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DIRECTION DEPENDENT f–STRUCTURES SATISFYING $f^3+f=0$

BY

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Introduction. In [8],[9] K. Yano has defined the notion of f–structure on a manifold satisfying $f^3+f=0$ and such that the rank of f is equal to a constant k everywhere. It is the purpose of the present paper to start a study of f–structures which depend on direction.

1. Preliminaries. Let N be a real differentiable manifold of dimension n. Denote by TN the tangent bundle over N and by π the canonical projection of TN to N. Also, denote by $d\pi : TTN \to TN$ the differential of π and define the vertical subbundle VTN of TTN as the kernel of $d\pi$. A complementary distribution HTN to VTN in TTN is called a nonlinear connection on TN (see M. Matsumoto [6]). Of course, the fibres of vector bundles VTN and HTN are of the same dimension n. It is well known, VTN is an integrable distribution on TN.

Let (x^i, y^i) be a coordinate system on TN and $\{\delta/\delta x^i, \partial/\partial y^i\}$ be a local field of frames on TN adapted to the decomposition $TTN = HTN \oplus VTN$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial y^j},$$

and $N_i^j(x,y)$ are n^2 differentiable functions locally defined on TN. The automorphism

$$P : \Gamma(TTN) \to \Gamma(TTN),$$

which is defined by

$$PX = X_V^i(x,y)\frac{\delta}{\delta x^i} + X_H^i(x,y)\frac{\partial}{\partial y^i},$$

for

$$X = X_H^i(x,y)\frac{\delta}{\delta x^i} + X_V^i(x,y)\frac{\partial}{\partial y^i},$$

is an almost product structure on TN, i.e, $P^2 = I$, where $\Gamma(TTN)$ is the $\mathcal{F}(TN)$–module of all differentiable cross sections of TTN. We keep the same notation for any other bundle. The almost product structure P will be called natural almost product structure on TN. If we denote by v and h the projection morphism of TTN to VTN and HTN respectively, we have

$$P \circ h = v \circ P.$$

The automorphism

$$JX = -X^i_V(x,y)\frac{\delta}{\delta x^i} + X^i_H(x,y)\frac{\partial}{\partial y^i},$$

for

$$X = X^i_H(x,y)\frac{\delta}{\delta x^i} + X^i_V(x,y)\frac{\partial}{\partial y^i},$$

is the natural almost complex structure on TN named by Matsumoto [5].

2.Finsler f-structure satisfying $f^3 + f = 0$.

Definition 2.2. *We call Finsler f-structure of rank r on N a non-null Finsler tensor field of type (1.1) and of class C^∞ such that $f^3 + f = 0$ and rank $f = r$, where r is constant everywhere.*

Definition 3.2. *We call horizontal f-structure of rank r on N a non null horizontal tensor field $\overline{f}$ on TN of type $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ satisfying $\overline{f}^3 + \overline{f} = 0$ and rank $\overline{f} = r$, where r is constant.*

Of course, an f-structure on TN of rank r is a non-null tensor field F of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ such that $F^3 + F = 0$, rank$F = r$, where r is constant everywhere.

For our study it is very convenient to consider f or $\overline{f}$ as morphism of vector bundles

$$f\colon \Gamma(VTN) \to \Gamma(VTN) \quad \text{and} \quad \overline{f}\colon \Gamma(HTN) \to \Gamma(HTN),$$

respectively. Let f be a Finsler f-structure of rank r on N. The morphism $\ell = -f^2$, $m = f^2 + I_{\Gamma(VTN)}$, where $I_{\Gamma(VTN)}$ denotes identity morphism on VTN, applied to the vertical vector bundle on TN are complementary projection morphisms. Then there exist complementary distributions VL and VM corresponding to the projection morphism ℓ ans m respectively such that dim$VL = r$ and dim$VM = n - r$.

For the tensor field f and the morphisms ℓ, m we have

$$f\ell = \ell f = f, \quad f^2\ell = -\ell \quad \text{and} \quad fm = mf = 0. \tag{2.1}$$

Proposition 2.1. *If a Finsler f-structure of rank r is defined on N, then an horizontal f-structure of rank r is defined on N by the natural almost product structure of TN or the natural almost complex structure of TN.*

Proof. If we put

$$f_1X = PfPX, \quad \forall X \in \Gamma(HTN), \tag{2.2}$$

$$f_2X = -JfJX, \quad \forall X \in \Gamma(HTN), \tag{2.3}$$

where P and J are the tensor fields of natural product and natural almost complex structure of TN respectively, it is easy to see that

$$f_1^3 + f_1 = 0 \quad \text{or} \quad f_2^3 + f_2 = 0, \tag{2.4}$$

and rankf_1 = rank$f_2 = r$.

