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A THEORY OF THERMOELASTIC MICROPOLAR MATERIALS WITH VOIDS

BY

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1.Introduction. The basic equations of the elastic porous solids in which the skeletal material is elastic and interstices are void of material was established by N u n z i a t o and C o w i n in [1]. This theory was extended to thermoelastic materials by I e ş a n [2]. In what follows such a kind of material will be called *elastic material with voids*. Our purpose is to establish the basic equations of nonlinear theory of thermoelastic micropolar materials with voids. Later we will give the basic equations of the linear theory and an uniqueness theorem.

2.Statement of the problem. Throughout this paper, a fixed rectangular Cartesian coordinate system $0x_i$ $(i = 1,2,3)$ is used. The usual summation and differentiation conventions are emploied: latin subscripts range over the integers $(1,2,3)$, summation over repeated subscripts is implied and a subscript preceded by a comma denotes partial differentiations with respect to the corresponding Cartesian coordinate.

Let the body occupy at time $t = 0$ a properly regular region B_0 of Euclidean three-dimensional space, bounded by a piece-wise smooth surface ∂B_0. The configuration of this time we call *reference configuration*. A generic point x of the body has in the reference configuration the coordinates X_A $(A = \overline{1,3})$. At the time t the body occupies the region B bounded by the surface ∂B (*actual configuration*) and the same point x has the coordinates x_i $(i = 1,2,3)$.

The motion of the body from the reference configuration to actual configuration is given by

$$x_i = x_i(X_A,t),\ \varphi_i = \varphi_i(X_A,t),\ T = T(X_A,t),\ \nu = \nu(X_A,t), \tag{2.1}$$

where φ_i are the components of the microrotation vector, T is the absolute temperature of the body and ν is a new kinematical variable which Nunziato and Cowin [1] called *volumetric distribution function* and we will call it simply *porosity*. It is designated as a ratio between the density of the body ρ and the density of the material skeletal γ, that is

$$\nu = \frac{\rho}{\gamma} \tag{2.2}$$

In the general case, ν depends of the coordonates of the particles and of the time.

All the functions involved in (2.1) are considered being of class C^p ($p \geq 2$). Such a deformation of the body is possible if and only if

$$J = \det(x_{i,A}) > 0. \tag{2.3}$$

This means that first three equations of (2.1) can be inverted and we have

$$X_A = X_A(x_i, t), \quad (\forall)\, t > 0. \tag{2.4}$$

Let us finally note that the components of the displacement vector $\bar{u}$ are given by

$$u_i = x_i - X_i. \tag{2.5}$$

3.Basic equations of the nonlinear theory. In [3] Green and Rivlin showed that we can obtain the balance of the mass, the balance of impulse and the balance of the kinetical moment from the balance of energy. Following their method, let us apply the balance of energy for an arbitrary, regular region V of the body in actual configuration, bounded by a piece-wise smooth surface ∂V. So we have

$$\begin{aligned} &\int_V \rho_0(v_i\dot{v}_i + I_{ij}\omega_j\dot{\omega}_i + k\dot{\nu}\ddot{\nu})dV + \int_V \rho_0\dot{\varepsilon}dV = \\ &= \int_V \rho_0(f_iv_i + \ell_i\omega_i + \ell\dot{\nu} + s)dV + \int_{\partial V}(t_iv_i + m_i\omega_i + h\dot{\nu} + q)d\sigma \end{aligned} \tag{3.1}$$

where

$$v_i = \dot{u}_i, \quad \omega_i = \dot{\varphi}_i \tag{3.2}$$

and ε is the internal energy, f_i - the components of the body forces, ℓ_i - the components of the body couples, ℓ - the extrinsic equilibrated body force,

t_i - the components of the stress vector, s - the heat supply per unit time, m_i - the components of couple-stress vector, h - the extrinsic equilibrated stress, q - the heat flux, k - the equilibrated inertia, ρ_0 - the density of the body in the reference configuration and I_{ij} - the components of the tensor of inertia which satisfies the symmetry relation

