.1. *Suppose that in equation (3.1) functions f_1,f_2 satisfy the assumptions (A_1),(A_2) and (A_3) with the positive constants c and d satisfying (A_6) and p satisfying (A_9).*

Suppose further that the supplementary conditions

$$\lim_{|u|\to\infty}\frac{1}{u^2}\left\{\int_0^u f_1(\xi)d\xi-\frac{1}{2}uf_1(u)\right\}\geq 0; \tag{3.2}$$

and

$$\lim_{|z|\to\infty}\frac{1}{z^2}\left\{\int_0^z f_2(\xi)d\xi-\frac{1}{2}zf_2(z)\right\}\leq 0; \tag{3.3}$$

are satisfied.
Then, equation (1.1) has uniformly dissipative solutions.

Remark 1. If $f_1(x''')=ax'''$, $a>0$ in (1.1),then we obtain the equation

$$x^{(iv)}+ax'''+f_2(x'')+cx'+dx=p(t,x,x',X'',X'''), \tag{3.4}$$

studied by Barbalat and Halanay in [8]; while if $f_2(x'')=bx''$, with $b>0$ in (1.1), we obtain the equation

$$x^{(iv)}+f_1(x''')+bx''+cx'+dx=p(t,x,x',x'',x'''), \tag{3.5}$$

studied by Halanay in [11].

Proof of Theorem 3.1. In order to be in the situation suitable for using Theorem A, we let $f_1(u)=au+\hat f_1(u)$ and $f_2(z)=bz+\hat f_2(z)$. Then, f_j, $(j=1,2)$ satisfy

$$0\leq \hat f_j(\xi)/\xi\leq\mu_j,\quad (\xi\neq 0)\text{ and } 0\leq \hat f_j(\xi)\leq\mu_j.$$

Setting $x=x_1$, $x_1'=x_2$, $x_2'=x_3$, $x_3'=x_4$ in (3.1) and

$$X=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix};\quad A_1=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-d&-c&-b&-a\end{pmatrix};\quad B_1=\begin{pmatrix}0&0\\0&0\\0&0\\1&1\end{pmatrix};$$

$$C_1=\begin{pmatrix}0&0\\0&0\\1&0\\0&1\end{pmatrix};\quad \Phi_1(\sigma)=\begin{pmatrix}\hat f_2(x_3)\\ \hat f_1(x_4)\end{pmatrix};\quad\text{and}\quad P_1(t,X)=\begin{pmatrix}0\\0\\0\\p(t,x_1,x_2,x_3,x_3')\end{pmatrix};$$

in (2.1).
With this setting, and assumption (A_6), we have matrix (A_1) stable, hence for all $\omega\in\overline{\mathbf{R}}$, $(i\omega I-a)^{-1}$ exists.
Moreover, the transfer matrix $G_1(i\omega)=C^*(i\omega I-A_1)^{-1}B_1$ is

$$G_1(i\omega)=\frac{-\omega^2}{\Delta(i\omega)}\begin{pmatrix}1&1\\i\omega&i\omega\end{pmatrix} \tag{3.7}$$

where $\Delta(i\omega)=(\omega^4-b\omega^2+d)+i\omega(c-a\omega^2)$.
By assumption (A_7), $\Delta(i\omega)\neq 0$ for all $\omega\in\overline{\mathbf{R}}$.
Now, when $D_1=\operatorname{diag}(\tau_j/\mu_j)$; $D_2=\operatorname{diag}(\theta_j/\mu_j)$; $(j=1,2)$ and $D_3\equiv 0$; the frequency–domain matrix inequality (2.2) becomes

$$\pi_1(\omega)=\begin{pmatrix}\pi_{11}(\omega)&\pi_{12}(\omega)\\\pi_{21}(\omega)&\pi_{22}(\omega)\end{pmatrix}>0, \tag{3.8}$$

with

$$\pi_{11}(\omega)=\frac{\tau_1}{\mu_1}-\frac{\omega^2}{|\Delta|^2}\left(\tau_1 s+\frac{\omega^2\theta_1 t}{\mu_1}\right),$$

$$\pi_{22}(\omega)=\frac{\tau_2}{\mu_2}-\frac{\omega^4}{|\Delta|^2}\left(\tau_2 t-\frac{\theta_2 s}{\mu_2}\right),$$

