Proposition 4.4. *The leaves of $D^{\perp}$ are totally geodesic in M if and only if*

$$< 2A_{JZ}Y - JS(Y,Z), W >= 0 \tag{4.4}$$

for each Y in D and Z, W in $D^{\perp}$.

In view of the above proposition, we may state:

Theorem 4.2. *Let M be a CR-submanifold of a symplectic manifold M, then M is a CR-product if and only if*

$$2A_{JD^{\perp}}D = [JS(D, D^{\perp})]_T \tag{4.5}$$

where subscript T denotes the tangential part.

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INFINITELY MANY RADIAL SOLUTIONS FOR A WEIGHTED SEMILINEAR ELLIPTIC EQUATION INVOLVING CRITICAL SOBOLEV EXPONENTS

BY

LIVIU I. NICOLAESCU

0.Introduction. In a previous related paper [7] we have studied the problem

$$(P_\lambda)\begin{cases} -\frac{1}{r^\gamma}\frac{d}{dr}\left(r^\gamma\frac{du}{dr}\right) = \lambda r^\alpha u + r^\sigma|u|^{p-1}u \ \ in \ \ (0,R) \\ u(R)=0, \ \int_0^R r^\gamma|u'|^2dr < \infty, \end{cases}$$

where $\gamma > 1, \sigma, \alpha < -2$ and $p = \frac{\gamma+3+2\sigma}{\gamma-1}$ corresponds to a critical Sobolev imbedding. There we have established the existence of a positive solution of (0,1), for λ in a suitable range (depending on α and γ).

The aim of this paper is to estabish the existence of infinitely many solutions for the problem (P_λ) for all $\lambda > 0$ if γ satisfies

$$\gamma > 2\alpha + 5. \tag{0.1}$$

This result extends a previous one due to S o l i m i n i [8] in which $\sigma = \alpha = 0$ and $\gamma = N-1$, $N \geq 7$ an integer.

The strategy of our proof resembles the one used by C e r a m i, S o l i m i n i, S t r u w e [7]. However, some modifications were needed since the main hypothesis of the paper [4] namely $0 < \lambda < \lambda_1$ (λ_1 is the first eigenvalue of the radial Laplacian in $B_R(0) \subset R^N$) is no longer satisfied.

With the occasion we give a new proof of the above quoted result of S o l i m i n i [8].

In order to describe our strategy we have to consider the following eigenvalue problem

$$\begin{cases} -\frac{1}{r^\gamma}\frac{d}{dr}\left(r^\gamma\frac{du}{dr}\right) = \lambda r^\alpha u \ \ in \ \ (0,R) \\ \int_0^R r^\gamma|u'|^2dr < \infty, \ u(R)=0, \end{cases} \tag{0.2}$$

In [7] we have proved that the problem (0.2) has a discrete spectrum $0<\lambda_1<\lambda_2\le\lambda_3\le\ldots\le\lambda_n\le L$ each eigenvalue having finite multiplicity. We denote by $\lambda_1(R)=\lambda_1(R;\gamma,\alpha)$ the first eigenvalue of (0.2). The main steps of the proof are the following:

1.For $0<\lambda<\lambda_1(R)$ and γ satisfying (0.1) we prove as in Cerami, Solimini, Struwe [4] the existence of a solution of (P_λ) changing sign.

2.For every $\lambda>0$ and $\gamma>2\alpha+5$ (P_λ) has infinitely many solutions changing sign.

Correspondingly the paper is divided in two sections. Section 1 deals with the case $0<\lambda<\lambda_1(R)$ while section 2 deals with the general case.

1. **The case $0<\lambda<\lambda_1(R)$.** We first recall some of the results established in [7].

Let $R\in(0,+\infty]$. E_γ^R is the closure of the set $S=\{u\in C^1[0,R]\mid u\equiv 0$ in a neighbourhood of $R\}$ in the norm $\|\cdot\|_{\gamma,R}$ defined by

$$\|u\|_{\gamma,R}=\left(\int_0^R r^\gamma|u'|^2dr\right)^{1/2}.$$

$L_\theta^q(0,R)$ is the weighted space $L^q(0,R;r^\theta dr)$. Then it is known the following result (see [6] for a proof).

Imbedding Lemma. Let $\gamma>1, R<\infty$, $\theta>\gamma-2$. Then

$$E_\gamma^R\hookrightarrow L_\theta^q(0,R)\ \text{ continuously}$$

if and only if

$$\frac{\theta+1}{q}\ge\frac{\gamma-1}{2}. \tag{1.1}$$

The imbedding is compact if

$$\frac{\theta+1}{q}>\frac{\gamma-1}{2}. \tag{1.1'}$$

Moreover, if $\frac{\theta+1}{q}=\frac{\gamma-1}{2}$ then also E_γ^∞ is continuously imbedded in $L_\theta^q(0,\infty)$.

Let

$$S=S(\gamma,\theta;R)=Inf\{\|u\|_{\gamma,R}^2\mid u\in E_\gamma^R,\ \ \|u\|_{L_\theta^{\gamma^*}(0,R)}=1,\ \gamma^*=\frac{2(\theta+1)}{\gamma-1}\} \tag{1.2}$$

Then, the following facts were estabhished in [7].

Proposition 1.1. *Let $\theta>\gamma-2$. Then, $S(\gamma,\theta)$ does not depend on R and the infimum in (1.2) is achieved if and only if $R=\infty$ in which case an extremal function is*

$$\tilde U(r)=\frac{C}{(1+2^{2+\sigma})^{\frac{\sigma-1}{2+\sigma}}}=C\cdot U(r),\ \sigma=\theta-\gamma \tag{1.3}$$

where C is a normalization constant such that $\|\tilde U\|_{L_\theta^{\gamma^*}}=1$.

We consider the problem

$$-\frac{1}{r^\gamma}\frac{d}{dr}\left(r^\gamma\frac{du}{dr}\right)-\lambda r^\alpha u=r^\sigma|u|^{p-1}u\ \ in\ \ (0,R) \tag{1.4}$$

$$u\in E_\gamma. \tag{1.5}$$

Here $p=\frac{\gamma+2+2\sigma}{\gamma-1}$. By a weak solution of (1.4),(1.5) we understand a function $u\in E_\gamma$ such that

$$\int_0^R r^\gamma u'v'dr=\lambda\int_0^R r^{\gamma+\alpha}uvdr=\int_0^R r^\theta|u|^{p-1}uvdr,\ \forall v\in E_\gamma, \tag{1.6}$$

where $\theta=\sigma+\gamma$. We notè that $p+1=\gamma^*=\frac{2(\theta+1)}{\gamma+1}$ is the critical Sobolev exponent so that the imbedding $E_\gamma\hookrightarrow L_\theta^{\gamma^*}$ is not compact.