$$I_{ij} = I_{ji} \tag{3.3}$$

Let us assume that in the actual configuration we can arrive from the reference configuration through a prescribed motion. We add to this motion a constant velocity translation. Because f_i, ℓ_i,, ℓ, s, $\dot{\varepsilon}$, t_i, m_i, h, q are not changed when v_i are replaced by $v_i + a_i$ (a_i being constants), we have

$$\begin{aligned} &\int_V [\rho_0(v_i + a_i)\dot{v}_i + I_{ij}\omega_j\dot{\omega}_i + k\dot{\nu}\ddot{\nu}]dV + \int_V \rho_0\dot{\varepsilon}dV = \\ &= \int_V \rho_0(f_i(a_i + v_i) + \ell_i\omega_i + \ell\dot{\nu} + s)dV + \\ &+ \int_{\partial V}(t_i(v_i + a_i) + m_i\omega_i + h\dot{\nu} + q)d\sigma. \end{aligned} \tag{3.4}$$

From (3.1),(3.4) and due to the fact that a_i are arbitrary we obtain the balance of impulse, that is:

$$\int_V \rho_0\dot{v}_i dV = \int_V \rho_0 f_i dV + \int_{\partial V} t_i d\sigma. \tag{3.5}$$

By a classical reasoning, from (3.5) it results:

$$t_i = t_{ji}n_j \tag{3.6}$$

and

$$t_{ji,j} + \rho_0 f_i = \rho_0\ddot{u}_i \tag{3.7}$$

where n_i are the components of outward unit normal to ∂V. Taking into account (3.6),(3.7) the balance of energy (3.4) becomes

$$\begin{aligned} &\int_V \rho_0(I_{ij}\dot{\omega}_i\omega_j + k\dot{\nu}\ddot{\nu})dV + \int_V \rho_0\dot{\varepsilon}dV + \int_V t_{ji}v_{i,j}dV = \\ &= \int_V \rho_0(\ell_i\omega_i + \ell\dot{\nu} + s)dV + \int_{\partial V}(m_i\omega_i + h\dot{\nu} + q)d\sigma. \end{aligned} \tag{3.8}$$

It is not difficult to obtain from (3.8) an equation similar to (3.7)

$$(m_i - m_{ji}n_j)\omega_i + (h - h_i n_i)\dot{\nu} + (q - q_i n_i) = 0 \tag{3.9}$$

where m_{ji} are the components of the couple stress tensor, h_i are the components of the equilibrated stress vector and q_i are the components of heat flux vector.

The divergence theorem and the relations (3.8),(3.9) give us the local form of the balance of energy:

$$\begin{aligned}(m_{ji,j} + \rho_0\ell_i - \rho_0 I_{ij}\dot{\omega}_j)\omega_i + (h_{i,i} + \rho_0\ell - \rho_0 k\ddot{\nu})\dot{\nu} + \\ + t_{ji}v_{i,j} + q_{i,i} + \rho_0 s + m_{ji}\omega_{i,j} + h_i\dot{\nu}_{,i} - \rho_0\dot{\varepsilon} = 0.\end{aligned} \tag{3.10}$$

Let us superpose on the the considered motion, a constant unghiular velocity rigid rotation. In (3.10) we will replace ω_i by $\omega_i + b_i$ (b_i constants) and v_i by $v_i + \varepsilon_{ijk}b_j x_k$. From (3.10) and the arbitrariety of b_i we have:

$$m_{ji,j} + \varepsilon_{irs}t_{rs} + \rho_0\ell_i = \rho_0 I_{ij}\ddot{\varphi}_j. \tag{3.11}$$