and

$$\pi_{12}(\omega)=\overline{\pi}_{21}(\omega)=\frac{-\omega^2}{2|\Delta|^2}\left\{\left[\left(\tau_1-\frac{\omega^2\theta_2}{\mu_2}\right)s+\omega^2\left(\tau_2+\frac{\theta_1}{\mu_1}\right)t\right]+\right.$$
$$\left.+i\omega\left[\left(\frac{\theta_1}{\mu_1}-\tau_2\right)s-\left(\tau_1+\frac{\omega^2\theta_2}{\mu_2}\right)t\right]\right\}$$

where $\Delta=\Delta(i\omega)=s+i\omega t$, and s with t as given earlier.
To complete the proof of Theorem 3.1, it suffices to show that there exist diagonal matrices $D_1>0$, D_2 and $D_3\geq 0$ such that inequality (3.8) holds for all $\omega\in\overline{\mathbf{R}}$, and for which the supplementary condition (2.3) holds.
By Sylvester's criteria, (cf [10]), inequality (3.8) will be true for all $\omega\in\overline{\mathbf{R}}$ if either

$$\pi_{11}(\omega)>0\text{ and }\det.\pi_1(\omega)>0,\text{ for all }\omega\in\overline{\mathbf{R}},$$

or

$$\pi_{22}(\omega)>0\text{ and }\det.\pi_1(\omega)>0,\text{ for all }\omega\in\overline{\mathbf{R}}.$$

Choose $\tau_j>0$, $(j=1,2)$. Then, $\pi_{11}(\omega)$ can be rewritten as

$$\tau_1\left\{\frac{1}{\mu_1}-\frac{\omega^2[(\omega^4-b\omega^2+d)+\omega^2\theta_1(c-a\omega^2)/\mu_1\tau_1]}{(\omega^4-b\omega^2+d)^2+\omega^2(c-a\omega^2)^2}\right\}. \tag{3.9}$$

This will be positive for all $\omega\in\overline{\mathbf{R}}$ by lemma 1, and for all

$$0\leq\mu_1<\frac{d(c-a\omega_0^2)(\omega_1^2-\omega_0^2)}{c\omega_0^4},$$

if

$$\frac{\theta_1}{\tau_1}<\left(\frac{a\omega_0^2-c}{\omega_0^2}\right)<0.$$

In a similar manner, $\pi_{22}(\omega)$ can be rewritten as

$$\tau_2\left\{\frac{1}{\mu_2}+\frac{\omega^4[\frac{\theta_2}{\mu_2\tau_2}(\omega^4-b\omega^2+d)-(c-a\omega^2)]}{(\omega^4-b\omega^2+d)^2+\omega^2(c-a\omega^2)^2}\right\}, \tag{3.10}$$

which will be positive for all $\omega\in\overline{\mathbf{R}}$ by lemma 2 and for

$$0\le\mu_2<(c-a\omega_0^2)/\omega_0^2 \tag{3.11}$$

if

$$0<\theta/\tau_2<A_4/c^2.$$

Next, a simplification of $\det.\pi_1(\omega)$ gives

$$\det.\pi_1(\omega)=\frac{\tau_1\tau_2}{\mu_1\mu_2}-\frac{\omega^2}{|\Delta|^2}\times\left\{\frac{\tau_2}{\mu_2}\left(\tau_1 s+\frac{\omega^2\theta_1 t}{\mu_1}\right)+\right. \tag{3.12}$$

$$\left.+\frac{\omega^2\tau_1}{\mu_1}\left(\tau_2 t-\frac{\theta_2 s}{\tau_2}\right)+\frac{\omega^2}{4}\left[\left(\tau_1+\omega^2\frac{\theta_2}{\tau_2}\right)^2+\omega^2\left(\tau_2 t-\frac{\theta_1}{\mu_1}\right)^2\right]\right\}.$$

For μ_1, μ_2 satifying (A_9) and τ_2 chosen large enough, this will be positive for all $\omega\in\overline{\mathbf{R}}$.

This becomes clearer, if we evaluate (3.12) at $\omega^2=0$, ω_0^2, c/a, ω_1^2, b and as ω^2 tends to ∞ respectively.