In [7] we have followed the device of Brézis - Nirenberg [3] to get the existence of a positive solution of (1.4),(1.5). Namely we have studied the minimization problem

(1.7)

$$S_\lambda(R)=S_\lambda(\gamma,\theta;R)=Inf\left\{\|u\|_\gamma^2-\lambda|u|_{L_{\gamma+\alpha}^2}^2\mid u\in E_\gamma^R,\ |u|_{L_\theta^{p+1}}=1\right\}.$$

(In the sequel, we will denote $|u|_2=|u|_{L_{\gamma+\alpha}^2}$ and $|u|_*=|u|_{L_\theta^{p+1}}$.)

We have established that if $0<S_\lambda(R)<S$ the minimization problem (1.7) has a solution which yields a positive solution of (1.4). This happens if $\gamma\ge\alpha+3$ and $0<\lambda<\lambda_1(R)$. $\lambda_1(R)$ denotes the first eigenvalue of the problem

$$\begin{cases}-\frac{1}{r^\gamma}\frac{d}{dr}\left(r^\gamma\frac{du}{dr}\right)=\lambda r^\alpha u\\ u\in E_\gamma.\end{cases} \tag{1.8}$$

By a weak solution of (1.8) it is meant a pair (u,λ) where $u \in E_\gamma \setminus \{0\}$, $\lambda \in \mathbb{R}$ such that

$$\int_0^R r^\gamma u'v'dr = \lambda \int_0^R r^{\gamma+\alpha} uvdr, \ \forall r \in E_\gamma \tag{1.9}$$

The problem thus formulated is self-adjoint and compact. It has a discrete spectrum. The smallest eigenvalue $\lambda_1(R)$ is positive and simple.

Let $\lambda \in (0, \lambda_1(R))$, $\gamma \geq \alpha + 3$. We consider the functionals $f_\lambda, I_\lambda : E_\gamma \to \mathbb{R}$ defined by

$$f_\lambda(u) = \begin{cases} \frac{|u|_*^{\gamma^*}}{\|u\|_\gamma^2 - \lambda |u|_2^2}, & u \neq 0, \\ 0, & u = 0 \end{cases}$$

$$I_\lambda(u) = \frac{1}{2}(\|u\|_\gamma^2 - \lambda|u|_2^2) - \frac{1}{\gamma^*}|u|_*^{\gamma^*}$$

f_λ is continuous and I_λ is C^1. We also note that $f_\lambda(u) = 1$ is equivalent to

$$u \neq 0, \ (\nabla I_\lambda(u), u)_{E_\gamma} = 0 \iff \|u\|_\gamma^2 - \lambda|u|_2^2 = |u|_*^{\gamma^*}. \tag{1.10}$$

The positive solution u_0 of (1.4),(1.5) described above can also be described by

$$I_\lambda(u_0) = C_0(R) = Inf\{I_\lambda(u) \mid u \in E_\gamma^R, \ u \geq 0, \ f_\lambda(u) = 1.\} \tag{1.11}$$

Indeed, for $u \geq 0$, such that $f_\lambda(u) = 1$, we have

$$I_\lambda(u) = \left(\frac{1}{2} - \frac{1}{\gamma^*}\right)(\|u\|_\gamma^2 - |u|_2^2) = \left(\frac{1}{2} - \frac{1}{\gamma^*}\right)|u_*|^{\gamma^*} \geq \left(\frac{1}{2} - \frac{1}{\gamma^*}\right) S_\lambda |u|_*^2$$

so that $|u|_*^{\gamma^*} \geq S_\lambda^{\gamma^*/(\gamma^*-2)}$. Hence

$$I_\lambda(u) \geq \left(\frac{1}{2} - \frac{1}{\gamma^*}\right) S_\lambda^{\gamma^*/(\gamma^*-2)}$$

Since u_0 is a solution of (1.4),(1.5), then $f_\lambda(u_0) = 1$ and

$$I_\lambda(u_0) = \left(\frac{1}{2} - \frac{1}{\gamma^*}\right)(\|u_0\|_\gamma^2 - \lambda|u_0|_2^2) = \left(\frac{1}{2} - \frac{1}{\gamma^*}\right)|u_0|_*^{\gamma^*} = $$
$$= \left(\frac{1}{2} - \frac{1}{\gamma^*}\right) S_\lambda |u_0|_*^2.$$

So that

$$I_\lambda(u_0) = \left(\frac{1}{2} - \frac{1}{\gamma^*}\right) S_\lambda^{\gamma^*/(\gamma^*-2)}.$$

Let us note the equality

$$C_0(R) = \left(\frac{1}{2} - \frac{1}{\gamma^*}\right) S_\lambda^{\gamma^*/(\gamma^*-2)}(R). \tag{1.12}$$

Set

$$U = U(R) = \{u \in E_\gamma^R : f_\lambda(u^\pm) = 1\}$$

Let

$$C = C(R) = Inf\,\{I_\lambda(u) \mid u \in U(R)\} \tag{1.13}$$

Then we have the following result

Proposition 1.2.

(a) *If $u_1 \in U(R)$ is such that $I_\lambda(u_1) = C(R)$ then u_1 is a weak solution of (1.4),(1.5) and moreover u_1 has a unique zero in $(0,R)$;*

(b) *If the infimum $C = C(R)$ is achieved for all R in an interval $J \subset (0, +\infty]$ such that:*

$$\lambda < \lambda_1(R), \ \forall R \in J \tag{1.14}$$

then C depends continuously on $R \in J$.

Remark 1.3. The first part of statement (a) is essentially due to S o l i m i n i [8]. For the reader's convenience, we give a proof of it.

Proof. (a) The statement that u_1 is a weak solution is equivalent to

$$\nabla I_\lambda(u_1) = 0.$$

Assume by contradiction that $\nabla I_\lambda(u_1) \neq 0$. Let us denote by the mapping

$$\eta(t,u) = u - t\nabla I(u).$$

Fix $\varepsilon > 0$ and consider the two dimensional surface σ parametrized by

$$\sigma(\alpha,\beta) = \alpha u_1^+ - \beta u_1^- \ \text{ with } \ (\alpha,\beta) \in [1-\varepsilon, 1+\varepsilon]^2.$$

Then set

$$\sigma_t(\alpha,\beta) = \eta(t, \sigma(\alpha,\beta)).$$

We notice that

$$\frac{d}{dt} I_\lambda(\sigma_t(\alpha,\beta))_{t=0} = -\|\nabla I_\lambda(\sigma(\alpha,\beta)\|_\gamma^2 < 0, \ \forall(\alpha,\beta) \in [1-\varepsilon, 1+\varepsilon]^2$$

for $\varepsilon > 0$ small enough since $\nabla I_\lambda(\sigma(1,1)) = \nabla I_\lambda(u_1) \neq 0$. Hence, for $t > 0$ small enough

$$I_\lambda(\sigma_t(\alpha,\beta)) < I_\lambda(\sigma(\alpha,\beta)), \ \forall(\alpha,\beta) \in [1-\varepsilon, 1+\varepsilon]^2. \tag{1.15}$$