Proposition 2.2. *If a Finsler f-structure of rank r is defined on N, then an f-structure is defined on TN of rank r by the natural almost product or natural almost complex structure of TN.*

Proof. If we put

$$F_1 = (P + I) \circ f \circ v, \tag{2.5}$$

where I is the identity morphism on TTN, we have

$$F_1X = PfvX + fvX.$$

Thus $hF_1X = PfvX$ and $vF_1X = fvX$.
It is easy to check that

$$F_1^2 = (P + I) \circ f^2 \circ v,$$

and

$$vF_1^2X = f^2vX.$$

Then

$$F_1^3X = (P + I)f_1^3vX = -(P + I)fvX = -F_1X,$$

and

$$F_1^3 + F_1 = 0.$$

From the definition of F_1 we get that

$$\ker F_1 = HTN \oplus \ker f.$$

Then $\operatorname{rank} F_1 = r$ and the morphism F_1 defines an f-structure satisfyng $F_1^3 + F_1 = 0$ on TN. Similarly if we put

$$F_2 = (J + I) \circ f \circ v,$$

we can see that F_2 defines an f-structure satisfyng $F_2^3 + F_2 = 0$ on TN.

If ℓ_1, m_1 are the complementary projection morphisms of the horizontal f-structure f_1 which is defined by the natural almost product structure of TN we have

(2.6) $\ell_1 X = -f_1^2 X = -PfPPfPX = -Pf^2PX = P\ell PX, \quad \forall X \in \Gamma(HTN),$

(2.7) $m_1 X = f_1^2 X + I_{\Gamma(HTN)} X = Pf^2PX + PI_{\Gamma(VTN)}PX = PmPX, \ \forall X \in \Gamma(HTN).$

Similary if ℓ_2, m_2 are the complementary projection morphisms of the horizontal f-structure f_2, which is defined by the natural almost complex structure of TN, we have

(2.8) $\ell_2 X = -J\ell JX \quad \text{and} \quad m_2 X = -JmJX, \quad \forall X \in \Gamma(HTN).$

Thus, if there is given a Finsler f-structure on N of rank r, then there exist complementary distributions HL_1, HM_1 or HL_2, HM_2 corresponding to the morphisms ℓ_1, m_1 and ℓ_2 , m_2 respectively such that

(2.9) $HL_1 = PVL, \qquad HM_1 = PVM, \qquad HL_2 = -JVL, \qquad HM_2 = -JVM.$

Thus we have the decompositions

$$TTN = HTN \oplus VTN = PVL \oplus PVM \oplus VL \oplus VM,$$

$$TTN = (-JVL) \oplus (-JVM) \oplus VL \oplus VM.$$

From (2.9) we have

(2.10) $VL = PHL_1, \quad VM = PHM_1, \quad VL = -JHL_2, \quad VM = -JHM_2.$

3. Existence theorems. Let $CVTM$ be the complexified vertical vector bundle on TN and Π be a complex subbundle of $CVTN$ such that $\Pi \cap \overline{\Pi} = 0$, where $\overline{\Pi}$ is the complex conjugate of Π. We denote K the complementary distribution of $\operatorname{Re}\Pi$ to VTN, that is $VTN = \operatorname{Re}\Pi \oplus K$. We can define a morphism of vector bundles by the relation

$$f(Z) = 0, \quad \forall Z \in \Gamma(K), \quad f(X) = \frac{1}{2}\sqrt{-1}(U - \overline{U}),$$

where $U = X + \sqrt{-1}Y$ is a cross section of $\prod$.

Then, it is easy to check that

$$f^3 + f = 0.$$

It is clear that $\operatorname{rank} f = \dim \operatorname{Re}\Pi_q$, where Π_q is the fibre of Π over $q \in TN$. Thus a Finsler f-structure is defined on N by the subbndle Π.

Converselly, let f be a Finsler f-structure on N. We can define a subbundle of $CVTN$ by

$$\Pi = \{X - \sqrt{-1}fX, \quad X \in \Gamma(VL)\}.$$

It is clear that $\operatorname{Re}\Pi = VL$ and $\Pi \cap \overline{\Pi} = 0$. Thus we have.