Finally, evaluation of (2.3) with $D_2=\operatorname{diag}(\theta_j/\mu_j)$, $(j=1,2)$ gives

$$\lim_{|x_3|\to\infty}\frac{\theta_1}{x_3^2}\left\{\int_0^{x_3}\hat f_2(\alpha)d\alpha-\frac{x_3}{2}\hat f_2(x_3)\right\}\ge 0 \tag{3.13}$$

and

$$\lim_{|x_4|\to\infty}\frac{\theta_2}{x_4^2}\left\{\int_0^{x_4}\hat f_1(\alpha)d\alpha-\frac{x_4}{2}\hat f_1(x_4)\right\}\ge 0, \tag{3.14}$$

with $\tau_j>0$, $(j=1,2)$.

Clearly, these inequalities hold with the conditions on θ_1 and θ_2. This completes the proof of Theorem 3.1.

The dual system equivalent to the system corresponding to equation (3.1), with the choice of matrices A_1, B_1, C_1 as in (2.1) is

$$\begin{aligned} &y_1'=-dy_4+p(t,y_1,y_2,y_3,y_4), && y_2'=y_1-cy_4,\\ &y_3'=y_2-by_4-l_2(y_4), && y_4'=y_3-ay_4-l_1(y_4)\end{aligned} \tag{3.15}$$

with the basic assumptions on functions $l_1(y_4)$ and $l_2(y_4)$ given by

(A_{10}) l_1 has continuous first, second and third derivatives such that

$$a+l_1'(y)>0,\qquad l_1''(y)\ge 0,\qquad l_1'''(y)\le 0$$

for large values of $|y|$.

(A_{11}) l_2 has continuous fist and second derivatives such that

$$b+l_2'(y)>0,\qquad l_2''(y)>0$$

for large values of $|y|$.

On setting $y_4=y$, system (3.15) is equivalent to the fourth order nonlinear differential equation

$$y^{iv}+\frac{d^3}{dt^3}[ay+l_1(y)]+\frac{d^2}{dt^2}[by+l_2(y)]+cy'+dy=p(t,y,y',y'',y'''). \tag{3.16}$$

A direct application of Theorem B yields the following:

Theorem 3.2. *Suppose that a, b, c, d are posirive constants in (3.16) such that assumption (A_6) holds and μ_1, μ_2 are nonnegative constants such that assumptions (A_8) hold. Suppose further that $l_1(y)$ and $l_2(y)$ are functions for which assumptions (A_{10}) and (A_{11}) hold and such that $0<l_1'(y)\le\mu_1$ and $0<l_2'(y)\le\mu_2$ for sufficiently large values of $|y|$.*

Then, equation (3.15) has uniformly dissipative solutions.

4. Results on equation (1.2) and its dual.

Consider the equation

$$x^{iv}+f_1(x''')+bx''+f_3(x')+dx=p(t,x,x',x'',x'''). \tag{4.1}$$

Theorem 4.1. *Suppose that in equation (4.1), functions f_1, f_2, satisfy assumptions (A_1), (A_2), and (A_4) with the positive constants b and d satisfying (A_6) and p satisfying (A_9). Suppose further that the supplementary conditions*

$$\lim_{|u|\to\infty}\frac{1}{u^2}\left\{\int_0^u f_1(\xi)d\xi-\frac12 uf_1(u)\right\}\ge 0 \tag{4.2}$$

and

$$\lim_{|y|\to\infty}\frac{1}{y^2}\left\{\int_0^y f_3(\xi)d\xi-\frac12 yf_3(y)\right\}\ge 0 \tag{4.3}$$

are satisfied. Then, equation (4.1) has uniformly dissipative solutions.

Proof of Theorem 4.1. We shall give the sketch of the proof for this theorem, in such a way that the equivalent frequency-domain inequality will be satisfied, for a system of the form (2.1).

Let $f_1(u) = au + \hat{f}_1(u)$ and $f_3(y) = cy + \hat{f}_3(y)$. Then,

$$0 \le \hat{f}_1(u)/u \le \mu_1, \qquad (u \ne 0), \qquad 0 \le \hat{f}_1(u) \le \mu_1$$

and

$$0 \le \hat{f}_3(y)/y \le \mu_3, \qquad (y \ne 0), \qquad 0 \le \hat{f}_3(y) \le \mu_3.$$