Moreover, by eventually taking a smaller t with ε fixed one has

$$\begin{aligned} &(i) \begin{cases} f_\lambda(\sigma_t(\alpha,\beta)^+) < 1 & if \quad \alpha = 1-\varepsilon \\ f_\lambda(\sigma_t(\alpha,\beta)^+) > 1 & if \quad \alpha = 1+\varepsilon \end{cases} \\ &(ii) \begin{cases} f_\lambda(\sigma_t(\alpha,\beta)^-) < 1 & if \quad \beta = 1-\varepsilon \\ f_\lambda(\sigma_t(\alpha,\beta)^-) > 1 & if \quad \beta = 1+\varepsilon \end{cases} \end{aligned} \tag{1.16}$$

Indeed, these inequalities hold for $t = 0$. Let us consider e.g. (1.16)(ii) $\alpha = 1-\varepsilon$. Then, for $t = 0$

$$f_\lambda(\sigma_0(\alpha,\beta)^+) = \frac{|\alpha u_1^+|_*^{\gamma^*}}{\|\alpha u_1^+\|_\gamma^2 - \lambda|u_1^+|_2^2} = \alpha^{\gamma^*-2} f_\lambda(u_1^+) = \alpha^{\gamma^*-2} < 1.$$

By continuity, we get (1.16).

From Miranda's theorem (see [5]), we deduce that there exists $(\alpha_t, \beta_t) \in [1-\varepsilon, 1+\varepsilon]^2$ such that

$$f_\lambda(\sigma_t(\alpha_t,\beta_t)^{\pm}) = 1$$

Hence, $u_t = \sigma_t(\alpha_t, \beta_t) \in U(R)$.

From (1.15) we infer

$$I_\lambda(u_t) = I_\lambda(\sigma_t(\alpha_t,\beta_t)) < I_\lambda(\sigma(\alpha_t,\beta_t)) \tag{1.17}$$

But

$$I(\sigma(\alpha,\beta)) \le I_\lambda(u_1), \quad \forall(\alpha,\beta) \in [1-\varepsilon, 1+\varepsilon]^2. \tag{1.18}$$

Indeed,

$$\begin{aligned} I(\sigma(\alpha,\beta)) &= \frac{\alpha^2}{2}(\|u_1^+\|_\gamma^2 - \lambda|u_1^+|_2^2) - \frac{\alpha^{\gamma^*}}{\gamma^*}|u_1^+|_*^{\gamma^*} + \\ &+ \frac{\beta^2}{2}(\|u_1^-\|_\gamma^2 - \lambda|u_1^-|_2^2) - \frac{\beta^{\gamma^*}}{\gamma^*}|u_1^-|_*^{\gamma^*} = \\ &= Ag(\alpha) + Bg(\beta) \le (A+B)g(1) = I_\lambda(u_1), \end{aligned}$$

where $A = \|u_1^+\|_\gamma^2 - \lambda|u_1^+|_2^2 = |u_1^+|_*^{\gamma^*}$, $B = \|u_1^-\|_\gamma^2 - \lambda|u_1^-|_2^2 = |u_1^-|_*^{\gamma^*}$, $g(t) = \frac{t^2}{2} - \frac{t^{\gamma^*}}{\gamma^*} \le g(1)$, $\forall t \in R$.

From (1.17) and (1.18) , we infer

$$I_\lambda(u_t) < I_\lambda(u_1)$$

which contradicts the choice of u_1.

To prove the second part of this statement let us assume the contrary, namely $Z = \{r \in (0,R) \mid u_1(r) = 0\}$ contains more than one point. u_1 is a weak solution of (1.4),(1.5) and by classical regularity results, u_1 satisfies (1.4) classically in $(0,R)$. If Z has a cluster point $\bar{r} \in (0,R)$, then $u_1(\bar{r}) = u_1(\bar{r}) = 0$ and by the uniqueness of the Cauchy problem

$$\begin{aligned} &-\frac{1}{r^\gamma}\frac{d}{dr}\left(r^\gamma \frac{du}{dr}\right) = \lambda r^\alpha u + r^\sigma |u|^{p-1}u \\ &u_1(\bar{r}) = u_1'(\bar{r}) = 0 \end{aligned}$$

we would infer $u_1 \equiv 0$. Hence, the points of Z can be arranged in a strictly decreasing manner

$$R = r_0 > r_1 > \dots$$

such that u_1 has constant sign on each (r_{i+1}, r_i).

Moreover, at each r_i, u_1 changes its sign. Indeed let L be the operator $u \mapsto -\frac{1}{r^\gamma}\frac{d}{dr}(r^\gamma \frac{du}{dr})$. If u_1 does not change sign at a point r_i then we can assume e.g. $u_1 > 0$ on (r_{i+1}, r_{i-1}). Hence

$$\begin{cases} Lu_1 = g \ge 0 & \text{on } (r_{i+1}, r_{i-1}) \\ u_1 = 0 & \text{for } r = r_{i-1}, r_i, r_{i+1}. \end{cases}$$

The strong maximum principle then yields a contradiction since it implies $u_1(r_i) > 0$.

Now let the function $v \in E_\gamma$ be defined by

$$v(r) = \begin{cases} u_1(r), & r \in (r_2, R) \\ -u_1(r), & r \in (0, r_2). \end{cases}$$

Then since the restriction u_1^i of u_1 to each (R_{i+1}, r_i) satisfies $f_\lambda(u_1^i) = 1$ we infer $f_\lambda(v^{\pm}) = 1$ and clearly $I_\lambda(v) = I_\lambda(u_1)$.

Consequently, v is a minimizing function so v satisfies (1.4),(1.5) and by the previous considerations v sould change sign at each r_i. However, this does not happen for $r_1 = r_2$.

(b) Let $R \in J$ and $R_n \to R$. Set $\mu_n = \frac{R_n}{R}$. Let $u \in U(R)$ such that $I_\lambda(u) = c(R)$ and let $\bar{r}$ be its unique zero in $(0, R)$. We may assume

$$u|_{(0,\bar{r})} = u^+ \quad and \quad u|_{(\bar{r},R)} = -u^-.$$

We consider the functions

$$u_n(r) = \alpha_n u^+(\mu_n r) - \beta_n u^-(\mu_n r)$$

where α_n and β_n are chosen so that $f_\lambda(u_n^\pm) = 1$ i.e. $u_n \in U(R)$.

Since $\mu_n \to 1$ one can see that $\alpha_n, \beta_n \to 1$.

Therefore, $c(R_n) \le I_\lambda(u_n) = I_\lambda(u) + o(1)$ so that

$$\limsup c(R_n) \le c(R). \tag{1.19}$$

Conversely, let $u_n \in U(R_n)$ such that $I_\lambda(u_n) = c(R_n)$ and let $\bar{r}_n$ be its unique zero. We claim that there exists positive constants c_1, c_2 such that

$$c_1 < \|u_n^\pm\|_\gamma < c_2 \tag{1.20}$$

Indeed, $I_\lambda(u_n) = c(R_n) \le c(R) + o(1)$ by (1.19)).

