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Tomul XXXVII, s. I. a. Matematică, 1991, f. 4

ON THE ORDER AND TYPE OF INTEGRAL FUNCTIONS OF SEVERAL COMPLEX VARIABLES

BY

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1. Introduction. In this paper we shall obtain the relations between two or more integral functions of several complex variables and study the relations between the coefficients in the Taylor expansion of integral functions and their orders and types. In this connection it may be mentioned that similar problems were investigated earlier in the papers [6], [2], and [5]. In what follows, we shall, for the sake of simplicity, consider only two variables though the results can easily be extended to several variables. Let

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n \tag{1.1}$$

be a function of two complex variables where $a_{mn} \in \mathbf{C}$ (the field of complex numbers). The series (1.1) represents an integral function of two variables z_1 and z_2 if it converges absolutely for all values of z_1 and z_2 such that $|z_1| < \infty$ and $|z_2| < \infty$. M.M. Dzrbasyan ([1], pp. 1–23) has shown that the necessary and sufficient condition for the series (1.1) to represent an integral function of variables z_1 and z_2, is

$$\limsup_{m+n\to\infty} |a_{mn}|^{\frac{1}{m+n}} = 0. \tag{1.2}$$

Let $\overline{G}_r$ be the family of closed polycircular domains in space (z_1, z_2) dependent on a parameter $r > 0$ and possess the property that $(z_1, z_2) \in \overline{G}_r$, if and only if $(z_1/r, z_2/r) \in \overline{G}_1$. The maximum modulus of the integral function $f(z_1, z_2)$ is denoted by

$$M_G(r, f) = \max_{(z_1,z_2)\in\overline{G}_r} |f(z_1, z_2)|$$

and the function will be called G-order and G-type respectively, if

$$\rho_G = \limsup_{r\to\infty} \frac{\log\log M_G(r, f)}{\log r} \tag{1.3}$$

(1.4) $$T_G = \limsup_{r\to\infty} \frac{\log M_G(r,f)}{r^{\rho_G}}.$$

Denote

(1.5) $$\Phi = \Phi_G(m,n) = \max_{(z_1,z_2)\in \overline{G}_r} (|z_1|^m |z_2|^n).$$

A.A. Goldberg ([3], p.145) has proved the following theorems.

Theorem A. *All orders ρ_G are equal and*

(1.6) $$\rho = \rho_G = \limsup_{m+n\to\infty} \frac{(m+n)\log(m+n)}{-\log|a_{mn}|}.$$

Theorem B. *The G-type T_G satisfies the condition*

(1.7) $$(e^\rho T_G)^{1/\rho} = \limsup_{m+n\to\infty} \left[(m+n)^{1/\rho}\{\Phi_G(m,n)|a_{mn}|\}^{1/(m+n)}\right].$$

2. The lower bar order $\underline{\lambda}$ is defined in [2] by

(2.1) $$\underline{\lambda} = \limsup_{m+n\to\infty} \frac{(m+n)\log(m+n)}{-\log|a_{mn}|}.$$

Theorem 1. *Let $f_s(z_1,z_2) = \sum_{m,n=0}^{\infty} a_{mn}^{(s)} z_1^m z_2^n$, where $s = 1,2,\ldots,k$ be k integral functions of non-zero finite orders $\rho_1,\rho_2,\ldots,\rho_k$ and non-zero lower bar orders $\underline{\lambda}_1,\underline{\lambda}_2,\ldots,\underline{\lambda}_k$ respectively. Then the function*

$$f(z_1,z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n,$$

where

(2.2) $$\log|a_{mn}|^{-p} \sim \left|\left(\prod_{s=1}^{k} \log|a_{mn}^{(s)}|^{-p_s}\right)^{\frac{1}{k}}\right|$$

(p's being some positive constants) is an integral function such that

(2.3) $$\left(\prod_{s=1}^{k-1} \frac{p_s}{\rho_s}\right)^{\frac{1}{k}} \le \frac{\left(\frac{p}{\rho}\right) / \left(\frac{p_k}{\rho_k}\right)^{\frac{1}{k}}}{\left(\frac{p}{\underline{\lambda}}\right) / \left(\frac{p_k}{\underline{\lambda}_k}\right)^{\frac{1}{k}}} \le \left(\prod_{s=1}^{k-1} \frac{p_s}{\underline{\lambda}_s}\right)^{\frac{1}{k}}$$

where ρ is the order and $\underline{\lambda}$ is the lower bar order of $f(z_1,z_2)$.