Set $x = x_1$, $x_1' = x_2$, $x_2' = x_3$, $x_3' = x_4$ we have

$$(4.4) \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \; A_2 = A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -d & -c & -b & -a \end{pmatrix}, \; B_2 = B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix};$$

$$C_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}; \; \Phi_2(\sigma) = \begin{pmatrix} \hat{f}_3(x_2) \\ \hat{f}_1(x_4) \end{pmatrix}; \text{ and } P_2(t, X) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ p(t, x_1, x_2, x_3, x_3') \end{pmatrix}.$$

Choosing the diagonal matrices $D_1 > 0$, D_2, $D_3 \ge 0$ as

$$D_1 = \begin{pmatrix} \tau_1/\mu_1 & 0 \\ 0 & \tau_3/\mu_3 \end{pmatrix}; \quad D_2 = \begin{pmatrix} \theta_1/\mu_1 & 0 \\ 0 & \theta_3/\mu_3 \end{pmatrix}; \quad D_3 \equiv 0,$$

the frequency domain inequality (2.2) becomes

$$(4.5) \qquad \pi(\omega) = \begin{pmatrix} \pi_{11}(\omega) & \pi_{13}(\omega) \\ \pi_{31}(\omega) & \pi_{33}(\omega) \end{pmatrix} > 0,$$

with

$$\pi_{11}(\omega) = \frac{\tau_1}{\mu_1} + \frac{\omega^2}{|\Delta|^2}\left(\tau_1 t - \frac{\theta_1 s}{\mu_1}\right)$$

$$\pi_{33}(\omega) = \frac{\tau_1}{\mu_3} + \frac{\omega^2}{|\Delta|}\left(\frac{\theta_3 s}{\mu_3} - \tau_3 t\right)$$

and

$$\pi_{13} = \overline{\pi}_{31} + \frac{1}{2|\Delta|^2}\left\{\omega^2\left[\left(\frac{\omega^2\theta_3}{\mu_3} - \frac{\theta_1}{\mu_1}\right)s + (\tau_1 - \omega^2\tau_3)\,t\right] + \right.$$
$$\left. + i\omega\left[(\tau_1 + \omega^2\tau_3)\,s + \omega^2\left(\frac{\theta_1}{\mu_1} + \omega^2\frac{\theta_3}{\mu_3}\right)t\right]\right\},$$

where as before, $s = \omega^4 - b\omega^2 + d$, $t = c - a\omega^2$ and $|\Delta|^2 = s^2 + \omega^2 t^2$.

We note that

$$(4.6) \qquad \det.\pi(\omega) = \frac{\tau_1\tau_3}{\mu_1\mu_3} + \frac{\omega^2}{|\Delta|^2} \times \left\{\left[\frac{\tau_3}{\mu_3}\left(\tau_1 t - \frac{\theta_1}{\mu_1}s\right) + \right.\right.$$

$$\left.\left. + \frac{\omega^2\tau_1}{\mu_1}\left(\frac{\theta_3}{\mu_3}s - \tau_3 t\right)\right] - \frac{1}{4}\left[(\tau_1 + \omega^2\tau_3)^2 + \omega^2\left(\frac{\theta_1}{\mu_1} + \omega^2\frac{\theta_3}{\mu_3}\right)^2\right]\right\}.$$

Inequalitity (4.5) will be valid for all $\omega \in \overline{\mathbf{R}}$, since by following the analogous analysis carried out in section 3, we can use lemma 3 and lemma 2 respectively, with

$$0 < \mu_1 < a\omega^2 - c \text{ when } \theta_1/\tau_1 > 0,$$

and

$$0 \le \mu_3 < (c\omega_1^2 - ad)/d, \text{ when } 0 \le \theta_3/\tau_3 < A_4/(ac).$$

Moreover, we also have

$$\omega_0^2 < \frac{\mu_1}{\mu_3} < \left(\frac{a\omega_1^2 - c}{c - a\omega_0^2}\right)\omega_0^2,$$

as given in (A_8), when $\theta_3/\theta_1 > a/c$.

Thus, the supplementary conditions (4.2) and (4.3) would follow. This completes the proof of the theorem.