By (1.14), we infer (since $f_\lambda(u_n) = 1$).

$$I_\lambda(u_n) = \left(\frac{1}{2} - \frac{1}{\gamma^*}\right)(\|u_n\|_\gamma^2 - |u_n|_2^2) \ge const.\|u_n\|_\gamma^2$$

$$\|u_n\|_\gamma^2 = \|u_n^+\|_\gamma^2 + \|u_n^-\|_\gamma^2 \le const.\ independent\ of\ n.$$

$$|u_n^\pm|_*^{\gamma^*} = \left(\|u_n^\pm\|_\gamma^2 - \lambda|u_n|_2^2\right) \ge const.\ |u_n^\pm|_\gamma^2\ const. \ge\ S|u_n^\pm|^2$$

so that $|u_n^\pm|_* \ge const.$. (We have implicitely used in these considerations the fact that $\lambda_1(R)$ depends continuously on R.)

We consider

$$v_n(r) = \alpha_n u_n^+\left(\frac{r}{\mu_n}\right) - \beta_n u_n^-\left(\frac{r}{\mu_n}\right)$$

where $\alpha_n = \alpha_n(\|u_n\|) > 0$, $\beta_n = \beta_n(\|u_n\|)$ are chosen such that

$$f_\lambda(v_n^\pm) = 1 \quad i.e. \quad v_n \in U(R).$$

Such a choice is always possible and from (1.20) we infer

$$\lim_n \alpha_n = \lim_n \beta_n = 1 \quad since \quad \lim_n \mu_n = 1.$$

Then, $I_\lambda(v_n) = I_\lambda(u_n) + o(1)$. Hence

$$C(R) \le I_\lambda(v_n) = I_n(u_n) + o(1) = c(R_n) + o(1)$$

$$C(R) \le \liminf c(R_n) \tag{1.21}$$

From (1.19) and (1.21), we deduce our result. q.e.d.

Let $U^1(R) = \{u \in E_\gamma^R \mid \exists r_1 \in (0, R) : u(r) \ge 0 \text{ on } (0, r_1), u(r) < 0$ on (r_1, R) and $f_\lambda(u^\pm) = 1\}$.

Let $C_1(R) = \inf\limits_{U^1(R)} I_\lambda(u)$. Clearly, $U^1(R) \subset U(R)$ so that

$$C(R) \le C_1(R)$$

If the infimum $C(R)$ were achieved then by Proposition 1.2(a), any minimizing function would belong to $U^1(R)$ in which case

$$C(R) = C_1(R)$$

The same arguments used by C e r a m i, S o l i m i n i, S t r u w e[4], sect.3, yield the following result.

Proposition 1.4. *If $0 < \lambda < \lambda_1(R)$ and*

$$C(R) < C_0(R) + \left(\frac{1}{2} - \frac{1}{\gamma^*}\right) S^{\frac{\gamma^*}{\gamma^*-2}} \tag{1.22}$$

then the infimum $C(R)$ is achieved and moreover, $C(R) = C_1(R)$.

Since the proof of the above results does not essentially differ from the proof of the above quoted result of [4] we omit it.

The aim of our next result is to prove that (1.22) always takes place if $\gamma > 2\alpha + 5$.

Proposition 1.5. *If $0 < \lambda < \lambda_1(R)$ and $\gamma > 2\alpha + 5$ then*

$$C(R) \le C_1(R) < C_0(R) + \left(\frac{1}{2} - \frac{1}{\gamma^*}\right) S^{\frac{\gamma^*}{\gamma^*-2}} \tag{1.22'}$$

The crucial fact that we will rely on is contained in

Lemma 1.6. *Let $\mu > 0$, $\gamma > \alpha + 3$ and $0 < \lambda < \lambda_1(\mu)$. Then*

$$\exists C_1, C_2 > 0 : S_\lambda(\mu) \le S + C_1 \varepsilon^{\gamma-1} - C_2 \mu^{\alpha+2} \varepsilon^{2+\alpha}, \ \forall \varepsilon \in (0,1) \tag{1.23}$$

Proof of Lemma 1.6. We consider the mapping

$$E^1_\gamma \overset{T_\mu}{\mapsto} E^\mu_\gamma$$

defined by

$$u(r) \mapsto v(\rho) = \mu^{-\frac{\gamma-1}{2}} u\left(\frac{\rho}{\mu}\right).$$

T_μ is an izometry between these two spaces and also between

$$L^{\gamma^*}_\theta(0,1) \xrightarrow{\sim} L^{\gamma^*}_\theta(0,\mu).$$

Via this mapping the minimization problem

$$S_\lambda(1) = \inf\{\|u\|^2_\gamma - \lambda |u|^2_2 \mid u \in E^1_\gamma, \ |u|_* = 1\} \tag{1.7$_1$}$$

is transformed in the problem

$$S_{\mu^{-\alpha-2}\lambda}(1) = Inf\{\|u\|^2_\gamma - \mu^{-\alpha-2}\lambda |u|^2_2 \mid u \in E^\mu_\gamma, \ |u|_* = 1\} \tag{1.7$_\mu$}$$

Moreover, if (u_n) is a minimizing sequence for $(1.7)_1$ then the sequence $(T_\mu u_n)$ is minimizing for (1.7). Hence, we infer $S_\lambda(1) = S_{\mu^{-\alpha-2}\lambda}(\mu)$ or

$$S_\lambda(\mu) = S_{\mu^{\alpha-2}\lambda}(\mu)(1) \tag{1.24}$$

In [7] we have established that for $\gamma > \alpha + 3$, $0 < \lambda < \lambda_1(1)$, $\exists C_1$, $C_2 > 0 : S_\lambda(1) \le S + C_1 \varepsilon^{\frac{\gamma-1}{2+\sigma}} - \lambda C_2 \varepsilon^{\frac{2+\alpha}{2+\sigma}}$.

Making the substitutions $\varepsilon \mapsto \varepsilon^{2+\sigma}$, $\lambda \mapsto \mu^{\alpha+2}\lambda$ we get (1.23), if we notice also $\lambda_1(\mu) = \mu^{-\alpha-2}\lambda_1(1)$ so that $0 < \mu^{\alpha+2}\lambda < \lambda_1(1) \iff 0 < \lambda < \lambda_1(\mu)$.

q.e.d.

