It will now follow, from (A_7) and (2.3.26), $F_1(\lambda, \tau) = F_2(\lambda, \tau)$ cannot have roots with zero or positive real parts and this completes the proof.

In conclusion we remark that the condition (3.7) means that the net reproduction rates of each species is greater than unity; a condition of this type is well known in age dependent populations. (3.8) can be interpreted to mean that the intraspecific competitive inhibition is higher than that of the interspecific interaction. The conditions of Theorem 3.2 are analytical and do not lend themselves for any worthwile interpretation in terms of the model parameters.

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A NOTE ON SEMI-INVARIANT SUBMANIFOLDS OF AN ALMOST CONTACT METRIC MANIFOLDS

BY

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0. Introduction. Semi-invariant submanifolds of an almost contact metric manifold were defined and studied by A. Bejancu and N. Papaghiuc [1] and by several others afterwards. Besides other results some necessary and sufficient conditions, for a submanifold of an almost contact metric manifold to be semi-invariant, are given by A. Bejancu in [2]. In this paper, we prove some more necessary and sufficient conditions for a submanifold of an almost contact metric manifold to be semi-invariant.

1. Preliminaries [1,2]. Let $\overline{M}$ be an almost contact metric manifold with structure tensors (φ, ξ, η, g) where φ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is an associated Riemannian metric on $\overline{M}$. These tensor fields are related by

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \varphi\xi = 0, \qquad \eta_0\varphi = 0, \tag{1.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(X), \qquad g(\xi, X) = \eta(X), \tag{1.2}$$

for arbitrary vector fields X, Y tangent to $\overline{M}$.

A submanifold M of $\overline{M}$, is called a *semi-invariant submanifold* of $\overline{M}$ if the following conditions are satisfied:

(*i*) $TM = D \oplus D^{\perp} \oplus \{\xi\}$; $\{\xi\}$ being the distribution spanned by ξ,
(*ii*) the distribution D is invariant by φ, that is, $\varphi(D_x) = D_x$ for each $x \in M$,
(*iii*) the distribution $D^{\perp}$ is anti-invariant by φ, that is, $\varphi(D_x^{\perp}) \subset T_xM^{\perp}$, for each $x \in M$,

where D, $D^{\perp}$ and $\{\xi\}$ are differentiable orthogonal distributions (with respect to the induced metric denoted by g itself) on M and $T_xM^{\perp}$ is the normal space to M at $x \in M$.

By $\Gamma(TM)$ [resp. $\Gamma(TM^{\perp})$] we mean the module of all differentiable vector fields tangent (resp. normal) to M. Similarly, $\Gamma(D)$ [resp. $\Gamma(D^{\perp})$] denotes the module of differentiable sections of D [resp. $D^{\perp}$], where $X \in \Gamma(TM)$ is a section of D [resp. $D^{\perp}$] if $X_x \in D_x$ [resp. $D_x^{\perp}$] for each $x \in M$.

For a submanifold M of $\overline{M}$ tangent to the structure vector field ξ, let us put for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$

$$\varphi X = \psi X + \omega X, \tag{1.3}$$

$$\varphi N = \pi N + fN, \tag{1.4}$$

where ψX and πN are the tangential parts while ωX and fN are the normal parts of φX and φN respectively.

If M is a semi-invariant submanifold of $\overline{M}$, then every $X \in \Gamma(TM)$ can be written as

$$X = UX + VX + \eta(X)\xi, \tag{1.5}$$

where U and V are projection morphisms of TM on D and $D^\perp$ respectively. Applying φ to (1.5) and using (1.3), one obtains

$$\psi X = \varphi UX, \qquad \omega X = \varphi VX. \tag{1.6}$$

In this case, the tensor field f given by (1.4) defines an f-structure [4] on the normal bundle [1].