Proof. Since $f_s(z_1,z_2)$ is an integral function, using (1.2) we have for arbitrary $\epsilon > 0$ and enough large R, $|a_{mn}^{(s)}|^{1/(m+n)} < 1/(R-\epsilon)$ for $m+n > N_s$ or $\log|a_{mn}^{(s)}|^{-p_s} > p_s(m+n)\log(R-\epsilon)$. Putting $s = 1,2,\ldots,k$ and multiplying out the k inequalities thus obtained, we have for all large $m+n$

$$\prod_{s=1}^{k} \log|a_{mn}^{(s)}|^{-p_s} > (p_1,p_2,\ldots,p_k)\{(m+n)\log(R-\epsilon)\}^k.$$

Therefore, if

$$\log|a_{mn}|^{-p} \sim \left|\left(\prod_{s=1}^{k} \log|a_{mn}^{(s)}|^{-p_s}\right)^{\frac{1}{k}}\right|$$

then for all large $m+n$

$$\log|a_{mn}|^{-p} > (p_1,p_2,\ldots,p_k)^{\frac{1}{k}}(m+n)\log(R-\epsilon)$$

or

$$|a_{mn}|^{\frac{1}{m+n}} < 1/(R-\epsilon)^{(p_1,p_2,\ldots,p_k)^{\frac{1}{k}}/p}.$$

Hence $\limsup_{m+n\to\infty}|a_{mn}|^{1/(m+n)} = 0$, since the p's are all positive constants. Hence $f(z_1,z_2)$ is an integral function.

Now using (1.5) and (2.1) for the function $f_s(z_1,z_2)$, we have

(2.4) $$\log|a_{mn}^{(s)}|^{-p_s} > \frac{p_s(m+n)\log(m+n)}{\rho_s+\epsilon} \quad \text{for } m+n > N_s \quad \text{and}$$

(2.5) $$\log|a_{mn}^{(s)}|^{-p_s} > \frac{p_s(m+n)\log(m+n)}{\underline{\lambda}_s+\epsilon}$$

for an infinite sequence of values of m,n. Taking $s = 1,2,\ldots,k$ in (2.4) and multiplying the k inequalities thus obtained, we have

$$\prod_{s=1}^{k} \log|a_{mn}^{(s)}|^{-p_s} > (p_1,p_2,\ldots,p_k)[(m+n)\log(m+n)]^k \cdot \prod_{s=1}^{k} \frac{1}{\rho_s+\epsilon}.$$

Thus using (2.2), we get $\log|a_{mn}|^{-p} > (m+n)\log(m+n)\left(\prod_{s=1}^{k}\frac{p_s}{\rho_s+\epsilon}\right)^{\frac{1}{k}}$. Taking the limit,

$$\limsup_{m+n\to\infty} \frac{(m+n)\log(m+n)}{\log|a_{mn}|^{-p}} \le \left(\prod_{s=1}^{k} \frac{p_s}{\rho_s}\right)^{-\frac{1}{k}}$$

$$\text{or} \quad \frac{p}{\rho} \ge \left(\prod_{s=1}^{k} \frac{p_s}{\rho_s}\right)^{\frac{1}{k}}, \quad \text{i.e.,} \quad \left(\prod_{s=1}^{k-1} \frac{p_s}{\rho_s}\right)^{\frac{1}{k}} \le \left(\frac{p}{\rho}\right) / \left(\frac{p_k}{\rho_k}\right)^{\frac{1}{k}},$$

which proves the first part of the left hand side of (2.3).

We now prove the other part of the left hand side of (2.3). Taking $s = 1, 2, \ldots, (k-1)$ in (2.4) and $s = k$ in (2.5) and multiplying out the inequalities thus obtained, we get

$$\prod_{s=1}^{k} \log |a_{mn}^{(s)}|^{-p_s} > (p_1, p_2, \ldots, p_k)[(m+n)\log(m+n)]^k \prod_{s=1}^{k-1} \frac{1}{\rho_s + \epsilon} \cdot \frac{1}{\lambda_k + \epsilon}.$$

Thus, using (2.2), we have

$$\log |a_{mn}|^{-p} > (m+n)\log(m+n) \left(\prod_{s=1}^{k-1} \frac{p_s}{\rho_s + \epsilon} \cdot \frac{p_k}{\lambda_k + \epsilon} \right)^{\frac{1}{k}}.$$

Taking the limit

$$\liminf_{m+n\to\infty} \frac{(m+n)\log(m+n)}{\log |a_{mn}|^{-p}} \le \left(\prod_{s=1}^{k-1} \frac{p_s}{\rho_s} \cdot \frac{p_k}{\lambda_k} \right)^{-\frac{1}{k}}$$

$$\text{or,} \quad \frac{p}{\lambda} \ge \left(\prod_{s=1}^{k-1} \frac{p_s}{\rho_s} \cdot \frac{p_k}{\lambda_k} \right)^{\frac{1}{k}} \quad \text{i.e.,} \quad \left(\prod_{s=1}^{k-1} \frac{p_s}{\rho_s} \right)^{\frac{1}{k}} \le (\frac{p}{\lambda})/(\frac{p_k}{\lambda_k})^{\frac{1}{k}},$$

which proves the other part of left hand side of (2.3).