Corollary 1.7. *Let $\mu > 0$, $\gamma > \alpha + 3$ and $0 < \lambda < \lambda_1(\mu)$. Then*

$$\exists C_1, C_2 > 0 : C_0(\mu) \le \left(\frac{1}{2} - \frac{1}{\gamma^*}\right) S^{\frac{\gamma^*}{\gamma^*-2}} + C_1 \varepsilon^{\gamma-1} - C_2 \mu^{\alpha+2} \varepsilon^{\alpha+2}, \ \forall \varepsilon \in (0,1). \tag{1.25}$$

Proof. It follows from (1.23) if we recall (1.12):
$C_0(\mu) = \left(\frac{1}{2} - \frac{1}{\gamma^*}\right) S^{\frac{\gamma^*}{\gamma^*-2}}(\mu)$. *q.e.d.*

Proof of Proposition 1.5. Let $u_0 \in E^R_\gamma$ be a solution of the minimization problem

$$C_0(R) = Inf\{I_\lambda(u) \mid u \in E^R_\gamma, \ u \ge 0, \ f_\lambda(u) = 1\}.$$

From our previous remarks such a solution always exists if $\gamma \ge \alpha + 3$ and $0 < \lambda < \lambda_1(R)$.

Let $\eta : R_+ \to R_+$ be the continuous function defined by $0 \le \eta \le 1$, $\eta \equiv 0$ on $[0,1]$, $\eta \equiv 1$ on $[2, +\infty)$. η is linear on $(1,2)$. Set $\eta_\varepsilon(t) = \eta\left(\frac{t}{\varepsilon}\right)$ and $u_\varepsilon(r) = n_\varepsilon(r) u_0(r)$. Then

$$u_\varepsilon(r) = \begin{cases} u_0(r), & r > 2\varepsilon \\ 0, & r < \varepsilon. \end{cases}$$

Let $\alpha_\varepsilon > 0$ such that $f_\lambda(\alpha_\varepsilon u_\varepsilon) = 1$. Since $u_\varepsilon \to_{\varepsilon searrow 0} u_0$ in E_γ and since $f_\lambda(u_0) = 1$ we infer $\alpha_\varepsilon \to 1$ as $\varepsilon \searrow 0$.

Set $v^1_\varepsilon = \alpha_\varepsilon u_\varepsilon$. An easy computation shows that for all $\varepsilon > 0$ small enough,

$$I_\lambda(v^1_\varepsilon) \le C \|u_0\|_\infty \varepsilon^{\gamma-1} + I_\lambda(u_0) \ (U_0 \in L^\infty \text{by the results of [7]}) \tag{1.26}$$

Now let us consider $v^0_\varepsilon \in E^\varepsilon_\gamma$ such that $v^0_\varepsilon \ge 0, f_\lambda(v^0_\varepsilon) = 1$ and $I_\lambda(v^0_\varepsilon) = C_0(\varepsilon)$. Let $v_\varepsilon \in U(R)$ be defined by

$$v_\varepsilon(r) = \begin{cases} -v^0_\varepsilon(r), & r \in (0,\varepsilon) \\ v^1_\varepsilon(r), & r \in (\varepsilon, R) \end{cases}$$

$$I_\lambda(v_\varepsilon) = I_\lambda(v^0_\varepsilon) + I_\lambda(v^1_\varepsilon) = C_0(\varepsilon) + I_\lambda(v^1_\varepsilon)$$

(by (1.25),(1.26)) $\left(\frac{1}{2} - \frac{1}{\gamma^*}\right) S^{\frac{\gamma^*}{\gamma^*-2}} + C_1 \varepsilon^{\gamma-1} - C_2 \varepsilon^{4+2\alpha} + C_0 \|u_0\|_\infty \varepsilon^{\gamma-1} + I_\lambda(u_0) \le \left(\frac{1}{2} - \frac{1}{\gamma^*}\right) S^{\frac{\gamma^*}{\gamma^*-2}} + C_0(R) + C_3 \varepsilon^{\gamma-1} - C_4 \varepsilon^{4+2\alpha}$. If $\gamma > 2\alpha + 5$ then for $\varepsilon > 0$ small enough we have

$$C_1(R) \le I_\lambda(v_\varepsilon) < \left(\frac{1}{2} - \frac{1}{\gamma^*}\right) S^{\frac{\gamma^*}{\gamma^*-2}} + C_0(R). q.e.d.$$

Remark 1.8. The idea of the proof is due to S o l i m i n i [8]. We can now state the main result of this section.

Theorem 1.9. *Let $0 < \lambda < \lambda_1(R)$ and $\gamma > 2\alpha + 5$. Then the infimum $C(R)$ is achieved by a function $u^1(r) \in U^1(R)$ which is also a solution changing sign of problem (1.4),(1.5).*

Remark 1.10. If one considers the following problem

$$(1.27)\qquad \begin{cases} -\frac{1}{r^\gamma}\frac{d}{dr}\left(r^\gamma \frac{du}{dr}\right) - \lambda r^\alpha u = r^\sigma |u|^{p-1}u, & r \in (a,b), \\ u(a) = u(b) = 0, & 0 < a < b < \infty \end{cases}$$

then we can define in a straightforward way the space $E_\gamma(a,b)$ which is now compactly imbedded in any $L^q_\theta(a,b)$, $1 \le q < \infty$.

Therefore, the exponent p is no longer critical and we can prove the following result.

Theorem 1.9'. *Let $0 < \lambda < \lambda_1(a,b)$. Then the infimum $C(a,b)$ is achieved by a function $u^1 \in U^1(a,b)$ which is also a solution of (1.21).*

Here $\lambda_1(a,b), C(a,b), U(a,b), U^1(a,b)$ are defined by a natural analogy with $\lambda_1(R), C(R), U(R)$ and $U^1(R)$. Moreover since p is not critical the proof of Theorem 1.9′ follows from a simple compactness argument couplet with an analogue version of Proposition 1.2(a), so we omit it.

2.The general case. Let $\lambda > 0$. For every integer $k \ge 1$ we set

$$(2.1)\quad M_k = \{u \in E^R_\gamma \mid \exists 0 = r_0 < r_1 < \ldots < r_m = R,\ m \le k,\ u(r_j) = 0,$$
$$f_\lambda(u_j) = 1,\ (-1)^j u_j \ge 0,\ r_{j+1} - r_{j-1} \le \frac{2R}{k},$$
$$j = 1, \ldots, m-1 \ \ where\ \ u_j = u|_{\Omega_j},\ \Omega_j = (r_{j-1}, r_j),\}$$

Let $C_k = \inf\limits_{M_k} I_\lambda$. An easy argument shows that $C_k > 0$. We will need a lower bound for $\lambda_1(a,b)$ which is the case of Laplacian is given by Faber-Krahn inequality (B a n d l e [2]).