Due to (3.11), the equation (3.10) becomes

$$\rho_0\dot{\varepsilon} = t_{ij}\dot{e}_{ij} + m_{ij}\dot{\chi}_{ij} + g\dot{\nu} + h_i\dot{\nu}_{,i} + g_{i,i} + \rho_0 s, \tag{3.12}$$

where

$$e_{ij} = u_{j,i} + \varepsilon_{ijk}\varphi_k, \quad \chi_{ij} = \varphi_{i,j} \tag{3.13}$$

and

$$g = h_{i,i} + \rho_0\ell - \rho_0 k\ddot{\nu}. \tag{3.14}$$

From (3.9), taking into account that ω_i and $\dot{\nu}$ are arbitrarily chosen, we obtain

$$m_i = m_{ji}n_j, \quad h = h_i n_i, \quad q = q_i n_i. \tag{3.15}$$

The entropy inequality applied on the region V provides us

$$\int_V \rho_0\dot{\eta}\,dV \geq \int_V \rho_0\frac{s}{T}dV + \int_{\partial V}\frac{q}{T}d\sigma \tag{3.16}$$

From (3.16) and (3.15) we have

$$\rho_0 T\dot{\eta} - \rho_0 s - q_{i,i} + \frac{1}{T}q_i T_{,i} \geq 0 \tag{3.17}$$

Like in the classical elasticity, from (3.17),(3.12) we obtain

$$\rho_0(\dot{\psi} + \eta\dot{T}) + t_{ij}\dot{e}_{ij} + m_{ij}\dot{\chi}_{ij} + g\dot{\nu} + h_i\dot{\nu}_{,i} + \frac{1}{T}q_i\theta_{,i} \geq 0 \tag{3.19}$$

where $\psi = \varepsilon - \eta T$ is free energy function, and $\theta = T - T_0$.

A thermoelastic micropolar material with voids is that material for which there exist the following constitutive relations:

$$\begin{cases} \psi = \hat{\psi}(e_{ij}, \chi_{ij}, \nu, \nu_{,i}, \theta, \theta_{,i}) \\ t_{rs} = \hat{t}_{rs}(e_{ij}, \chi_{ij}, \nu, \nu_{,i}, \theta, \theta_{,i}) \\ m_i = \hat{m}_i(e_{jk}, \chi_{jk}, \nu, \nu_{,j}, \theta, \theta_{,j}, n_j) \\ g = \hat{g}(e_{ij}, \chi_{ij}, \nu, \nu_{,i}, \theta, \theta_{,i}) \\ \rho_0\eta = \rho_0\hat{\eta}(e_{ij}, \chi_{ij}, \nu, \nu_{,i}, \theta, \theta_{,i}) \\ h = \hat{h}(e_{ij}, \chi_{ij}, \nu, \nu_{,i}, \theta, \theta_{,i}, n_i) \\ q = \hat{q}(e_{ij}, \chi_{ij}, \nu, \nu_{,i}, \theta, \theta_{,i}, n_i) \end{cases} \tag{3.20}$$

The entropy inequality (3.19) is possible only if

$$\psi = \hat{\psi}(e_{ij}, \chi_{ij}, \nu, \nu_{,i}, \theta) \tag{3.21}$$

and

$$t_{ij} = \frac{\partial\hat{\sigma}}{\partial e_{ij}}, \quad m_{ij} = \frac{\partial\hat{\sigma}}{\partial\chi_{ij}}, \quad g = \frac{\partial\hat{\sigma}}{\partial\nu}, \quad \rho_0\eta = -\frac{\partial\hat{\sigma}}{\partial\theta}, \quad h_i = \frac{\partial\hat{\sigma}}{\partial\nu_{,i}}, \tag{3.22}$$

where

$$\hat{\sigma} = \rho_0\hat{\psi}. \tag{3.23}$$

So, the inequality (3.19) reduces to

$$q_i\theta_{,i} \geq 0. \tag{3.24}$$

The basic equations of the nonlinear theory of thermoelastic micropolar materials with voids are: (3.7),(3.11),(3.12),(3.14) and (3.24), with the corresponding boundary and initial conditions.