The proof of the right hand side follows in a similar manner.

Corollary. *If any k L's out of $L, L_1, L_2, \ldots, L_k$ are equal to unity, then all the $(k+1)$ L's are equal to unity and we have*

$$\frac{p}{\rho} = (\prod_{s=1}^{k} \frac{p_s}{\rho_s})^{\frac{1}{k}},$$

where $L = \rho/\lambda$ and $L_s = \rho_s/\lambda_s$.

3. Theorem 2.

Let

$$f_s(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn}^{(s)} z_1^m z_2^n$$

where $s = 1, 2, \ldots, k$ be k integral functions of finite non-zero orders $\rho_1, \rho_2, \ldots, \rho_k$ and non-zero lower bar order $\lambda_1, \lambda_2, \ldots, \lambda_k$ respectively. Then the function

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$$

where

$$\log |a_{mn}|^{-p} \sim \log\{\prod_{s=1}^{k} |a_{mn}^{(s)}|^{-p_s}\} \tag{3.1}$$

(p's being some positive constants) is an integral function such that

$$\sum_{s=1}^{k-1} \frac{p_s}{\rho_s} \le \frac{\left(\frac{p}{\rho} - \frac{p_k}{\rho_k}\right)}{\left(\frac{p}{\lambda} - \frac{p_k}{\lambda_k}\right)} \le \sum_{s=1}^{k-1} \frac{p_s}{\lambda_s} \tag{3.2}$$

where ρ is the order and λ is the lower bar order of $f(z_1, z_2)$.

Proof. Since $f_s(z_1, z_2)$ is an integral function, we have, using (1.2), for arbitrary $\epsilon > 0$ and enough large R

$$|a_{mn}^{(s)}|^{1/(m+n)} < \frac{1}{R-\epsilon} \quad \text{for } m+n > N_s \quad \text{or } \log |a_{mn}^{(s)}|^{-p_s} > p_s(m+n)\log(R-\epsilon).$$

Putting $s = 1, 2, \ldots, k$ and adding the k inequalities thus obtained, we get for all large $m+n$

$$\log \left(\prod_{s=1}^{k} |a_{mn}^{(s)}|^{-p_s} \right) > (p_1 + p_2 + \ldots + p_k)(m+n)\log(R-\epsilon).$$

Therefore, using (3.1)

$$\log |a_{mn}|^{-p} > (p_1 + p_2 + \ldots + p_k)(m+n)\log(R-\epsilon)$$

$$\text{or} \quad |a_{mn}|^{-p} > (R-\epsilon)^{(m+n)(p_1+\cdots+p_k)} \quad \text{or} \quad |a_{mn}| < (R-\epsilon)^{-(m+n)\sum p_s/p}.$$

Hence $\limsup_{m+n\to\infty} |a_{mn}|^{1/(m+n)} = 0$, since p's are all positive. Hence $f(z_1, z_2)$ is an integral function. The rest follows as in Theorem 1, and this proves the theorem.

4. Theorem 3.

Let $f_s(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn}^{(s)} z_1^m z_2^n$ where $s = 1, 2, \ldots, k$ be k integral functions of finite non-zero orders $\rho_1, \rho_2, \ldots, \rho_k$ and finite non-zero types[1] $T_1, T_2, \ldots, T_k$ respectively. Then the function $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ where

$$\log(\Phi \cdot |a_{mn}|)^{-p} \sim \left(\prod_{s=1}^{k} \log(\Phi |a_{mn}^{(s)}|)^{-p_s} \right)^{\frac{1}{k}} \tag{4.1}$$

[1] The types $T_1, T_2, \ldots, T_k$ correspond to the same family of closed polycircular domains G_r.