Lemma 2.1. *$\lambda_1(a,b) \to \infty$ as $b - a \to 0$, $0 < a < b < R$.*

Proof. We will derive a lower bound for $\lambda_1(a,b)$ using the usual variational characterizations of eigenvalues. For $u \in E_\gamma(a,b)$

$$u(r) = \int_a^r u'(s)ds$$

$$u^2(r) = (\int_0^r u'(s)ds)^2 \le (b-a)^2 \int_a^b |u'|^2 dr \le$$

$$\le (b-a)\int_a^b r^{-\gamma} dr \|u\|^2_\gamma$$

$$\int_a^b r^{\gamma+\alpha} u^2(r) dr \le (b-a)^2 \int_a^b r^{-\gamma} dr \cdot \int_a^b r^{\gamma+\alpha} dr \|u\|^2_\gamma =$$
$$= C(a,b)|u\|^2_\gamma$$

where $C(a,b) \to 0$ as $a, b \to \ell_0(0,R)$. Hence

$$(2.2)\qquad \lambda_1(a,b) \ge \frac{1}{C(a,b)}$$

If $a, b \to \ell_0 \in (0,R)$ then $\lambda_1(a,b) \to \infty$ (cf. (2.2)). If $a, b \to 0$ that $(a,b) \subset (0,b)$, and therefore

$$\lambda_1(a,b) \ge \lambda_1(b) = b^{-\alpha-2}\lambda_1(1) \to \infty . q.e.d.$$

Remark 2.2. From the proof of the above lemma it is easily seen that given $\lambda > 0$ in (2.1) we always can choose $k(\lambda)$ large enough so that

$$(2.3)\qquad 0 < 2\lambda < \lambda_1(r_{j-1}, r_{j+1}),\ \forall\, j = 1, \ldots, m-1.$$

We want to prove that for $k \ge k(\lambda)$ the infimum C_k is achieved and any minimizing function $u_k \in M_k$ is a solution of (1.4),(1.5). This device then yields infinitely many solutions of (1.4),(1.5) changing sign.

Let $(u^n) \subset M_k$ be a minimizing sequence for C_k. We may assume that all u^n have the same number of nodes corresponding to the definition of M_k, else we can extract a subsequence with this property. Hence we have a sequence of partitions

$$(2.4)\qquad \Delta^n : 0 = r^n_0 < r^n_1 < \ldots < r^n_m = R,\ m \le k.$$

Set $\Omega^n_j = (r^n_{j-1}, r^n_j)$, $S^n_j = (r^n_{j-1}, r^n_{j+1})$, $u^n_j = u^n|_{\Omega^n_j}$. We also may assume $r^n_j \to_n r_j$ and hence we have a limit partition

$$(2.5)\qquad \Delta : 0 = r_0 \le r_1 \le r_2 \le \ldots \le r_m = R,\ \Omega_j = (r_{j-1}, r_j).$$

Definition 2.3. *The sequence u^n is* **non-concentrating** *if $0 = r_0 < r_1 < \ldots \le r_m = R$.*

Lemma 2.4. *Let $k \ge k(\lambda)$. If there exists a non-concentrating minimizing sequence for C_k then there exists $u \in M_k$ such that $I_\lambda(u) = C_k$ and u is a solution of (1.4),(1.5).*

Proof. Let (u^n) as above and assume $\{u^n\}$ is non-concentrating. Then, there exist partitions (2.4) and (2.5).

Let u_j be a solution of the minimizing problem

$$(2.6)\qquad Inf\{I_\lambda(u) \mid u \in E_\gamma(\Omega_j),\ u \ge 0,\ f_\lambda(u) = 1\}$$

Such u_j exists cf. [7], Remark 1.10 and Remark 2.2.

Given two bounded open intervals $I, J \subseteq R$ there exists an unique increasing affine function $A : I \to J$ which is one-to-one and onto.

Then, we take as above the affine mappings

$$S_i^n : (r_{i-1}, r_i) \to (r_{i-1}^n, r_i^n)$$

We consider the functions

$$\bar{u}(r) = (-1)^j u_j(r),\ r \in (r_{j-1}, r_j),\ j = 1, \ldots, m \tag{2.7}$$

$$v_j^n(r) = \alpha_j^n u^n(S_j^n(r)),\ r \in \Omega_j^n,\ j = 1, \ldots, m. \tag{2.8}$$

where α_j^n are chosen such that $f_\lambda(v_j^n) = 1$.

Since $\Delta^n \to \Delta$ we infer as in Proposition 1.2 (b), that $\alpha_j^n \to 1$ as $n \to \infty$, $\forall j = 1, \ldots, m$. Set

$$v^n(r) = v_j^n(r),\ r \in \Omega_j^n,\ j = 1, \ldots, m.$$

Then, $v^n \in M_k$ and from the definition of u_j we infer

$$I_\lambda(u_j) \leq I_\lambda(v_j^n) \tag{2.9}$$

so that

$$I_\lambda(\bar{u}) = \sum_{j=1}^{m} I_\lambda(u_j) < \sum_{j=1}^{m} I_\lambda(v_j^n) = I_\lambda(v^n). \tag{2.10}$$

Again, the same reasoning used in Proposition 1.2 (b), leads to

$$I_\lambda(u_j^n) = I_\lambda(v_j^n) + o(1),\ j = 1, \ldots, m$$

Hence

$$I_\lambda(u^n) = I_\lambda(v^n) + o(1) \text{ as } n \to \infty \tag{2.11}$$

From (2.10) and (2.11), we deduce

$$I_\lambda(u) \leq \lim I_\lambda(u^n) = C_k$$

and since $\bar{u} \in M_k$ we infer

$$I_\lambda(u) = C_k$$

By theorems 1.9, 1.9′, (which can be applied since for $k \geq k(\lambda)$, $\lambda < \frac{1}{2}\lambda_1(S_i)$ for every $S_i = (r_{i-1}, r_{i+1})$, $i = 1, \ldots, m-1$) we infer that on each S_i u satisfies weakly the equation

$$-\frac{1}{r^\gamma}\frac{d}{dr}\left(r^\gamma \frac{du}{dr}\right) - \lambda r^\alpha \bar{u} = r^\sigma |u|^{p-1} u$$
$$u \in E_\gamma(S_i).$$

A standard globalization argument (using the partition of unity subordinated to (S_i)) shows that u satisfies weakly (1.4),(1.5). q.e.d.

Our next result shows that non-concentrating minimizing sequences really exist.

Proposition 2.5. *Let $\lambda > 0$, $\gamma > 2\alpha + 5$ and $k \geq k(\lambda)$ such that (2.3) holds. Then, there exists a non-concentrating minimizing sequence for C_k.*

Proof. Let as above $(u^n) \in M_k$ be a minimizing sequence for C_k and let

$$\Delta^n : 0 = r_0^n < r_1^n < \ldots < r_m^n = R,\ m \leq k \tag{2.4}$$

the corresponding sequence of nodes. We may assume $\Delta^n \to \Delta$ where

$$\Delta : 0 = r_0 < r_1 < \ldots < r_m = R \tag{2.5}$$

The index j is said to be *degenerate* if $r_{j-1} = r_j$.

The degenerate indexes if they exist have the following important property.