The dual to the equivalent system corresponding to system (2.1) with the choice of A_2, B_2, C_2 and Φ_2 in (4.4) is

$$(4.7) \qquad \begin{aligned} y_1' &= -dy_4 + p(t, y_1, y_2, y_3, y_3'), & y_2' &= y_1 - cy_4 - l_3(y_4), \\ y_3' &= y_2 - by_4, & y_4' &= y_3 - ay_4 - l_1(y_4) \end{aligned}$$

with the basic assumptions on functions $l_1(y_4)$ as (A_{10}) and $l_3(y_4)$ given by (A_{12}) l_3 has continous first derivative such that

$$c + l_3'(y) \ge 0, \text{ for large values of } |y|.$$

Thus, using Theorem B, we have the following result for the equation

$$(4.8) \qquad y^{iv} + \frac{d^3}{dt^3}[ay + l_1(y)] + by'' + \frac{d}{dt}[cy + l_3(y)] + dy = p_1(t, y, y', y'', y''').$$

Theorem 4.2. *Suppose that a, b, c and d are positive constants such that assumption (A_6) holds, and μ_1, μ_3 are nonnegative constants such that the corresponding assumption in (A_8) holds. Suppose further that functions $l_1(y)$ and $l_3(y)$ satisfy asumptions (A_{10}) and (A_{12}) and for sufficiently large values of $|y|$, we have*

$$0 < l_1'(y) \le \mu_1 \quad \text{and} \quad 0 < l_3'(y) \le \mu_3.$$

Then, equation (4.8) has uniformly dissipative solutions.

5. Results on equation (1.3) and its dual. Consider the equation

$$(5.1) \qquad x^{iv} + f_1(x''') + bx'' + cx' + f_4(x) = p(t, x, x', x'', x''').$$

Theorem 5.1. *Suppose that functions f_1, f_4 satisfy the assumptions (A_1), (A_2), and (A_5), the positive constants b and c satisfying assumption (A_6) and function p satisfies assumptions (A_9). Suppose further that the supplementary conditions*

$$\lim_{|u|\to\infty} \frac{1}{u^2}\left\{\int_0^u f_1(\xi)d\xi - \frac{u}{2}f_1(u)\right\} \geq 0 \tag{5.2}$$

and

$$\lim_{|x|\to\infty} \frac{1}{x^2}\left\{\int_0^x f_4(\xi)d\xi - \frac{x}{2}f_4(x)\right\} \geq 0 \tag{5.3}$$

are satisfied. Then, equation (5.1) has uniformly dissipative solutions.

Proof of Theorem 5.1. Let $f_1(u) = au + \hat{f}_1$ and $f_4(x) = dx + \hat{f}_4(x)$. Then, for u and x sufficiently large , by assumptions (A_2) and (A_5), we have

$$0 \leq \frac{\hat{f}_1(u)}{u} \leq \mu_1, \qquad (u \neq 0), \qquad 0 \leq \hat{f}_1'(u) \leq \mu_1,$$

and

$$0 \leq \frac{\hat{f}_4(x)}{x} \leq \mu_4, \qquad (x \neq 0), \qquad 0 \leq \hat{f}_4'(x) \leq \mu_4.$$

Set $x = x_1$, $x_1' = x_2$, $x_2' = x_3$, $x_3' = x_4$. Then for

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}; \qquad A_3 = A_1, \qquad B_3 = B_1, \qquad P_3 = P_1 \text{ (as in section 3);}$$

$$\text{and} \quad C_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{we have} \quad \Phi_3(\sigma) = \begin{pmatrix} \hat{f}_4(x) \\ \hat{f}_1(x) \end{pmatrix}.$$

Thus,

$$G_3(i\omega) = C_3^*(i\omega I - A_3)^{-1}B_3 = \frac{1}{\Delta(i\omega)}\begin{pmatrix} 1 & 1 \\ -i\omega^3 & -i\omega^3 \end{pmatrix}.$$

Let us now choose diagonal matrices $D_1 > 0$, D_2, $D_3 \geq 0$ in the frequency–domain inequality (2.2) as

$$D_1 = \begin{pmatrix} \tau_1/\mu_1 & 0 \\ 0 & \tau_4/\mu - 4 \end{pmatrix}; \qquad D_2 = \begin{pmatrix} \theta_1/\mu_1 & 0 \\ 0 & \theta_4/\mu_4 \end{pmatrix}; \qquad D_3 = 0.$$