2. Some characterizations of semi-invariant submanifolds. Let M be a semi-invariant submanifold of an almost contact metric manifold $\overline{M}$. Then

$$\varphi^2 X = -X; \qquad X \in \Gamma(D \oplus D^\perp \oplus TM^\perp). \tag{2.1}$$

Let us put $\overline{D}_x \equiv \varphi(D_x^\perp)$ for each $x \in M$ and let $\widetilde{D}$ be the complementary orthogonal subbundle to $\overline{D}$ in $TM^\perp$. Thus,

$$TM^\perp \equiv \overline{D} \oplus \widetilde{D}. \tag{2.2}$$

If $X \in \Gamma(D^\perp)$, then in view of (2.1), $\varphi(\varphi X) = -X$ and consequently, $\varphi(\overline{D}_x) \subset T_xM$ for each $x \in M$. Next, for $N \in \Gamma(\widetilde{D})$, $N' \in \Gamma(\overline{D})$, $X \in \Gamma(TM)$, since $\varphi X \in \Gamma(D \oplus \overline{D})$ and $\varphi N' \in \Gamma(TM)$, we get $g(\varphi N, N') = -g(N, \varphi N') = 0$ and $g(\varphi N, X) = -g(N, \varphi X) = 0$, which implies $\varphi(\widetilde{D}_x) \subset \widetilde{D}_x$ for each $x \in M$. Moreover, since $\varphi : \widetilde{D}_x \to \widetilde{D}_x$ is non-singular, $\dim\varphi(\widetilde{D}_x) = \dim\widetilde{D}_x$ and thus $\varphi(\widetilde{D}_x) = \widetilde{D}_x$ for each $x \in M$. Now, taking account of (2.2), each $N \in \Gamma(TM^\perp)$ can be written as

$$N = PN + QN \tag{2.3}$$

where P and Q are projection morphisms of $TM^\perp$ on $\overline{D}$ and $\widetilde{D}$ respectively. Applying φ to (2.3) and comparing tangential and normal parts with (1.4), we get

$$\pi N = \varphi PN, \qquad fN = \varphi QN. \tag{2.4}$$

In view of (1.6) and (2.4), we obtain

$$\begin{aligned} &\psi X \in \Gamma(D), \qquad \omega X \in \Gamma(\overline{D}); \qquad X \in \Gamma(TM),\\ &\pi N \in \Gamma(D^\perp), \qquad fN \in \Gamma(\widetilde{D}); \qquad N \in \Gamma(TM^\perp). \end{aligned}$$

From (2.4), it follows that

$$\pi N = \varphi N; \quad N \in \Gamma(\overline{D}), \qquad \pi N = 0; \quad N \in \Gamma(\widetilde{D}), \tag{2.5}$$

$$fN = \varphi N; \quad N \in \Gamma(\widetilde{D}), \qquad fN = 0; \quad N \in \Gamma(\overline{D}). \tag{2.6}$$

Now we prove the following

Lemma 2.1. *Let M be a semi-invariant submanifold of an almost contact metric manifold $\overline{M}$. Then we have*

$$\begin{aligned} &(2.7)\quad (\varphi\circ\pi)N = (\omega\circ\pi)N = -PN, \qquad (\psi\circ\pi)N = 0;\\ &(2.8)\quad (\varphi\circ f)N = f^2N = -QN, \qquad (\pi\circ f)N = 0; \end{aligned} \qquad N \in \Gamma(TM^\perp).$$

Proof. Applying φ to $(2.4)_1$ [resp. $(2.4)_2$], using (1.3) [resp. (1.4)] and (2.1), we get the following two relations

$$\varphi^2(PN) = -PN = (\psi\circ\pi)N + (\omega\circ\pi)N = (\varphi\circ\pi)N,$$

$$\varphi^2(QN) = -QN = (\pi\circ f)N + f^2N = (\varphi\circ f)N,$$

which, on equating tangential and normal parts, yield (2.7) and (2.8) respectively. Consequently, we have the following

Corollary 2.1. *On a semi-invariant submanifold M of an almost contact metric manifold $\overline{M}$, we have*

$$(\omega\circ\varphi)N = (\omega\circ\pi)N = -N, \qquad (\psi\circ\varphi)N = 0; \qquad N \in \Gamma(\overline{D}), \tag{2.9}$$

$$(f\circ\varphi)N = f^2N = -N, \qquad (\pi\circ\varphi)N = 0; \qquad N \in \Gamma(\widetilde{D}). \tag{2.10}$$

Remark 2.1. From $(2.10)_1$, it follows that f is an almost complex structure on $\widetilde{D}$.