Lemma 2.6. *If j is degenerate, then $r_j = 0$.*

This fact is essentially proved in S o l i m i n i [8], Lemma 6. we don't give a proof of it (which is simple), but we merely point out its underlying philosophy: if $(r_{j+1}^n), (r_j^n)$ concentrate at a point $r_j \in (0, R]$ then for n large the exponent p is no longer critical in $(r_{j-1}^n), (r_j^n)$ and the norm of the imbedding

$$E_\gamma(\Omega_j^n) \hookrightarrow L_\theta^{\gamma^*}(\Omega_j^n) \text{ tends to zero.}$$

If there exist degenerate indexes we define

$$j = \max\{i > 0 \mid i \text{ is a degenerate index}\} \tag{2.12}$$

We claim that for n large

$$r_{j+1}^n \le \frac{2R}{k} \tag{2.13}$$

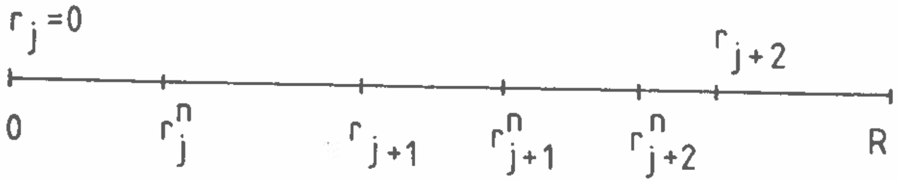

Fig.1.

Indeed, (see fig.1),

$$r_{j+1}^n = r_{j+1} + o(1) = r_{j+1} - r_j + o(1) <$$
$$< r_{j+2} - r_j + o(1) \le \frac{2R}{k} + o(1).$$

Let $\bar{u}$ be a solution of the minimization problem

$$Inf\{I_\lambda(u) \mid u \in U^1(r_{j+1})\}. \tag{2.14}$$

Since $r_{j+1} \le \frac{2R}{k}$, $k \ge k(\lambda)$, it follows by (2.3) that $0 < \lambda < \lambda_1(r_{j+1})$. Also $\gamma > 2\alpha + 5$, and from Theorem 1.9, we infer that such a solution exists. We may assume $(-1)^{j+1}\bar{u}$ is positive in a neighbourhood of 0, else we make the substitution $\bar{u} \mapsto -\bar{u}$.

Let $\mu_n = \frac{r_{j+1}^n}{r_{j+1}}$. We consider the functions $v^n : (0, r_{j+1}^n) \to I\!R$ defined by

$$v^n(r) = \begin{cases} \alpha_n \bar{u}(\mu_n r), & r < \frac{S_1}{\mu_n} \\ \beta_n u(\mu_n r), & r > \frac{S_1}{\mu_n} \end{cases}$$

where S_1 is the unique zero of u in $(0, r_{j+1})$ and the constants α_n, β_n are chosen so that

$$f(v^{n\pm}) = 1.$$

Such a choice is always possible since $f_\lambda(u^\pm) = 1$ and in a usual manner we deduce

$$I_\lambda(v^n) = I_\lambda(\bar{u}) + o(1) = C(r_{j+1}) + o(1).$$

(by Proposition 1.2 (b)) $= C(r_{j+1}^n) + o(1) \le I_\lambda(u^n|_{(0,r_{j+1}^n)}) + o(1)$ i.e.

$$I_\lambda(v^n) \le I_\lambda(u^n|_{(0,r_{j+1}^n)}) + o(1). \tag{2.15}$$

We define a sequence $(w^n) \subset M_k$ by

$$D^n : 0 = S_0^n < S_1^n < S_2^n < \ldots < S_t^n = R,\ t = m - j + 1$$
$$S_1^n = \frac{S_1}{\mu_n},\ S_2^n = r_{j+1}^n,\ S_3^n = r_{j+2}^n, \ldots, S_t^n = r_m^n = R \ \ (fig.2)$$

$$\begin{cases} w^n(r) = u^n(r), & \text{if } r \in (S_2^n, R) \\ w^n(r) = v^n(r), & \text{if } r \in (0, S_2^n) \end{cases} \tag{2.16}$$

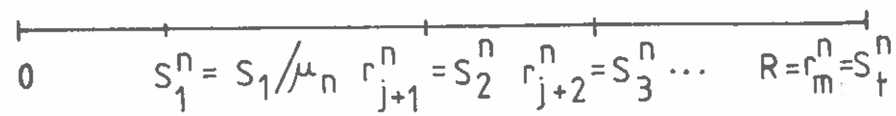

Fig.2.

By (2.13), $S_2^n - S_0^n = r_{j+1}^n \le \frac{2R}{k}$. Also, $S_3^n - S_1^n = r_{j+2}^n - S_1^n = r_{j+2}^n - S_1 + o(1) < r_{j+2} = r_{j+2} - r_j \le \frac{2R}{k}$ and $S_{i+1}^n - S_{i-1}^n \le \frac{2R}{k}$, $i = 3, \ldots, t-1$ for n large.

From our choice of v^n we infer (taking into account the above remarks) that $w^n \in M_k$ and

$$\begin{aligned} I_\lambda(w^n) &= I_\lambda(w^n|_{(0,r_{j+1}^n)}) + I_\lambda(w^n|_{(r_{j+1}^n,R)}) = \\ &= I_\lambda(v^n|_{(0,r_{j+1}^n)}) + I_\lambda(u^n|_{(0,r_{j+1}^n,R)}) \le \\ &\le I_\lambda(u^n|_{(0,r_{j+1}^n)}) + I_\lambda(u^n|_{(0,r_{j+1}^n,R)}) + o(1) = \\ &= I_\lambda(u^n) + o(1). \end{aligned} \tag{2.15}$$

Since (u^n) is minimizing we deduce that (w^n) is also minimizing. Moreover, $D^n \to D$ where

$$D : 0 = S_0 < S_1 = r_{j+1} < r_{j+2} < \ldots < r_m = R$$

so that (w^n) is also non-concentrating. q.e.d.

We can now state the following result

Corollary 2.7. *Let $\lambda > 0$, $\gamma > 2\alpha + 5$ and $k \geq k(\lambda)$. Then the infimum C_k is achieved and any extremal function $u_k \in M_k$ is a solution of (1.4),(1.5). Now let $\lambda > 0$, $\gamma > 2\alpha + 5$ and $k_0 > k(\lambda)$. We can find a solution of (1.4),(1.5) $u_0 \in M_{k_0}$. Its zeros are*

$$0 = r_1^0 < r_2^0 < \ldots < r_m^0 = R.$$

Let $d_0 = \min\{r_i^0 - r_{i-1}^0 \mid i = 1, \ldots, m\}$. Next we choose $k_1 > k_0$ such that $\frac{2R}{k_1} < d_0$. Applying Corollary 2.7 for this k_1 we get a solution $u_1 \in M_k$ of (1.4),(1.5) and a corresponding $d_1 < d_0$.

