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ON THE INTEGRABILITY OF THE DISTRIBUTIONS ON A CR - SUBMANIFOLD

BY

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0.Introduction. A symplectic manifold basically provides a setting for mechanics. Moreover, if it is equiped with a Hermitian metric it becomes rich enough to be studied for its geometric aspects which in turn may be helpful in interpreting several phenomena relating to mechanics. More specifically it helps to study the geometric behaviour of isotropic, Lagrangian and Symplectic parts of the space. This line of thinking leads to the investigation of CR-submanifolds (or some time even more general class of submanifolds) of Symplectic (almost Kähler) manifold. In fact, if M is an isometrically immersed submanifold of a Symplectic form Ω and if Ω induces a symplectic form Ω_M of constant rank over M, then the tangent bundle is partitioned into two distributions: one Isotropic and the other Symplectic [7]. Geometrically $M \hookrightarrow \bar{M}$ is said to be a CR-submanifold of $\bar{M}$ if its tangent bundle $T(M)$ is a direct sum of two distributions D and $D^{\perp}$ such that D is holomorphic ($JD = D$) and $D^{\perp}$ is totally real ($JD^{\perp} \subseteq T^{\perp}(M)$) [1]. It is interesting to observe that D is Symplectic and $D^{\perp}$, being Ker(Ω_M), isotropic. Now it is natural to ask: under what conditions these distributions are integrable so that M is foliated by their leaves and is locally the Riemannian product of these leaves.

These considerations motivate us to study the geometry of CR-submanifold of Symplectic manifolds and thus make an endeavour to geometrize the mechanics obtained on a Symplectic (or almost Symplectic) manifolds. In pursuit of this as a first step in section 2, we have established some integrability conditions for the distribution D and $D^{\perp}$ of a CR-submanifold of an almost Hermitian manifold and have studied the geometry of the leaves of these distributions leading to a characterization for the CR-submanifold to be a CR-product in a hermitian manifold. In section

3 after having obtained the integrability conditions for the distributions of a CR-submanifold of a locally conformal Kähler manifold we again obtain a characterization for CR-product in this setting. We have also obtained a characterization for CR-product in a nearly Kähler manifold which as it, turns out, is the same as obtained by Chen in case of Kähler manifolds [3]. In our concluding section, we have studied the CR-submanifolds of symplectic manifolds and have obtained a condition for CR-product submanifolds. The two canonical distributions in this case are found to be symplectic and isotropic.

1.Preliminaries. Let M be an almost Hermitian manifold with almost complex structure J and Hermitian metric $<,>$. To describe the geometry of an almost Hermitian manifold two tensors are of fundamental importance, one is a 2-form Ω called the symplectic form and is defined as

$$\Omega(U,V) =< JU, V >, \tag{1.1}$$

where U, V are vector fields in $\bar{M}$, and the other is $(1,2)$ tensor S called Nijenhuis tensor defined as

$$S(U,V) = [U,V] + J[JU,V] + J[U,JV] - [JU,JV]. \tag{1.2}$$

For an almost Hermitian manifold the following are well known relations

$$(\bar{\nabla}_U\Omega)(V,M) =< (\bar{\nabla}_U J)V, W >, \tag{1.3}$$

$$2(\bar{\nabla}_U\Omega)(V,M) = d\Omega(U,V,W) - d\Omega(U,JV,JW) - < U, S(V,JW) >, \tag{1.4}$$

for U,V and W in $T(\bar{M})$, the tangent bundle of $\bar{M}$, where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$. We call an almost Hermitian manifold $\bar{M}$ Kählerian if $(\bar{\nabla}_U J) = 0$, for every U in $T(\bar{M})$, symplectic (or almost Kählerian) if $d\Omega = 0$ and nearly Kähler if $(\bar{\nabla}_U J)U = 0$. We call $\bar{M}$ to be locally conformal Kähler if $\bar{M}$ is Hermitian and $d\Omega = \Omega \wedge \eta$ where η is a closed one form called the Lee-form. It is well known that $\bar{M}$ is hermitian if and only if $S = 0$ and that on a Hermitian manifold $(\bar{\nabla}_U J) = 0$ and $d\Omega = 0$ are equivalent.