4.Basic equations of the linear theory. In the linear theory we take the free internal energy in the form

$$\begin{aligned}\hat{\sigma} = \frac{1}{2}A_{ijrs}e_{ij}e_{rs} + B_{ijrs}e_{ij}\chi_{rs} + \frac{1}{2}C_{ijrs}\chi_{ij}\chi_{rs} - \\ - D_{ij}e_{ij}\theta - E_{ij}\chi_{ij}\theta - \frac{1}{2}a\theta^2 + F_{ij}e_{ij}\nu + G_{ij}\chi_{ij}\nu + H_{ijk}e_{ij}\nu_{,k} + \\ + K_{ijr}\chi_{ij}\nu_{,r} + L_i\nu\nu_{,i} + \frac{1}{2}M_{ij}\nu_{,i}\nu_{,j} + \frac{1}{2}b\nu^2 - c\nu\theta - d_i\nu_{,i}\theta.\end{aligned} \tag{4.1}$$

The constitutive relations (3.22) become

$$
(4.2)\qquad \begin{cases} t_{ij} = A_{ijrs}e_{rs} + B_{ijrs}\chi_{rs} + F_{ij}\nu + H_{ijk}\nu_{,k} - D_{ij}\theta \\ m_{ij} = B_{rsij}e_{rs} + C_{ijrs}\chi_{rs} + G_{ij}\nu + K_{ijr}\nu_{,r} - F_{ij}\theta \\ \rho_0\eta = D_{ij}e_{ij} + E_{ij}\chi_{ij} + a\theta + c\nu + d_i\nu_{,i} \\ g = F_{ij}e_{ij} + G_{ij}\chi_{ij} + L_i\nu_{,i} + b\nu - c\theta \\ h_i = H_{rsi}e_{rs} + K_{rsi}\chi_{rs} + L_i\nu + M_{ij}\nu_{,j} - d_i\theta \\ q_i = k_{ij}\theta_{,j} \end{cases}
$$

the characteristic coefficients of the material satisfying the following conditions:

$$
(4.3)\qquad A_{ijrs} = A_{rsij}, \quad C_{ijrs} = C_{rsij}, \quad M_{ij} = M_{ji}.
$$

In the linear theory, the energy equation reduces to

$$
(4.4)\qquad \rho_0 T_0 \dot{\eta} = q_{i,i} + \rho_0 s.
$$

Basic equations of the linear theory of thermoelastic micropolar materials with voids are: (3.7),(3.11),(4.2) and (4.4).

The boundary conditions may be chosen in different ways. So, if S_i $(i = \overline{1,8})$ are parts of the boundary ∂B of the region B, which satisfy:

$$
(4.5)\qquad \begin{cases} \bar{S}_1 \cup S_2 = \bar{S}_3 \cup S_4 = \bar{S}_5 \cup S_6 = \bar{S}_7 \cup S_8 = \partial B \\ S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = S_7 \cap S_8 = \emptyset \end{cases}
$$

we can have:

$$
(4.6)\qquad \begin{cases} u_i = \tilde{u}_i \text{ on } S_1 \times [0,t_0), \quad t_i = \tilde{t}_i \text{ on } S_2 \times [0,t_0) \\ \varphi_i = \tilde{\varphi}_i \text{ on } S_3 \times [0,t_0), \quad m_i = \tilde{m}_i \text{ on } S_4 \times [0,t_0) \\ \nu = \tilde{\nu}_i \text{ on } S_5 \times [0,t_0), \quad h = \tilde{h} \text{ on } S_6 \times [0,t_0) \\ \theta = \tilde{\theta} \text{ on } S_7 \times [0,t_0), \quad q = \tilde{q} \text{ on } S_8 \times [0,t_0) \end{cases}
$$

where $\tilde{u}_i$, $\tilde{t}_i$, $\tilde{\varphi}_i$, $\tilde{m}_i$, $\tilde{\nu}$, $\tilde{h}$, $\tilde{\theta}$, $\tilde{q}$ are given and t_0 is a fixed moment.