From $(2.9)_1$ and $(2.10)_1$, we deduce that $\overline{D}_x \subset \omega(T_xM)$ and $\widetilde{D}_x \subset f(T_xM^\perp)$ for each $x \in M$. Since $\omega X \in \Gamma(\overline{D})$ and $fN \in \Gamma(\widetilde{D})$, we can state the following

Lemma 2.2. *If an almost contact metric manifold $\overline{M}$ admits a semi-invariant submanifold M, then*

$$\omega(T_xM) = \overline{D}_x, \qquad f(T_xM^\perp) = \widetilde{D}_x, \tag{2.11}$$

for each $x \in M$ and consequently

$$rank(\omega) = dim\overline{D}, \qquad rank(f) = dim\widetilde{D}. \tag{2.12}$$

Now we state and prove the main theorem of the paper.

Theorem 2.1. *For a submanifold M of an almost contact metric manifold $\overline{M}$ such that the structure vector field ξ is tangential to M, the following assertions are equivalent:*
(A) M is a semi-invariant submanifold of $\overline{M}$,
(B) $TM^\perp = \overline{D} \oplus \widetilde{D}$, where $\overline{D}$ and $\widetilde{D}$ are complementary orthogonal subbundles of $TM^\perp$ such that $\varphi(\overline{D}_x) \subset T_xM$ and $\varphi(\widetilde{D}_x) = \widetilde{D}_x$ for each $x \in M$.
(C) $\mathrm{rank}(\omega) =$ constant and $(f \circ \omega) = 0$,
(D) $\mathrm{rank}(f) =$ constant and $(\pi \circ f) = 0$,
(E) ψ is an f-structure, that is, $\psi^3 + \psi = 0$ on M and $(\omega \circ \psi) = 0$,
(F) f is an f-structure, that is, $f^3 + f = 0$, on the normal bundle, and $(\pi \circ f) = 0$.

Proof. The pattern of the proof will be as follows: $(A) \Leftrightarrow (E)$, $(A) \Leftrightarrow (B)$, $(A) \Rightarrow (C) \Rightarrow (B)$, $(A) \Rightarrow (D) \Rightarrow (B)$, and $(A) \Rightarrow (F) \Rightarrow (B)$.

(i)(A) $\Leftrightarrow$ (E). Let (A) be true. Applying φ to $(1.6)_1$ [resp. $(1.6)_2$] and using (1.3) [resp. (1.4)] and (2.1) we get respectively

$$\begin{aligned} \varphi^2(UX) &= -UX = \psi^2 X + (\omega \circ \psi)X = (\varphi \circ \psi)X; \\ \varphi^2(VX) &= -VX = (\pi \circ \omega)X + (f \circ \omega)X = (\varphi \circ \omega)X; \end{aligned} \qquad X \in \Gamma(TM).$$

Equating tangential and normal parts in the above relations, we obtain

$$(\varphi \circ \psi)X = \psi^2 X = -UX, \qquad (\omega \circ \psi)X = 0. \tag{2.13}$$

$$(\varphi \circ \omega)X = (\pi \circ \omega)X = -VX, \qquad (f \circ \omega)X = 0. \tag{2.14}$$