Let M be a Riemannian manifold isometrically immersed in $\bar{M}$. For the submanifold M of $\bar{M}$, we denote by $<,>$ the Riemannian metric on M as well as that induced on M. Let ∇ be the Riemannian connection on M induced by $\bar{V}$. The Gauss and Weingartan formula for M are given respectively by

$$\bar{\nabla}_U V = \nabla_U + h(U,V), \tag{1.5}$$

$$\bar{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{1.6}$$

for vector fields U,V tangent to M and N normal to M. h and A, are second fundamental forms related by

$$< h(U,V), N >=< A_N U, V > . \tag{1.7}$$

M is said to be totally umbilical in $\bar{M}$ if

$$h(U,V) =< U, V > H, \tag{1.8}$$

where $H = \frac{1}{n}Tr(h)$ is the mean curvature vector n being the dimension of M. M is said to be a CR-submanifold of $\bar{M}$ if it admits a differentiable distribution D such that

(1) D is invariant under J i.e., $JD_x + D_x$ for each x in M and

(2) The orthogonal complementary distribution $D^{\perp}$ is totally real i.e., $JD^{\perp}_X \subseteq T^{\perp}_X(M)$ for each $x \in M$, $T^{\perp}_X(M)$ being the normal space at x [1].

M is said to be a CR-product if it is locally the Riemannian product of the leaves of holomorphic and totally real distribution [3]. The orthogonal complement of $JD^{\perp}$ in $T^{\perp}(M)$ is and invariant subbundle of $T^{\perp}(M)$ under J and it is denoted by u. The transforms JU and JN are decomposed into tangential and normal components as

$$JU = PU + FU, \tag{1.9}$$

$$JN = tN + fN, \tag{1.10}$$

We now define

$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \tag{1.11}$$

$$(\bar{\nabla}_U F)Y = \nabla^{\perp}_U FV - F\nabla_U V, \tag{1.12}$$

$$(\bar{\nabla}_U t)N = \nabla_U tN - t\nabla^{\perp}_U N, \tag{1.13}$$

$$(\bar{\nabla}_U f)N = \nabla^{\perp}_U fN - f\nabla^{\perp}_U N, \tag{1.14}$$

for U,V tangent to M and N normal to M.

2. ***CR*-submanifolds of almost Hermitian manifolds.** Let M be a submanifold of an almost Hermitian manifold $\bar{M}$. For U, V in $T(M)$,

$$(\bar{\nabla}_U J)V = \bar{\nabla}_U JV - J\bar{\nabla}_U V,$$

which on using (1.5),(1.6),(1.9),(1.10),(1.11) and (1.12) takes the form

$$(\bar{\nabla}_U J)V = (\bar{\nabla}_U P)V - A_{FV}U - th(U,V) + (\bar{\nabla}_U F)V + h(U,PV) - fh(U,V). \tag{2.1}$$

Denoting the tangential and normal parts of $(\bar{\nabla}_U J)V$ in (2.1) by $\mathcal{P}_U V$ and $\mathcal{Q}_U V$ respectively, we get

$$\mathcal{P}_U V = (\bar{\nabla}_U P)V - A_{FV}U - th(U,V), \tag{2.2}$$

$$\mathcal{Q}_U V = (\bar{\nabla}_U F)V + h(U,PV) - fh(U,V). \tag{2.3}$$

Similarly, if $\mathcal{P}_U N$ and $\mathcal{Q}_U N$ denote the tangential and normal parts of $(\bar{\nabla}_U J)N$ for N in $T^{\perp}(M)$, then

$$\mathcal{P}_U N = (\bar{\nabla}_U t)N + PA_N U - A_{fN}U, \tag{2.4}$$

$$\mathcal{Q}_U N = (\bar{\nabla}_U f)N + h(tN,U) + FA_N U. \tag{2.5}$$

It is easy to observe that $\mathcal{P}$ and $\mathcal{Q}$ satisfy the following properties:

(p_1) $\mathcal{P}_{U+V}W = \mathcal{P}_U W + \mathcal{P}_V W$; $\mathcal{Q}_{U+V}W = \mathcal{Q}_U W + \mathcal{Q}_V W$

(p_2) $\mathcal{P}_U(V+W) = \mathcal{P}_U V + \mathcal{P}_U W$; $\mathcal{Q}_U(V+W) = \mathcal{Q}_U V + \mathcal{Q}_U W$

(p_3) $< \mathcal{P}_U V, W > = - < V, \mathcal{P}_U W >$

(p_4) $< \mathcal{Q}_U V, N > = - < V, \mathcal{Q}_U >$

(p_5) $\mathcal{P}_U JV + \mathcal{Q}_U JV = -J(\mathcal{P}_U V + \mathcal{Q}_U V)$

for all U, V and W in $T(M)$ and N in $T^{\perp}(M)$.

We now establish the integrability conditions for holomorphic and totally real distributions of a CR-submanifold of an almost Hermitian manifold in terms of the tensor field $\mathcal{P}$ and $\mathcal{Q}$.

Proposition 2.1. *Let M be a CR-submanifold of an almost Hermitian manifold $\bar{M}$. Then the holomorphic distribution D is integrable if and only if*

$$\mathcal{Q}_X Y - \mathcal{Q}_Y X = h(X,JY) - h(JX,Y) \tag{2.6}$$

for each X, Y in D.

Proof. For $\xi \in JD^{\perp}$,

$$\begin{aligned} &< h(X,JY) - h(JX,Y), \xi > = < \bar{\nabla}_X JY - \bar{\nabla}_Y JX, \xi > = \\ &= < J(\bar{\nabla}_X Y - \bar{\nabla}_Y X) + \mathcal{Q}_X Y - \mathcal{Q}_Y X, \xi > = \\ &= < F[X,Y] + \mathcal{Q}_X Y - \mathcal{Q}_Y X, \xi > \text{ or} \\ &< h(X,JY) - h(JX,Y) + \mathcal{Q}_Y X - \mathcal{Q}_X Y, \xi > = < F[X,Y], \xi > \end{aligned}$$

Hence D is integrable if and only if

$$h(X,JY) - h(Y,JX) + \mathcal{Q}_Y X - \mathcal{Q}_X Y \in u \tag{2.7}$$

for all $X, Y \in D$. But for $N \in u$

$$0 = < J(\nabla_X Y - \nabla_Y X), N > = < h(X,JY) - h(JX,Y) - \mathcal{Q}_X Y + \mathcal{Q}_Y X, N > .$$

This shows that

$$h(X,JY) - h(JX,Y) + \mathcal{Q}_Y X - \mathcal{Q}_X Y \in JD^{\perp}.$$

In view of (2.7) and the above, the proposition clearly follows.

Proposition 2.2. *Let M be a CR-submanifold of an almost Hermitian manifold $\bar{M}$. Then the totally real distribution $D^{\perp}$ is integrable if and only if*

$$\mathcal{P}_Z W - \mathcal{P}_W Z = A_{JZ}W - A_{JW}Z, \tag{2.8}$$

for each Z, W in $D^{\perp}$.

Proof. For $X \in D$, and $Z, W \in D^{\perp}$

$$\begin{aligned} &< A_{JZ}W - A_{JW}Z, X > = < \bar{\nabla}_Z JW - \bar{\nabla}_W JZ, X > = \\ &= < J(\bar{\nabla}_Z W - \bar{\nabla}_W Z) + \mathcal{P}_Z W - \mathcal{P}_W Z, X > \end{aligned}$$

or

$$< A_{JZ}W - A_{JW}Z + \mathcal{P}_W Z - \mathcal{P}_Z W, X > = < J[W,Z], X >$$

which implies that $D^{\perp}$ is integrable if and only if

$$A_{JZ}W - A_{JW}Z + \mathcal{P}_W Z - \mathcal{P}_Z W \in D^{\perp}.$$

But $A_{JZ}W - A_{JW}Z + \mathcal{P}_W Z - \mathcal{P}_Z W$ lies in D for all $Z, W \in D^\perp$ as for Z' in $D^\perp$

$$\begin{aligned} 0 &= < J(\nabla_Z W - \nabla_W Z), Z' >= \\ &= < \bar{\nabla}_Z JW - \bar{\nabla}_W JZ + \mathcal{P}_W Z - \mathcal{P}_Z W, Z' >= \\ &= < A_{JZ}W - A_{JW}Z + \mathcal{P}_W Z - \mathcal{P}_Z W, Z' > . \end{aligned}$$

This proves the proposition.