(p's being some positive constants) is an integral function such that

$$pT \leq (\prod_{s=1}^{k} p_s T_s)^{\frac{1}{k}} \tag{4.2}$$

where ρ and T are the order and type of $f(z_1, z_2)$ respectively, provided

$$\frac{p}{\rho} = (\prod_{s=1}^{k} \frac{p_s}{\rho_s})^{\frac{1}{k}}.$$

The proof requires the following:

Lemma. *Let*

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n \tag{1}$$

be a function of two complex variables z_1, z_2, where $a_{mn} \in C$. The series (1) represents an integral function of two variables z_1, z_2 if and only if

$$\limsup_{m+n\to\infty} (\Phi|a_{mn}|)^{1/(m+n)} = 0 \tag{2}$$

where Φ has the same meaning as given in (1.5).

Proof of the lemma. First, let (1) define an integral function. Then (1) converges absolutely for all values of $z_1, z, |z_1| < \infty$ and $|z_2| < \infty$ and hence in any closed polycircular domain $G_r (0 < r < \infty)$. Let $(z_1', z_2') \in \delta G_r$ be such that $\Phi = |z_1'|^m |z_2'|^n = r_1^m r_2^n$, taking $|z_1'| = r_1$ and $|z_2'| = r_2$. We consider points with coordinates $z_1 = r_1 t$, $z_2 = r_2 t (t > 0)$. Then proceeding as in ([4], p. 72), we obtain

$$\limsup_{m+n\to\infty} (|a_{mn}| r_1^m r_2^n)^{1/(m+n)} = 0 \text{ i.e., } \limsup_{m+n\to\infty} (\Phi|a_{mn})^{1/(m+n)} = 0.$$

Next, let the given condition (2) be satisfied. Let $(z_1', z_2') \in \overline{G}_r$ be such that $|z_1'|, |z_2'| > 1$ and choose $\Phi = \Phi_G(m,n) = |z_1'|^m |z_2'|^n$. Then by the given condition $\Phi|a_{mn}| < 1/r^{m+n} (r > 1)$ for all large $m+n$, i.e., $|a_{mn}||z_1'|^m |z_2'|^n < 1/r^{m+n}, r > 1$. Now the series $\sum_{m,n=0}^{\infty} \frac{1}{r^{m+n}}$ being convergent, it follows that the series (1) converges absolutely for all (z_1', z_2'), where $|z_1'|, |z_2'| > 1$. But G_r being a polycircular domain, the series (1) converges absolutely for every (z_1, z_2) such that $|z_1| < |z_1'|$ and $|z_2| < |z_2'|$ and the lemma is thus proved.

Proof of the theorem. Since $f_s(z_1, z_2)$ is an integral function, by the lemma, we have for arbitrary $\epsilon > 0$ and enough large R, $(\Phi|a_{mn}^{(s)}|)^{1/(m+n)} < \frac{1}{R-\epsilon}$ for

$m+n > N_s$, $\log(\Phi|a_{mn}^{(s)}|)^{-p_s} > p_s(m+n)\log(R-\epsilon)$. Putting $s = 1, 2, \ldots, k$ and multiplying the k inequalities thus obtained, we have

$$\prod_{s=1}^{k} \log(\Phi|a_{mn}^{(s)}|)^{-p_s} > \{(m+n)\log(R-\epsilon)^k (\prod_{s=1}^{k} p_s)\}.$$

Therefore, using (4.1) $\log(\Phi|a_{mn}|^{-p}) > (\prod_{s=1}^{k} p_s)^{1/k} (m+n) \log(R-\epsilon)$ or $(\Phi|a_{mn}|)^{1/(m+n)} < (R-\epsilon)^{-(\prod_{s=1}^{k} p_s)^{1/k}/p}$ for all large $m+n$. Therefore $\limsup_{m+n\to\infty} (\Phi|a_{mn}|)^{1/(m+n)} = 0$ (since p's are all positive). Hence by the lemma $f(z_1, z_2)$ is an integral function.

The proof of the rest viz., the relation (4.2) is patterned after [5].

Using (1.6) for the function $f_s(z_1, z_2)$, we have

$$\limsup_{m+n\to\infty} [(m+n)^{1/\rho_s} \{\Phi|a_{mn}^{(s)}|\}^{1/(m+n)}] = (e\rho_s T_s)^{1/\rho_s}. \tag{4.3}$$

From (4.3), we get for arbitrary $\epsilon_s > 0$ and for all large $m+n$

$$(m+n)^{1/\rho_s} \{\Phi|a_{mn}^{(s)}|\}^{1/(m+n)} < \{e\rho_s(T_s + \epsilon_s)\}^{1/\rho_s}$$

for $m+n > N_s$. Setting $e\rho_s(T_s + \epsilon_s) = A_s$, $(m+n)^{1/\rho_s} \{\Phi|a_{mn}^{(s)}|\}^{1/(m+n)} < (A_s)^{1/\rho_s}$ for $m+n > N_s$, or

$$p_s \log \frac{1}{\Phi|a_{mn}^{(s)}|} > \frac{p_s}{\rho_s}(m+n) \log \frac{m+n}{A_s}.$$