In view of (1.5), $(2.13)_1$ and $(2.14)_1$, we have

$$X - \eta(X)\xi = -\psi^2 X - (\pi \circ \omega)X,$$

from which using $(2.7)_2$, it follows that $\psi^3 + \psi = 0$. Thus, in view of $(2.12)_2$, (E) holds. Conversely, let (E) hold, that is

$$\psi^3 + \psi = 0, \qquad \omega \circ \psi = 0. \tag{2.15}$$

Defining D and $D^\perp$ by projection operators U and V given by

$$U = -\psi^2, \qquad V = I + \psi^2 - \eta \otimes \xi,$$

we get $TM = D \oplus D^\perp \oplus \{\xi\}$. For $X \in \Gamma(TM)$, in view of $(2.15)_2$, we get

$$\varphi(-\psi^2 X) = -\psi^2(\psi X) \in \Gamma(D).$$

Thus, D is invariant by φ. Similarly, in view of (2.15), for $X \in \Gamma(TM)$, we get

$$\varphi(X + \psi^2 X - \eta(X)\xi) = \psi X + \omega X + \psi^3 X + (\omega \circ \psi^2)X = \omega X \in \Gamma(TM^\perp).$$

Hence $D^\perp$ is anti-invariant by φ, and therefore $(E) \Rightarrow (A)$.

(ii)(A) $\Leftrightarrow$ (B). Here $(A) \Rightarrow (B)$ is obvious. For converse part, let (B) hold, that is

$$TM^\perp = \overline{D} \oplus \widetilde{D}$$

where $\overline{D}$ and $\widetilde{D}$ are complementary orthogonal subbundles of $TM^\perp$ such that $\varphi(\overline{D}_x) \subset T_xM$ and $\varphi(\widetilde{D}_x) = \widetilde{D}_x$ for each $x \in M$. Let us write a vector field N normal to M as

$$N = PN + QN \tag{2.3}$$

where P and Q are projection morphisms of $TM^\perp$ on $\overline{D}$ and $\widetilde{D}$ respectively. We define a distribution $D^\perp$ on TM by $D^\perp_x \equiv \varphi(\overline{D}_x)$, for each $x \in M$. Since $g(\varphi N, \xi) = 0$ for each $N \in \Gamma(\overline{D})$, $D^\perp$ is orthogonal to $\{\xi\}$. Let D denote the complementary orthogonal distribution of $D^\perp \oplus \{\xi\}$ on M. Now, for every $N \in \Gamma(\overline{D})$, $\varphi(\varphi N) = \varphi^2 N = -N \in \Gamma(\overline{D})$, and hence $D^\perp$ is anti- invariant by φ. For $X \in \Gamma(D)$, $N \in \Gamma(\overline{D})$ and $N' \in \Gamma(TM^\perp)$, it follows that

$$\begin{aligned} &g(\varphi X, \varphi N) = -g(X, \varphi^2 N) = g(X, N) = 0, \\ &g(\varphi X, \xi) = 0, \\ &g(\varphi X, N') = -g(X, \varphi PN') - g(X, \varphi QN') = 0 \end{aligned}$$

which makes D invariant by φ, and thus $(B) \Rightarrow (A)$. Hence, $(A) \Leftrightarrow (B)$.

(iii)(A) $\Rightarrow$ (C) $\Rightarrow$ (B) From $(2.12)_1$ and $(2.14)_2$, it follows that $(A) \Rightarrow (C)$. Assuming (C), we define the subbundle $\overline{D}$ of $TM^\perp$ by

$$\overline{D}_x \equiv \omega(T_xM),$$

for each $x \in M$. Since $f \circ \omega = 0$, we get

$$\varphi(\omega X) = \pi(\omega X) \in \Gamma(TM),$$

and thus $\overline{D}$ becomes anti-invariant by φ. If $\widetilde{D}$ is orthogonal complement of $\overline{D}$ in $TM^\perp$, then for $N \in \Gamma(\widetilde{D})$, $N' \in \Gamma(\overline{D})$, and $X \in \Gamma(TM)$, the following relations

$$g(\varphi N, N') = -g(N, \varphi N') = 0, \qquad g(\varphi N, X) = -g(N, \psi X + \omega X) = 0,$$

imply that $\widetilde{D}$ is invariant by φ and consequently $(C) \Rightarrow (B)$.