Now under these conditions Frobenius theorem guarantees the foliation of the leaves of F and $D^\perp$.

Proposition 2.3. *The leaves of D are totally geodesic in M if and only if any of the following holds.*

(a) $\mathcal{P}_X Y + th(X, Y) \in D$

(b) $\mathcal{P}_X Z + A_{JZ}X \in D^\perp$

(c) $Q_X Y - h(X, JY) \in u$

for each X, Y in D and Z in $D^\perp$.

The first part immediately follows from equation (2.2) where as the parts (b) and (c) can be seen to be equivalent to (a) using the properties (p_3) and (p_5).

Proposition 2.4. *The leaves of $D^\perp$ are totally geodesic in M if and only if any of the following holds*

(a') $\mathcal{P}_W Z + A_{JZ}W \in D^\perp$

(b') $\mathcal{P}_Z X + th(X, Z) \in D$

(c') $Q_W X - h(W, JX) \in u$

for each X in D and Z, W in $D^\perp$.

Similarly, here the first part follows from equation (2.2) and parts (b') and (c') are equivalent to (a') in view of properties (p_3) and (p_5).

Consequently, we may state:

Theorem 2.1. *Let M be a CR-submanifold of an almost Hermitian manifold $\bar{M}$. Then the following statements are equivalent*

(i) M is a CR-product.

(ii) $\mathcal{P}_U X + th(U, X) \in D$.

(iii) $\mathcal{P}_U Z + A_{JZ}U \in D^\perp$.

(iv) $Q_U X - h(U, JX) \in u$.

for each X in D, Z in $D^\perp$ and U in $T(M)$.

In view of the above theorem if the ambient space, in particular, is taken to be Kähler, these conditions reduce to the condition $A_{JZ}X = 0$ for each X in D and Z in $D^\perp$. This coincides with the condition given by Chen [3] for the characterization of CR-products in a Kähler manifold. Thus our result can be treated as the extension of Chen's result in a more general setting.

We pause to remark that as a consequence of the above discussion, a characterization for the CR-product in nearly Kähler can be obtained analogously to the one in the Kähler setting. In fact, we prove the following:

Theorem 2.2. *Let M be a CR-submanifold of a nearly Kähler manifold $\bar{M}$ with integrable distributions. Then M will be a CR-product if and only if $A_{JZ}X = 0$ for each X in D and Z in $D^\perp$.*

Proof. In view of the equation $(\bar{\nabla}_U J)V = \mathcal{P}_U V + Q_U V$, the nearly Kähler condition yields

$$\mathcal{P}_U V = -\mathcal{P}_V U \quad \text{and} \quad Q_U V = -Q_V U \tag{2.9}$$

for all U, V tangent to M. Since D is integrable, $(\bar{\nabla}_X J)Y \in D$ and $h(X, JY) = h(JX, Y)$ for X, Y in D [8]. That implies $Q_X Y = 0$. Now using Proposition 2.3 we obtain that leaves of D are totally geodesic in M if and only if

$$h(X, Y) \in u, \tag{2.10}$$

for $X, Y \in D$. On the other hand $D^\perp$ is integrable if and only if

$$< (\bar{\nabla}_W J)Z, X >= 0, \tag{2.11}$$

for Z, W in $D^\perp$ and X in D [8]. In other words,

$$< \mathcal{P}_W Z, X >= 0 \tag{2.12}$$

which on using Proposition 2.4 gives that the leaves of $D^\perp$ are totally geodesic in M if and only if

$$< h(X, W), JZ >= 0 \tag{2.13}$$

The theorem follows immediately by combining (2.10) and (2.13).