Putting $s = 1, 2, \ldots, k$ and multiplying out the k inequalities thus obtained, we have

$$\prod_{s=1}^{k} p_s \log \frac{1}{\Phi|a_{mn}^{(s)}|} > \prod_{s=1}^{k} \frac{p_s}{\rho_s} (m+n)^k \prod_{s=1}^{k} \log(\frac{m+n}{A_s})$$

for all large $m+n$. Therefore, using (4.1) and (4.2), we have

$$p \log \frac{1}{\Phi|a_{mn}|} > \frac{p}{\rho}(m+n) \left\{ \prod_{s=1}^{k} \log(\frac{m+n}{A_s}) \right\}^{1/k}$$

$$\text{or} \quad \log \frac{1}{\Phi|a_{mn}|} > \frac{m+n}{\rho} \left(\prod_{s=1}^{k} \left\{ 1 - \frac{\log A_s}{\log(m+n)} \right\} \right)^{1/k} \log(m+n)$$

$$> \frac{m+n}{\rho} \left[1 - \sum_{s=1}^{k} \frac{\log A_s{}^{1/k}}{\log(m+n)} + O(\log(m+n)^{-2}) \right] \log(m+n) \quad \text{or}$$

$$(m+n)\{\Phi|a_{mn}|\}^{\rho/(m+n)} < (m+n)^{\beta}\text{with} \quad \beta = \sum_{s=1}^{k} \frac{\log A_s^{1/k}}{\log(m+n)} + O(\log(m+n)^{-2}).$$

Since $\lim_{m+n\to\infty}(m+n)^{\beta} = \prod_{s=1}^{k} A_s'^{1/k}$, where $A_s' = e\rho_s T_s$ taking lim sup, we get $\limsup_{m+n\to\infty}[(m+n)(\Phi|a_{mn}|)^{\rho/(m+n)}] \leq \prod_{s=1}^{k}(e\rho_s T_s)^{1/k}$.
Hence $e\rho T \leq \prod_{s=1}^{k}(e\rho_s T_s)^{1/k} = e\prod_{s=1}^{k}(\rho_s)^{1/k}(\prod_{s=1}^{k} T_s)^{1/k} = \frac{\rho}{p} e\prod_{s=1}^{k}(p_s T_s)^{1/k}$ or $pT \leq \prod_{s=1}^{k}(p_s T_s)^{1/k}$, and the theorem is proved.

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Received 22.VI.1990

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ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL. I. CUZA" IAŞI
Tomul XXXVII, s. I. a. Matematică, 1991, f. 4

ON THE DEFICIENCIES OF DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS

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In this paper we use the usual notations of the Nevanlinna theory of meromorphic functions as in [2]. By a meromorphic function we shall always mean a function meromorphic in the complex plane.

If f is a non-constant meromorphic function we shall denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = O(T(r,f)) \quad \text{as} \quad r\to\infty \tag{1}$$

through all values if f is of finite order and as $r\to\infty$ possibly outside a set of finite linear measure if f is of infinite order.

Also we shall always denote by α, α_0, $\alpha_1,\ldots$ etc. meromorphic functions satisfying

$$T(r,\alpha), T(r,\alpha_i(z)) = S(r,f) \tag{2}$$

Let P be the differential polynomial in f witch is the sum of a finite number of terms of the form $f^{(1)^{l_1}} f^{(2)^{l_2}} \ldots f^{(m)^{l_m}}$ where $f^{(1)}$, $f^{(2)},\ldots$ are the successive derivatives of f and l_1, $l_2,\ldots l_m$ are non-negative integers.

If $l_1+l_2+l_3+\ldots+l_m = n$ (a fixed positive integer) in every term of P then P is called a homogeneous differential polynomial in f of degree n.

In this paper we prove the results on homogeneous differential polynomials in f. Some results on differential polynomials in f are already proved by K.L.Hiong [1].

To prove the results in this paper we require the following lemma.

Lemma. *If P is a homogeneous differential polynomial in the derivatives of f of degree $n\geq 1$ then*

(3) (i) $$m(r,P/(f-a)^n) = S(r,f)$$
(4) (ii) $$m(r,P^1/(f-a)^n) = S(r,f)$$
(5) (iii) $$m(r,P^1/(P-b)) = S(r,f)$$

Proof. It is known [2] that

$$m(r,f^{(i)}/(f-a)) = S(r,f) \text{ for } i=1,2,3,\ldots$$