Theorem 3.1. *A necessary and sufficient condition for an n-dimensional manifold N to admit a Finsler f-structure is that there is a complex subbundle Π of $CVTN$ such that $\Pi \cap \Pi = \{0\}$. Then $\operatorname{rank} f = \dim \operatorname{Re}\Pi$.*

If there is a subbundle of $CVTN$ such that $\Pi \cap \overline{\Pi} = \{0\}$, then according to Proposition 2.1 and 2.2 an horizontal f-structure is defined on N and an f-structure on TN. Thus the existence of a subbundle Π of $CVTN$, such that $\Pi \cap \overline{\Pi} = \{0\}$ is a necessary and sufficient condition to define a Finsler f-structure, an horizontal f-structure on N and an f-structure on TN, of rank equal to $\dim \operatorname{Re}\Pi$.

Now we suppose that the vertical bundle VTN on TN is endowed with a pseudo-Riemannian structure, i.e. a mapping

$$h: \Gamma(VTN) \times \Gamma(VTN) \to \mathcal{F}(TN),$$

which is symmetric, $\mathcal{F}(TN)$-bilinear and non-degenerate on each fibre of VTN. Of course, a Finsler structure, or a Lagrange structure are examples of a pseudo-Riemannian structure [1].

The mapping

$$\alpha: \Gamma(VTN) \times \Gamma(VTN) \to \mathcal{F}(TN),$$

which is defined by

$$\alpha(X, Y) = \frac{1}{2}(h(\ell X, \ell Y) + h(mX, mY)), \quad \forall X, Y \in \Gamma(VTN),$$

is a pseudo-Riemannian structure on TN such that

$$\alpha(X, Y) = 0, \qquad \forall X \in \Gamma(VL), \qquad Y \in \Gamma(VM).$$

It is easy to show

$$\alpha(X, Y) = \alpha(\ell X, \ell Y) + \alpha(mX, mY), \tag{3.1}$$

$$\forall X, Y \in \Gamma(VTN).$$

Then we have

Theorem 3.2. *If a Finsler f-structure of rank r is defined on N, then there exists a peudo-Riemannian structure on TN with respect to the complementary distributions VL and VM are orthogonal and the f is an isometry on VL. The $\operatorname{rank} f = r$ must be even.*

Proof. If we put

$$g(X, Y) = \frac{1}{2}(\alpha(X, Y) + \alpha(fX, fY)),$$

we can easy see

$$g(X,Y)=0, \qquad \forall X\in\Gamma(VL), \qquad \forall Y\in\Gamma(VM),$$

that is VL, VM are orthogonal distributions with respect to g.

For every $X,Y\in\Gamma(VL)$ we have

$$g(fX,fY)=\frac{1}{2}(\alpha(fX,fY)+\alpha(f^2X,f^2Y)).$$

Using (2.1) we get

$$g(fX,fY)=\frac{1}{2}(\alpha(fX,fY)+\alpha(X,Y))$$

and

$$g(X,Y)=g(fX,fY). \tag{3.2}$$

Thus f is an isometry with respect to g.

If X is a differentiable section of VL then the fX is also a differentiable section of VL and orthogonal to X with respect to g, because from (3.2) we have

$$g(X,fX)=g(fX,f^2X)=-g(fX,X)=0.$$

Consequently we can choose in $\Gamma(VL)$ $r=2m$ mutualy orthogonal vector fields such that

$$X_{m+1}=fX_1, \qquad X_{m+2}=fX_2, \qquad X_{2m}=fX_m.$$

Thus the rank$f=r$ must be even, equal to $2m$. The proof is complete. Using (3.1) we get the relation

$$g(X,Y)=g(\ell X,\ell Y)+g(mX,mY), \qquad \forall X,Y\in\Gamma(VTN). \tag{3.3}$$

An adapted frame of the Finsler f-structure is the orthogonal frame (X_b,X_B), $b=1,\dots,2m$, $B=2m+1,\dots,n$, where $X_{m+i}=fX_i$, $\forall i=1,\dots,m$ and X_B an orthogonal frame of VM.

Theorem 3.3. *A necessary and sufficient condition for an n-dimensional manifold, whose vertical bundle has a pseudo-Riemannian structure, to admit a Finsler f-structure of rank r is that r is even $r=2m$ and the group of the vertical bundle of the manifold be reduced to the group $U_{(m)}\times 0_{(n-2m)}$.*

The proof is analogous of Theorem 2.2 of Yano (see [9], p. 106).