3.CR-submanifolds of locally conformal Kähler manifolds. In this section, we consider CR-submanifolds of locally conformal Kähler manifold $\bar{M}$. By a locally conformal Kähler manifold, we mean a Hermitian manifold equiped with a nondegenerate 2-form Ω such that $d\Omega = \Omega \wedge \eta$ and $d\eta = 0$ for some 1-form η, called the Lee-form [5]. A Lee vector field λ is a vector field metrically equivalent to η i.e. $< \lambda, U >= \eta(U)$ for U in $T(\bar{M})$.

An easy computation throughout equations (1.1),(1.4),(2.2) and (2.3) gives the following relations for the CR-submanifolds of locally conformal Kähler manifold

$$\mathcal{P}_U V = \frac{1}{2}[\Omega(U, V)\lambda_T - \eta(V)PU + < U, V > (J\lambda)_T + \eta(JV)U], \tag{3.1}$$

$$Q_U V = \frac{1}{2}[\Omega(U,V)\lambda_N - \eta(V)FU + < U, V > (J\lambda)_N] \tag{3.2}$$

for all U, V in $T(M)$ where the subscripts T and N denote the tangential and normal parts respectively. In view of the above relations, we obtain the integrability conditions for the leaves of holomorphic and totally real distributions of the CR-submanifold M of a locally conformal Kähler manifold $\bar{M}$ as follows:

Proposition 3.1. *Let M be a CR-submanifold of a locally conformal Kähler manifold $\bar{M}$. Then the holomorphic distribution D is integrable if and only if*

$$\Omega(X,Y)\lambda_N = h(X, JY) - h(JX, Y), \tag{3.3}$$

for all X, Y in D, λ_N being the normal part of the Lee vector field.

Proof. using Proposition 2.1 in equation (3.2), the proposition follows immediately.

Proposition 3.2. *For a CR-submanifold M of a locally conformal Kähler manifold $\bar{M}$, $D^{\perp}$ is always integrable.*

Proof. For Z, W in $D^{\perp}$, consider $\mathcal{P}_Z W - \mathcal{P}_W Z$ which by equation (3.1) is given by

$$\mathcal{P}_Z W - \mathcal{P}_W Z = \frac{1}{2}[\eta(JW)Z - \eta(JZ)W].$$

A little bit of computation shows that the right hand side of above equation is in fact equal to $[A_{JZ}W - A_{JW}Z]$. Now using Proposition 2.2, the assertion follows.

Remark. The same fact has also been established by Blair and Chen [2].

Now for a totally umbilical submanifolds, we have $h(U,V) = < U, V > H$. Therefore, a totally umbilical CR-submanifolds M of a locally conformal Kähler manifold $\bar{M}$ will satisfy

$$h(JX, Y) - h(X, JY) = 2 < JX, Y > H$$

for all X, Y in D. Further if M is foliate i.e. D is involutive, then equation (3.3) yields

$$\Omega(X,Y)\lambda_N = -2 < JX, Y > H. \tag{3.4}$$

Thus as an immediate and useful consequence of Proposition 3.1, we conclude

Proposition 3.3. *If M is a totally umbilical foliate CR-submanifold of a locally conformal Kähler manifold then*

$$H = -\frac{1}{2}\lambda_N. \tag{3.5}$$

Remark. The above result agrees with the result in Kähler setting as in that case $\lambda = 0$ and the totally umbilical submanifold become totally geodesic when D is involutive.

Now for the totally geodesicness of the leaves of D and $D^{\perp}$ we have

Proposition 3.4. *The leaves of D are totally geodesic in M if and only if*

$$2 < A_{JZ}X, Y > = \Omega(X,Y)\eta(Z) - < X, Y > \eta(JZ) \tag{3.6}$$

for every X, Y in D and Z in $D^{\perp}$.