Then, the frequency–domain inequality becomes

$$\pi(\omega) = \begin{pmatrix} \pi_{11} & \pi_{14} \\ \pi_{41} & \pi_{44} \end{pmatrix} > 0 \tag{5.6}$$

with

$$\pi_{11}(\omega) = \frac{\tau_1}{\mu_1} + \frac{1}{|\Delta|^2}\left(\tau_1 s + \omega^2\frac{\theta_1}{\mu_1}\right);$$

$$\pi_{44}(\omega) = \frac{\tau_4}{4} + \frac{\omega^4}{|\Delta|^2}\left(\frac{\theta_4 s}{\mu_4} - \tau_4 t\right)$$

and

$$\pi_{14} = \overline{\pi}_{41} = \frac{1}{2|\Delta|^2}\left\{\left[\left(\tau_1 + \frac{\omega^4\theta_4}{\mu_4}\right)s + \omega^2\left(\frac{\theta_1}{\mu_1} - \omega^2\tau_4\right)t\right]\right\} +$$
$$+ i\omega\left\{\left[\left(\frac{\theta_1}{\mu_1} + \omega^2\tau_4\right)s + \left(\frac{\omega^4\theta_4}{\mu_4} - \tau_1\right)t\right]\right\}.$$

Inequality (5.6) will be true for all $\omega \in \overline{\mathbf{R}}$, folowing the analysis in section 3 and that in [5], we have

$$0 \leq \mu_1 \leq A_4/a^2, \text{ when } 0 < \theta_1/\tau_1 < (a\omega_1^2 - c) \text{ (by lemma 4)}$$

and

$$0 \leq \mu_4 < (c - a\omega_0^2)/\omega_0^2, \text{ when } 0 < \theta_4/\tau_4 < A_4/c^2 \text{ (by lemma 2).}$$

The determinant of $\pi(\omega)$ is given by

$$\det.\pi(\omega) = \frac{\tau_1\tau_4}{\mu_1\mu_4} + \frac{1}{|\Delta|^2} \times \left\{\left[\frac{\tau_4}{\mu_4}\left(\tau_1 s + \frac{\omega^2\theta_1 t}{\mu_1}\right)\right.\right. \tag{5.7}$$
$$\left.\left. + \frac{\tau_1}{\mu_1}\left(\frac{\theta_4 s}{\mu_4} - \tau_4 t\right)\right] - \frac{1}{4}\left[\left(\tau_1 - \frac{\omega^4\theta_4}{\mu_4}\right)^2 + \left(\frac{\theta_1}{\mu_1} + \omega^2\tau_4\right)\right]\right\}.$$

Inequality (5.6) is fullfiled for all $\omega \in \overline{\mathbf{R}}$ if μ_4 is chosen such that

$$\mu_4 < \frac{\theta_1}{\tau_1}\tilde{\omega}^2, \quad \text{with } \tilde{\omega}^2 \text{ either } \omega_0^2 \text{ or } \omega_1^2$$

which holds whenever $c/a < b/2$. The supplementary conditions (5.2) and (5.3) clearly follow by these choices of θ_1 and θ_4. This completes the proof of the theorem.

The dual to the equivalent system corresponding to system (2.1) with the choice of A_3, B_3, C_3 and Φ_2 in (5.5) is

$$y_1' = -dy_4 - l_4(y_4) + p(t, y_1, y_2, y_3, y_3'), \quad y_2' = y_1 - cy_4, \tag{5.8}$$
$$y_3' = y_2 - by_4, \qquad y_4' = y_3 - ay_4 - l_1(y_4).$$

The basic assumptions on the function $l_1(y_4)$ is as before given by (A_{10}) and on function $l_4(y_4)$ given by
(A_{13}) $l_3(y)$ is continous and satisfies

$$d + l_4(y) \geq 0, \quad \text{for large values of } \mid y \mid .$$

On using Theorem B, we have the following result for the equation

$$y^{iv} + \frac{d^3}{dt^3}[ay + l_1(y)] + by'' + cy' + [dy + l_4(y)] = p(t, y, y', y'', y''') . \tag{5.9}$$

Theorem 5.2. *Let a, b, c and d be positive constants in (5.9) such that assumption (A_6) holds. Suppose that μ_1, μ_4 are nonnegative constants such that assumption in (A_8) hold.*
Suppose further that functions $l_1(y)$ and $l_4(y)$ satisfy assumptions (A_{10}) and (A_{13}), and such that for sufficiently large values of $\mid y \mid$,

$$0 < l_1'(y) < \mu_1 \quad \text{and} \quad 0 < l_4(y) < \mu_4$$

hold. Then, equation (5.9) has uniformly dissipative solutions.

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