By means of the pseudo-Riemannian structure g on VTN, we can define a mapping $g_P\colon\Gamma(HTN)\times\Gamma(HTN)\to\mathcal{F}(TN)$ such that

$$g_P(X,Y)=g(PX,PY), \qquad \forall X,Y\in\Gamma(HTN).$$

The mapping g_P has the properties of g and defines a metric structure on HTN.

Then, using (2.10) the distributions HL_1 and the horizontal f-structure f_1, which is defined by natural almost product structure P of TN in Propotition 2.1 is an isometry on HL_1 with respect to g_P.

Similarly is we define a metric on HTN by the relation

$$g_J(X,Y)=g(JX,JY), \qquad \forall X,Y\in\Gamma(HTN),$$

then using (2.10) we have that the distributions HL_2 and HM_2 are orthogonal with respect to g_J, and the horizontal f-structure f_2, which is defined by natural almost complex structure of TN in Propotition 2.1, is an isometry on HL_2 with respect to g_J.

If (X_b,X_B), $b=1,\dots,2m$, $B=2m+1,\dots,n$ is an adapted frame of a given Finsler f-structure on N we have

$$g_P(PX_i,f_1PX_i)=g(P^2X_i,Pf_1PX_i), \qquad \forall i=1,\dots,m.$$

Using (2.2) implit

$$g_P(PX_i,f_1PX_i)=g(X_i,fX_i)=0, \qquad \forall i=1,\dots,m.$$

Similarly

$$g_J(JX_i,f_2JX_i)=g(J^2X_i,Jf_2JX_i)=-g(X_i,fX_i)=0, \quad \forall i=1,\dots,m.$$

Thus we have

Proposition 3.1. *If (X_b,X_B) is an adapted frame of a given Finsler f-structure f on N with respect to g, then the frame (PX_b,PX_B) is an adapted frame of the horizontal f-structure f_1 with respect to g_P and the frame (JX_b,JX_B) is an adapted frame of the horizontal f-structure f_2 with respect to g_J.*

It is clear that the frames (PX_b,PX_B,X_b,X_B) and (JX_b,JX_B,X_b,X_B) are adapted frames to the decomposition $TTN=HL_1\oplus HM_1\oplus VL\oplus LM$ and $TTN=HL_2\oplus HM_2\oplus VL\oplus LM$ respectively.

Thus we have

Theorem 3.4. *If a Finslerf-structure is defined on an n-dimensional manifold whose vertical bundle has endowed with a pseudo-Riemannian structure, then the elements of the structure-group of the tangent bundle on TN have the form*

$$A=\begin{bmatrix} Gm & Hm & 0 & 0 & 0 & 0\\ -Hm & Gm & 0 & 0 & 0 & 0\\ 0 & 0 & On-2m & 0 & 0 & 0\\ 0 & 0 & 0 & Am & Bm & 0\\ 0 & 0 & 0 & -Bm & Am & 0\\ 0 & 0 & 0 & 0 & 0 & On-2m \end{bmatrix}$$

and thus the structure group reduces to

$$U(m) \times O(n-2m) \times U(m) \times On - 2m.$$

If we consider A as

$$A = \begin{bmatrix} Gm & Hm & 0 \\ -Hm & Gm & 0 \\ 0 & 0 & O2n-2m \end{bmatrix},$$

then, according to the Theorem 2.2 ([9], p. 106) the tangent space TN admits a tensor field $F \neq 0$ of type (1.1) such that $F^3 + F = 0$ and rank$F = r = 2m$.

4. On integrability conditions of a Finsler f–structure. Let f be a Finsler f–structure on the manifold n. The Nijenhuis tensor N_f of f is given by

$$N_f(X,Y) = [fX, fY] - \ell[X,Y] - f[fX,Y] - f[X,fY], \qquad \forall X, Y \in \Gamma(VTN).$$

Theorem 4.1. *A necessary and sufficient condition for a Finsler f–structure to be integrable is that*

$$N_f(X,Y) = 0, \qquad \forall X, Y \in \Gamma(VTN). \tag{4.1}$$

The Proof is analogous to the case of an f–structure on a manifold.