We iterate this device. We get a sequence of integers

$$k_0 < k_1 < \ldots < k_n < \ldots \to \infty \text{ a decreasing sequence}$$

$$d_0 > d_1 > \ldots > d_n > \ldots \to 0$$

and a sequence of solutions of (1.4),(1.5) $u_n \in M_k$ of (1.4) such that d_n is the distance between the closest consecutive zeros of u_n. Clearly, u_n are pairwise distinct. We can now state the main result of our paper.

Theorem 2.8. *Let $\lambda > 0$, $\gamma > 2\alpha + 5$. Then problem (1.4) has an infinity of solutions (u_n) changing sign.*

Remark 2.9.

a) It can be shown as in Proposition 1.2 (b), that $\lim_n I_\lambda(u_n) = +\infty$.

b) Corollary 2.7 allows us to apply the arguments used in C e r a m i, S o l i m i n i, S t r u w e [4]. Namely we consider the subset $\bar{M}_k \subset M_k$. $\bar{M}_k = \{u \in M_k \mid$ the partition corresponding to u contains exactly k nodes $0 = r_0 < r_1 < \ldots < r_k = R\}$. Then let

$$\bar{c}_k = Inf\{I_\lambda(u) \mid u \in \bar{M}_k\}$$

There exists by Corollary 2.7 an integer $k_0 > k(\lambda)$ and a function $u_0 \in \bar{M}_k$ such that

$$I_\lambda(u_0) = \bar{c}_{k_0}$$

Then, as in Lemma 4.1, of [4], we infer

$$\bar{c}_{k+1} < \bar{c}_k + \left(\frac{1}{2} - \frac{1}{r^\lambda}\right) S^{\frac{\gamma^*}{\gamma^* - 2}}, \ \forall k \geq k_0 \tag{2.17}$$

Condition (2.17) will then allow us to establish as in [4] that any minimizing sequence for $\bar{c}_k$ is *non concentrating* and will produce a solution of (1.4),(1,5) in $\bar{M}_k$, $\forall k \geq k_0$.

Thus, we can state a more precise result.

Theorem 2.10. *Let $\lambda > 0$, $\gamma > 2\alpha + 5$. Then, for every $k \geq k(\lambda)$ the problem (1.4),(1.5) has at least one solution u_k with exactly k zeros. Moreover,*

$$\lim_k I_\lambda(u_k) = +\infty.$$

c) The previous result is optimal in the sense that given an integer $k \geq 1$, one can prove the existence of a real number $\bar{\lambda}_k > 0$, such that if $\gamma > 2\alpha + 5$ and $\lambda > \bar{\lambda}_k$ each solution of (1.4),(1.5) cannot have less than k zeros.

Indeed for any $d \in (0, R)$ let us define

$$F(d) = \sup\{\lambda_1(a,b) \mid 0 \leq a < b \leq R, \ b - a \geq d\}$$

Then, $F : (0, R) \to (0, +\infty]$ is a decreasing function.

Let us prove that $F(d) < \infty$, $\forall d \in (0, R)$. Given $0 \leq a < b \leq R$, $b - a \geq d$. We consider $u \in E_\gamma(a,b)$ as in Fig.3.

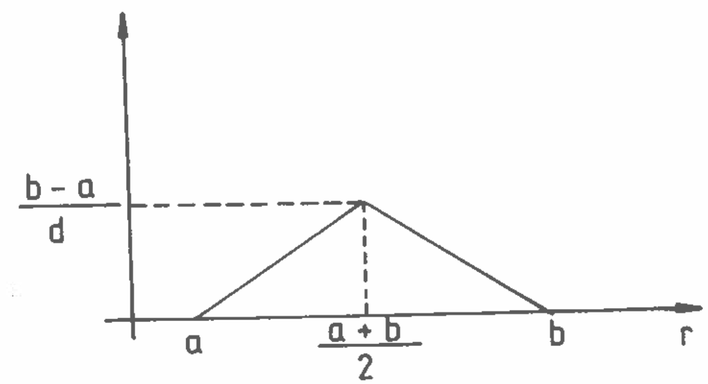

Fig.3

Then

$$\lambda_1(a,b)\frac{\|u\|_\gamma^2}{|u|^2_{L^2_{\gamma+\alpha}}} =$$

$$= \frac{\int_a^b r^\gamma dr}{\int_a^{\frac{a+b}{2}} r^{\gamma+\alpha}(r-a)dr + \int_{\frac{a+b}{2}}^b r^{\gamma+\alpha}\left(r - \frac{a+b}{2}\right)^2 dr} =$$

$$= G(a,b)$$

$G(a,b)$ is continuous and finite on the compact

$$K = \{(a,b) \mid 0 \leq a < b \leq R, \ b - a \geq d\}$$

Therefore, $F(d) \le \max_k G(a,b) < \infty$.

For any $k \ge 1$, let $\bar\lambda_k = F(\frac{R}{k})$. If there exists $\lambda > \lambda_k$ such that (1.4),(1.5) (with this λ) has a solution with less than k zeros, say

$$r_0 = 0 < r_1 < r_2 < \ldots < r_m = R,\ m \le k$$

then there exists $i = 1, \ldots, m$ such that $r_i - r_{i-1} \ge \frac{R}{k}$ so that

$$\lambda_1(r_{i-1}, r_i) \le \lambda_k < \lambda$$

Thus, we discover that the problem

$$\begin{cases} -\frac{1}{r^\gamma}\frac{d}{dr}\left(r^\gamma \frac{du}{dr}\right) - \lambda r^\alpha u = r^\sigma |u|^{p-1}u, & u \in (r_{i-1}, r_i) \\ u \in E_\gamma(r_{i-1}, r_i), \quad \lambda > \lambda_1(r_{i-1}, r_i) \end{cases}$$

has a positive solution.

However, testing this equations with φ_1 - a positve eigenfunction corresponding to λ_1 it is easily seen that is not possible for $\lambda > \lambda_1(r_{i-1}, r_i)$.

d) The following extensions of a result due to A t k i n s o n, B r é z i s, P e l e t i e r [1] seems extremely probable.

Conjecture 1. If $1 < \gamma \le 2\alpha + 5$, there exists $\lambda^* = \lambda^*(\gamma)$ such that if $0 < \lambda < \lambda^*$ (1.4),(1.5) has not solutions changing sign.

Conjecture 2. Given $k \ge 1$, there exists $\bar\gamma = \bar\gamma(\alpha)$, such that if $1 < \bar\gamma < \bar\gamma(\alpha)$ there exists $\lambda_k^* = \lambda_k^*(\gamma)$ with the following property: for every $\lambda \in (0, \lambda_k^*)$ (1.4),(1.5) has not solutions with more than k zeros.

e) The results of this paper can be translated in a P.D.E. language as we did in [6] and [7]. We leave to the reader the task of this translation.

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