Proof. Using the condition of totally geodesicness of the leaves of D obtained in Proposition 2.3 and equation (3.1), we get

$$< \Omega(X,Y)\lambda_T - \eta(Y)PX + < X, Y > (J\lambda)_T + \eta(JY)X + 2th(X,Y), Z > = 0$$

for X, Y in D and Z in $D^{\perp}$ or,

$$\Omega(X,Y) < \lambda, Z > + < X, Y > < J\lambda, Z > + 2 < Jh(X,Y), Z > = 0$$

or,

$$\Omega(X,Y)\eta(Z) - < X, Y > \eta(JZ) = 2 < h(X,Y), JZ > .$$

This proves the proposition.

Proposition 3.5. *the leaf of $D^{\perp}$ is totally geodesic in M if and only if*

$$2 < A_{JZ}X, W > = < Z, W > \eta(JX) \tag{3.7}$$

for each X in D and Z, W in $D^{\perp}$.

Proof. Using equation (3.1) and equations (2.6) and (2.7), we get

$$< Z, W > < J\lambda, X > + 2 < A_{JZ}W, X > = 0$$

or

$$2 < h(X,W), JZ > = < Z, W > \eta(JX)$$

which proves the proposition.

Propositions 3.4 and 3.5, immediately yield the following.

Theorem 3.1. *Let M be a CR-submanifold of a locally conformal Kähler manifold $\bar{M}$. Then M will be a CR-product if and only if*

$$2A_{JZ}X = \Omega(\lambda, Z)X + \Omega(\lambda, X)Z - \Omega(\lambda, JZ)JX \tag{3.8}$$

for X, Y in D and Z, W in $D^{\perp}$.

For the CR-product, equation (3.8) can also be established using Theorem 2.1, part (iii). The following is an interesting consequence of the above theorem.

Corollary 3.6. *Let M be a CR-submanifold of a locally conformal Kähler manifold $\bar{M}$ such that λ lies in the invariant part of the normal bundle, then M will be a CR-product if and only if $A_{JD^{\perp}}D = 0$.*

4.CR-submanifolds of symplectic manifolds. Throughout this section we take the ambient space $\bar{M}$ to be a symplectic manifold and M, as usual will denote the CR-submanifold of $\bar{M}$. That is $\bar{M}$ is equiped with a nondegenerate 2-form such that $d\Omega = 0$. We further assume that there exists a Hermitian metric $<,>$ over it such that $\Omega(X,Y) =< JX, Y >$ where J is the almost complex structure adapted with Ω. Let Ω_M be the restriction of Ω over M. Then the Ker(Ω_M) is the set of all vectors X in $T(M)$ such that $\Omega(X,Y) = 0$ for each Y in $T(M)$, which coincides with the totally real distribution is isotropic. This observation jutifies the investigation of CR-submanifolds of symplectic manifolds. It is also interesting to observe that even if M is just a submanifold of $\bar{M}$ and Ker(Ω_M) has a constant rank, the orthogonal complementary distribution of Ker(Ω_M) is symplectic which in case of CR-submanifold, is holomorphic.

The integrability of totally real distribution in this setting follows immediately from the following theorem.

Theorem 4.1. [7; p.408] *If ϕ is a differential p-form on M such that $d\phi = \phi \wedge \alpha$ where α is pfaffian form then the distribution generated by the set of sections of* Ker(ϕ) *is completely integrable.*

This means that totally real distribution on $D^{\perp}$ of M in a symplectic (even in l.C.K.) manifold is always integrable.

Lemma 4.1. *Let M be a CR-submanifold of $\bar{M}$. Then for any U, V and W in $T(M)$*

$$2 < \mathcal{P}_U V, W >=< U, JS(V,W) > . \tag{4.1}$$

Proof. Using the fact that $d\Omega = 0$ and $(\bar{\nabla}_U J)V = \mathcal{P}_U V + \mathcal{Q}_U V$, an easy computation through equations (1.3) and (1.4) immediately yields equation (4.1).