Let $CTTN$ be the complexified tangent bundle to TN. A CR–structure on TN is a complex subbundle H of $CTTN$, such that $H \cap \overline{H} = \{0\}$, where $\overline{H}$ is the complex conjugate of H and H is involutive.

We suppose that f is an integrable Finsler f–structure. Then we can define the complex subbundle H of $CVTN$ by

$$H = \{X - \sqrt{-1}fX, \qquad X \in \Gamma(VL)\}.$$

Then Re$H = VL$ and $H \cap \overline{H} = \{0\}$.
If $U = X - \sqrt{-1}fX$ and $V = Y - \sqrt{-1}fY$ are two cross sections of H, by direct computation we obtain:

$$[U,V] = [X,Y] - [fX,fY] - \sqrt{-1}([X,fY] + [fX,Y]). \tag{4.2}$$

Using the Theorem 4.1 we have

$$[fX,fY] - \ell[X,Y] = f([X,fY] + [fX,Y]),$$

$$f[fX,fY] + f^3[X,Y] = f^2([X,fY] + [fX,Y]),$$

and finally

$$f([X,Y] - [fX,fY]) = \ell([X,fY] + [fX,Y]). \tag{4.3}$$

From (4.1) and (3.1) implit

$$mN(X,Y) = m[fX,fY] = 0,$$

that is VL is integrable.
Then $[X,fY] \in \Gamma(VL)$ and $[fX,Y] \in \Gamma(VL)$, $\forall X, Y \in \Gamma(VL)$.
Consequently

$$\ell([X,fY] + [fX,Y]) = [X,fY] + [fX,Y]. \tag{4.4}$$

From (4.2), (4.3) and (4.4) we have

$$[U,V] = [X,Y] - [fX,fY] - \sqrt{-1}f([X,Y] - [fX,fY]),$$

that is $[U,V] \in \Gamma(H)$ and H is involutive.
Conversely, let H be a CR–structure on TN such that Re$HCVTN$ and K be the complementary distribution of ReH to VTN that is

$$VTN = \text{Re}H \oplus K.$$

We define a morphism of vector bundles

$$f : \Gamma(VTN) \to \Gamma(VTN),$$

given by

$$f(Z) = 0, \qquad \forall Z \in \Gamma(K),$$

and

$$f(X) = \frac{1}{2}\sqrt{-1}(U - \overline{U}), \tag{4.5}$$

where

$$U = X + \sqrt{-1}Y \in \Gamma(H).$$

Then it is easy to check that the morphism f define a Finsler f–structure, satisfying

$$f^3 + f = 0.$$

The distributions VL and VM, which are defined by f coincide to ReH and K respectively.
From (4.5) we have

$$f(X) = -Y \quad \text{and} \quad f^2X = -X. \tag{4.6}$$

Consequently every cross section of H can be written in the form

$$U = X - \sqrt{-1}fX, \ \forall X \in \Gamma(\text{Re}H).$$

If $U = X - \sqrt{-1}fX$ and $V = Y - \sqrt{-1}fY$ are two sections of H, taking account of H is involutive, we find

$$f([X,Y] - [fX,fY]) - [X,fY] + [fX,Y],$$

$$-f^2[fX,fY] - \ell[X,Y] = f([X,fY] + f[fX,Y]).$$

But $[fX,fY] \in \Gamma(H)$, $fX, fY \in \mathrm{Re}H$ and consequently $[fX,fY] \in \Gamma(\mathrm{Re}H)$. From (4.6) implit

$$-f^2[fX,fY] = [fX,fY],$$

and

$$[fX,fY] - \ell[X,Y] = f([X,fY] + f[fX,Y]).$$

Thus

$$N_f(X,Y) = 0.$$

Gathering the results, we have

Theorem 4.2. *A necessary and sufficient contition for an n-dimentional manifold N to admit an integrable Finsler f–structure is that TN has a CR–structure H such that*

$$\mathrm{Re}H \subset VTN$$

In similar way we can prove that a necessary and sufficient condition for an n-dimentional manifold N to admit an integrable horizontal f–structure is that TN has a CR–structure Π such that $\mathrm{Re}\Pi \subset HTN$.

Remark. If f is an integrable Finsler f–structure then f the horizontal f-structure, which are defined by Propotition 2.1, are not integrable because

$$P[X,Y] \neq [PX,PY], \qquad \forall X,Y \in \Gamma(VLN).$$

and

$$J[X,Y] \neq [JX,JY].$$

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