The initial conditions have the form

$$
(4.7)\qquad \begin{cases} u_i(x,0) = u_i^\circ(x), \ \dot{u}_i(x,0) = v_i^\circ(x), \ x \in B \cup \partial B \\ \varphi_i(x,0) = \varphi_i^\circ(x), \ \dot{\varphi}_i(x,0) = \omega_i^\circ(x), \\ \nu(x,0) = \nu^\circ(x), \ \dot{\nu}(x,0) = \gamma^\circ(x), \\ \theta(x,0) = \theta^\circ(x) \end{cases}
$$

where u_i°, v_i°, φ_i°, ω_i°, ν°, γ°, θ° are given functions on $B \cup \partial B$.

5. Uniqueness theorem. In the linear theory of thermoelastic micropolar materials with voids, let (3.7),(3.11),(3.14),(4.2),(4.4),(4.6),(4.7) a boundary value problem.

Theorem. *If*

$$
(5.1)\qquad \rho_0 > 0, \quad k > 0, \quad a > 0, \quad T_0 > 0,
$$

$$
(5.2)\qquad \rho_0 I_{ij}\xi_i\xi_j > 0, \quad (\forall)\ \xi_i \neq 0
$$

$$
(5.3)\qquad \begin{aligned} 2W &= A_{ijrs}e_{ij}e_{rs} + 2B_{ijrs}e_{ij}\chi_{rs} + C_{ijrs}\chi_{ij}\chi_{rs} + 2F_{ij}e_{ij}\nu + \\ &+ 2G_{ij}\chi_{ij}\nu + 2H_{ijk}e_{ij}\nu_{,k} + 2K_{ijr}\chi_{ij}\nu_{,r} + 2L_i\nu\nu_{,i} + b\nu^2 + \\ &M_{ij}\nu_{,i}\nu_{,j} \geq 0 \ (\forall)\ e_{ij} = e_{ji}, \ \chi_{ij}, \ \nu, \ \nu_{,i} \end{aligned}
$$

then the boundary and initial value problem has at most a solution.

Proof. From (4.2) we obtain

$$
(5.4)\qquad t_{ij}\dot{e}_{ij} + m_{ij}\dot{\chi}_{ij} + h_i\dot{\nu}_{,i} + g\dot{\nu} + \rho_0\dot{\eta}\theta = \frac{d}{dt}\left(W + \frac{1}{2}a\theta^2\right).
$$

On the other hand, from (3.7),(3.11),(3.14),(4.4) we have

$$
(5.5)\qquad \begin{aligned} & t_{ij}\dot{e}_{ij} + m_{ij}\dot{\chi}_{ij} + h_i\dot{\nu}_{,i} + g\dot{\nu} + \rho_0\dot{\eta}\theta = \rho_0(f_i - \ddot{u}_i)\dot{u}_i + \\ & + (\rho_0\ell_i - \rho_0 I_{ij}\ddot{\varphi}_j)\dot{\varphi}_i - \rho_0(\ell + k\ddot{\nu})\dot{\nu} + \frac{1}{T_0}\rho_0 s\theta + (t_{ji}v_i)_{,j} + \\ & + (m_{ij}\omega_j)_{,i} - (h_i\dot{\nu})_{,i} + \frac{1}{T_0}(q_i\theta)_{,i} - \frac{1}{T_0}q_i\theta_{,i} \end{aligned}
$$

From divergence theorem, (3.6) and (3.15) it results:

$$
(5.6)\qquad \begin{aligned} & \int_B (t_{ij}\dot{e}_{ij} + m_{ij}\dot{\chi}_{ij} + h_i\dot{\nu}_{,i} + g\dot{\nu} + \rho_0\dot{\eta}\theta)dV = \\ & \doteq \int_B \rho_0(f_i\dot{u}_i + \ell_i\dot{\varphi}_i - \ell\dot{\nu} + \frac{1}{T_0}s\theta)dV + \\ & + \int_{\partial B}(t_i\dot{u}_i + m_i\dot{\varphi}_i - h\dot{\nu} + \frac{1}{T_0}q\theta)d\sigma - \int_B \rho_0(\dot{u}_i\ddot{u}_i + k\dot{\nu}\ddot{\nu})dV - \\ & - \int \rho_0 I_{ij}\dot{\varphi}_i\dot{\varphi}_j dV - \frac{1}{T_0}\int_B q_i\theta_{,i}dV \end{aligned}
$$