(iv)(A) $\Rightarrow$ (D) $\Rightarrow$ (B). In view of $(2.12)_2$ and $(2.8)_2$, it is evident that $(A) \Rightarrow (D)$. Now, if (D) holds good, then defining

$$\widetilde{D}_x = f(T_x M^\perp); \qquad x \in M,$$

in $TM^\perp$ and $\overline{D}$ as the complementary subbundle orthogonal to $\widetilde{D}$ in $TM^\perp$, we can show that $\varphi(\overline{D}_x) \in T_x M$ and $\varphi(\widetilde{D}_x) = \widetilde{D}_x$ for each $x \in M$, which proves that $(D) \Rightarrow (B)$.

(v)(A) $\Rightarrow$ (F) $\Rightarrow$ (B). The proof of $(A) \Rightarrow (F)$ is given in [1]. For $(F) \Rightarrow (B)$, let (F) be true. Defining $\overline{D}$ and $\widetilde{D}$ on $TM^\perp$ by projection operators

$$P = I + f^2; \qquad Q = -f^2$$

respectively, it follows that for each $N \in \Gamma(TM^\perp)$

$$\varphi N + \varphi(f^2 N) = \pi N + fN + (\pi \circ f^2)N + f^3 N = \pi N \in \Gamma(TM),$$

$$\varphi(-f^2 N) = -f^2(fN) \in \Gamma(\widetilde{D}).$$

Thus $(F) \Rightarrow (B)$. This completes the proof of the theorem.

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CURVATURE THEORY OF RECURRENT HAMILTON SPACES WITH GENERALIZED MIRON'S d-CONNECTION

BY

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Introduction.

Let m be an n dimensional and E^* an $n+m$ dimensional differentiable manifold and let (E^*, π^*, M) be a dual vector bundle, such that $\pi^*(E^*) = M$. If $u \in E^*$, then $u = (x^1, \ldots, x^n, p_1, \ldots, p_m) = (x^i, p_a)$. In this paper we shall use two kinds of indices:

$$i, j, h, k, l, m = \overline{1,n} \text{ and } a, b, c, d, e, f = \overline{1,m}.$$

The Einstein summation convention will be used for both kinds of indices. We shall consider the following transformation of coordinate system. If $(x^{i'}, p_{a'})$ are the coordinates of the same point u in the new coordinate system, then

$$(1.1) \qquad (a) \quad x^{i'} = x^{i'}(x^1, \ldots, x^n), \ \text{rank}\,[\partial x^{i'}/\partial x^i] = n$$
$$(b) \quad p_{a'} = M^a_{a'}(x^1, \ldots, x^n) p_a, \ \text{rank}\,[\partial p_{a'}/\partial p_a] = m.$$

If (1.1) is valid, then exists an inverse transformation

$$(1.2) \qquad (a) \quad x^i = x^i(x^{1'}, \ldots, x^{n'}) \qquad (b) \quad p_a = M^{a'}_a(x^{1'}, \ldots, x^{n'}) p_{a'}$$

The elements $\partial_i = \partial/\partial x^i$ and $\partial^a = \partial/\partial p_a$ of $T(E^*)$ are transforming in the following way

$$(1.3) \qquad \begin{bmatrix} \partial_j \\ \partial^a \end{bmatrix} = \begin{bmatrix} \frac{\partial x^{i'}}{\partial x^i} & (\partial_i M^a_{a'}) p_a \\ 0 & M^a_{a'} \end{bmatrix} \begin{bmatrix} \partial_{i'} \\ \partial^{a'} \end{bmatrix}$$

$$(1.4) \qquad \begin{bmatrix} \partial i' \\ \partial^{a'} \end{bmatrix} = \begin{bmatrix} \frac{\partial x^j}{\partial x^{i'}} & (\partial_{i'} M^{b'}_b) p_{b'} \\ 0 & M^{a'}_b \end{bmatrix} \begin{bmatrix} \partial_j \\ \partial^b \end{bmatrix}$$

Substituting (1.4) into (1.3) we obtain

$$(1.5) \qquad (a) \quad \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x^{i'}} = \delta^j_i \qquad (b) \quad M^a_{a'} M^{a'}_a = \delta^a_b$$