Proposition 4.2. *The holomorphic distribution D is integrable if and only if for each X, Y in D and Z in $D^{\perp}$*

$$< 2A_{JZ}Y - JS(Y,Z), JX >=< 2A_{JZ}X - JS(X,Z), JY > . \tag{4.2}$$

Proof. Using (2.7) and the property (p_5) with Lemma 4.1 we conclude that D is integrable if and only if for each X, Y in D and for each Z in $D^{\perp}$,

$$< h(X, JY) - h(JX, Y), JZ >=< \mathcal{Q}_X Y - \mathcal{Q}_Y X, JZ >$$

or

$$< A_{JZ}JY, X > -2 < A_{JZ}JX, Y >=< X, S(Y,Z) > - < Y, S(X,Z) > .$$

Replacing Y by JY and X by JX in the above equation, we get

$$< 2A_{JZ}Y - JS(Y,Z), JX >=< 2A_{JZ}X - JS(X,Z), JY > .$$

This proves the proposition.

Proposition 4.3. *The leaves of D are totally geodesic in M if and only if*

$$2 < A_{JZ}Y - JS(Y,Z), X >= 0 \tag{4.3}$$

for each X, Y in D and Z in $D^{\perp}$.

Proof. From Lemma 4.1 and equation (2.6) we find that the leaves of D are totally geodesic in M if and only if

$$< X, JS(Y,Z) > + < 2th(X,Y), Z >= 0$$

or

$$< X, JS(Y,Z) > - < 2A_{JZ}Y, X >= 0$$

which proves the proposition.

Similarly for the totally geodesicness of the leaves of $D^{\perp}$ in M, we obtain

Proposition 4.4. *The leaves of $D^\perp$ are totally geodesic in M if and only if*

$$< 2A_{JZ}Y - JS(Y,Z), W >= 0 \tag{4.4}$$

for each Y in D and Z, W in $D^\perp$.

In view of the above proposition, we may state:

Theorem 4.2. *Let M be a CR-submanifold of a symplectic manifold $\bar{M}$, then M is a CR-product if and only if*

$$2A_{JD^\perp}D = [JS(D, D^\perp)]_T \tag{4.5}$$

where subscript T denotes the tangential part.

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INFINITELY MANY RADIAL SOLUTIONS FOR A WEIGHTED SEMILINEAR ELLIPTIC EQUATION INVOLVING CRITICAL SOBOLEV EXPONENTS

BY

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0.Introduction. In a previous related paper [7] we have studied the problem

$$(P_\lambda)\begin{cases} -\frac{1}{r^\gamma}\frac{d}{dr}\left(r^\gamma\frac{du}{dr}\right) = \lambda r^\alpha u + r^\sigma|u|^{p-1}u \ \ in \ \ (0,R) \\ u(R)=0, \ \int_0^R r^\gamma|u'|^2dr < \infty, \end{cases}$$

where $\gamma > 1, \sigma, \alpha < -2$ and $p = \frac{\gamma+3+2\sigma}{\gamma-1}$ corresponds to a critical Sobolev imbedding. There we have established the existence of a positive solution of (0,1), for λ in a suitable range (depending on α and γ).

The aim of this paper is to estabish the existence of infinitely many solutions for the problem (P_λ) for all $\lambda > 0$ if γ satisfies

$$\gamma > 2\alpha + 5. \tag{0.1}$$

This result extends a previous one due to S o l i m i n i [8] in which $\sigma = \alpha = 0$ and $\gamma = N-1$, $N \geq 7$ an integer.

The strategy of our proof resembles the one used by C e r a m i, S o l i m i n i, S t r u w e [7]. However, some modifications were needed since the main hypothesis of the paper [4] namely $0 < \lambda < \lambda_1$ (λ_1 is the first eigenvalue of the radial Laplacian in $B_R(0) \subset R^N$) is no longer satisfied.

With the occasion we give a new proof of the above quoted result of S o l i m i n i [8].

In order to describe our strategy we have to consider the following eigenvalue problem

$$\begin{cases} -\frac{1}{r^\gamma}\frac{d}{dr}\left(r^\gamma\frac{du}{dr}\right) = \lambda r^\alpha u \ \ in \ \ (0,R) \\ \int_0^R r^\gamma|u'|^2dr < \infty, \ u(R)=0, \end{cases} \tag{0.2}$$