If U is total energy in $[0, t_0)$, given by

$$U = \frac{1}{2}\int_B (\rho_0 \dot{u}_i \dot{u}_i + \rho_0 I_{ij}\dot{\varphi}_i\dot{\varphi}_j + \rho_0 k\dot{\nu}\dot{\nu} + 2W + a\theta^2)dV \tag{5.7}$$

then from (5.7),(5.6),(5.5) we have

$$\begin{aligned}&\dot{U} - \int_B \rho_0(f_i\dot{u}_i + \ell_i\dot{\varphi}_i - \ell\dot{\nu} + \frac{1}{T_0}s\theta)dV - \int_{\partial B}(t_i\dot{u}_i + m_i\dot{\varphi}_i - \\ &- h\dot{\nu} + \frac{1}{T_0}s\theta)d\sigma = -\frac{1}{T_0}\int_B q_i\theta_{,i}dV \le 0\end{aligned} \tag{5.8}$$

If the boundary value problem has two different solutions $(u_i, \varphi_i, \nu_i, T)$ corresponding to the same boundary conditions and initial values, then their difference will satisfy the same system with homogeneous boundary and initial conditions. If $U_1(t)$ is the total energy corresponding to the difference of the solutions we have

$$\dot{U}_1 - \int_B \rho_0(f_i^*\dot{u}_i^* + \ell_i^*\dot{\varphi}_i^* - \ell^*\nu^* + \frac{1}{T_0}s^*\theta^*)dV = -\frac{1}{T_0}\int_B q_i\theta^*_{,i}dV.$$

But $f_i^* = 0$, $\ell_i^* = 0$, $\ell^* = 0$, $s^* = 0$ and

$$\dot{U}_1^* \le 0 \tag{5.9}$$

that means that $U_1(t) \le U_1(0) = 0$. From (5.1)-(5.3) it follows that $U_1(t) = 0$, $\forall t \in [0, t_0)$, $u_i^* = 0$, $\varphi_i^* = 0$, $\nu^* = 0$, $\theta^* = 0$ and the proof is completed.

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HERMITIAN OPERATORS ON C^*-ALGEBRAS

BY

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1.Introduction. We first introduce some basic notions with which we work.

Definition 1.1. *Let E be an complex Banach algebra with unit element I and dual space E'. An element a of E is called* **hermitian** *if its numerical range $V(E,a)$, where*

$$V(E,a) = \{f(a) : f \in E',\ f(I) = 1 = \|f\|\}$$

is a subset of the real field $\mathbb{R}$, or equivalently, if $\|exp(i\alpha a)\| = 1$ for all $\alpha \in \mathbb{R}$.

Definition 1.2. *A C^*-algebra is a complex Banach algebra X with an involution $*$, satisfying $\|x^*x\| = \|x\|^2$ for all x in X. X is said to be an* **irreducible** *or a* **primitive C^*-algebra** *if there exists a faithful irreducible representation from X into $BL(X)$, the Banach algebra of all bounded opertors on a suitable Hilbert space K. An ideal P in X is called* **primitive** *if it is the kernel of an irreductible representation. A* **structure C^*-algebra** *is a C^*-algebra X such that the set, Prim(X), of all primitive ideals in X is a Hausdorff space when endowed with the Jacobson topology.*

For details, refer to [3],[10].

Definition 1.3. *A* **derivation** *on a C^*-algebra X is a linear mapping D of X to itself such that $D(ab) = aDb + (Da)b$, for any $a, b \in X$. D is called* **an inner derivation** *if there exists an element $c \in X$ such that $Da = ca - ac$, for all $a \in X$. A derivation D is said to be a* **star derivation** *if $Da^* = -(Da)^*$, (a^* is the adjoint of a).*

Throughout this paper, unless mentioned otherwise, X denotes a complex unital C^*-algebra with all derivations inner and the algebraic element